جامعة ديالى

كلية التربية المقداد

المادة: تحليل عقدي

المرحلة : الرابعة

اعداد

م. وفاء فايق غيدان

Chaptor(1)

Complex number

(1-1) Sum and prodacts

Defintion: A complex number is an ordered pairs (x,y) of real numbers x and y.

Nototions:- 1- we denode to the complex number by z.

2- we denode to the set of all complex number by $\mathbb{C}=\{(x,y): x,y \subseteq R\}$

Defintion:- Let $z=(x,y)$ be complex number.

- x is called the real part of z and denot by $Re(z)=x$.
- y is called the Imaginary parts of the z denoded By $Im(z)=y$

Defintion:- If $x=0$ Then $z=(0,y)$ is called apure Imaginary number

Example:- Let $z=(-3,1)$.

$Re\, z=-3$, $Im\, z=1$

Remarks:- For all $x \in R$, $x=(x,0)\in\mathbb{C}$, Then $R \in \mathbb{C}$.

Definitions:- We define the complex $i=(0,1)$.

Propostion: A- Let $i=(0,1)$ Then $i^2=-1$

B- Eevery complex number $z=(x,y)$ can be written as $z=x+iy$.

Defintion:- The two operation on $\mathbb{C}$ as follow

1- $(X_1, y_1) + (X_2, y_2) = (X_1 + X_2, y_1 + y_2)$

2- $(X_1, y_1) \cdot (X_2, y_2) = (X_1 X_2 - y_1 y_2, X_1 y_2 + X_2 y_1)$

3)- $Z_1 = Z_2$, $X_1 = X_2$, $y_1 = y_2$

Broof- pro.(1 - A):- $i^2 = -1$?

Let $i = (0,1)(0,1) = (0-1, 0+0) = -1$

Proof - pro (1-B):- $Z = (x,y) = x + iy$?

Let $Z = (x,y)$ Then

$x \in R$ $\quad x = (x,0)$

$y \in R$ $\quad y = (y,0)$

$iy = (0,1) \cdot (y,0) = (0, 0+y) = (0,y)$

$x + iy = (x,0) + (0,y) = (x,y)$

Example:- Find $Z_1 + Z_2$ and $Z_1 \cdot Z_2$ iff

$Z_1 = 2 - 3i$, $Z_2 = -2 + i$

$(2,-3)$ or $(-2,1)$

$Z_1 + Z_2 = (2 - 3i) + (-2 + i) = (0 - 2i) = (0,2)$

$Z_1 \cdot Z_2 = (2 - 3i) \cdot (-2 + i) = -1 + 8i = (-1, 8)$

(1-2)- Algebrac properties:-

Theorem -1- :- The following algebraic properties hold for all Z_1, Z_2 and Z_3 be a complex number:-

1- $Z_1 + Z_2 = Z_2 + Z_1$ (Commutative Low +)

2- $Z_1 \cdot Z_2 = Z_2 \cdot Z_1$ (Commutative low $\cdot$)

③

3- $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$ (associative +)

4- $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$ (associative of $\cdot$)

5- $z_1 \cdot (z_2+z_3) = z_1 \cdot z_2 + z_1 z_3$ [Distribution]

6- $z + 0 = 0 + z = z,\ 0 = (0,0)$ (identity of +)

7- $z \cdot 1 = 1 \cdot z = z,\ 1 = (1,0)$ (identity of $\cdot$)

8- The additive invers of $z = x+iy \Rightarrow -z = -x-iy$ and denoted by z^*. $z \cdot z^* = z^* \cdot z = 0$

Not :- every complex number has an additive invers.

9- The multiplicative invers of $z = x+iy,\ z \neq 0$ is $[z \cdot z^{-1} = z^{-1} \cdot z]$ on con show that

$$z^{-1} = \left(\frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2} \right)$$

Not :- every non zero complex number has a multipicative inverse.

Proof :- 2- $z_1 \cdot z_2 = z_2 \cdot z_1$

$L.H.S = (x_1+iy_1)(x_2+iy_2)$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$

$= (x_2+iy_2)(x_1+iy_1)$

$= z_2 \cdot z_1$

R.H.S = H.W

5) $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$

$L.H.S = z_1 \cdot (z_2+z_3)$ ~~$= z_1 z_2 + z_1 \cdot z_3$~~

$= (x_1+iy_1)((x_2+iy_2)+(x_3+iy_3))$

$= x_1(x_2+iy_2) + x_1(x_3+iy_3) + iy_1(x_2+iy_2)$

$+ iy_1(x_3+iy_3)$

$= x_1 x_2 + i x_1 y_2 + x_1 x_3 + i x_1 y_3 + i y_1 x_2 + (i)^2 y_1 y_2 + i y_1 x_3$

$+ (i)^2 y_1 y_3$

$= x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3 + i(x_1y_1 + x_1y_3 + y_1x_2$
$+ y_1x_3)$

R.H.S = H.w

proof. 9 $- zz^{-1} = z z^{-1} = 1$

$L.H.S = (x+iy)\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)$

$= \frac{x^2}{x^2+y^2} - \frac{ixy}{x^2+y^2} + \frac{iyx}{x^2+y^2} - \frac{(i)^2y^2}{x^2+y^2}$

$= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = 1$

R.H.S = H.w

and 1, ~~6~~, 3, 4, 7, 8 H.w

Example 1: Find z^{-1} if $z = i$?

Sol:- $z^{-1} = i^{-1} = \frac{1}{i} \cdot \frac{-i}{-i} = -i$

$\therefore z^{-1} = -i$

2- Find z^{-1} if $z = 2-3i$?

Sol: $z^{-1} = \frac{1}{2-3i} \cdot \left(\frac{2+3i}{2+3i}\right) = \frac{2+3i}{4+6i-6i-9(i)^2} = \frac{2}{13} + i\frac{3}{13}$

3- Find $\frac{-1+2i}{-3+4i}$?

Sol:- $\frac{-1+2i}{-3+4i} \cdot \frac{(-3-4i)}{(-3-4i)} = \frac{3+4i-6i-8(i)^2}{9+12i-12i-16(i)^2} = \frac{11-2i}{25}$

Proposition:- Let z_1 and z_2 be two complex number if $z_1z_2 = 0$ Then either $z_1 = 0$ or $z_2 = 0$.

proof:- Assume $z_1z_2 = 0$, $z_1 \neq 0 \Rightarrow z^{-1}$ exists

$z^{-1}(z_1z_2) = 0 \Rightarrow (z_1^{-1}z_1)z_2 = 0$

$1 \cdot z_2 = 0$ C!.

Example:- $Z^2+3iZ-2=0$

Sol $(Z+2i)(Z+i)=0$ H.i $[-2 = 2i \cdot i]$

$(Z+2i)=0 \Rightarrow Z=-2i$

or $(Z+i)=0 \Rightarrow Z=-i$

Ex Show that $Z=-1-2i$ satisfies the equation $Z^2+2Z+5=0$

Sol:- $(-1-2i)+2(-1-2i)+5=0$

$1+4(i)-4-2-4i+5=0$

H.w - Show that when $Z_1Z_2Z_3=0$ at least Z_1 or Z_2 or $Z_3=0$?

Example:- Show that $Im(iZ)=Re(Z)=X$?

Sol $Z=X+iy$

$R(Z)=X$ -- ①

$iZ=i(x+y)=iX-y=-y+iX$

$Im(iZ)=X$ -- ②

from ① and ② $Re(Z)=Im(iZ)=X$

H.w $Re(iZ)=-Im(Z)$

Remark :- $Z=0$ iff $(x,y)=(0,0)$ iff $X=ReZ=0$ an $y=Im(Z)=0$

Example:- Show that the additive and multiplicative invers are unique ?

Proof:- 1- We will show that if Z_1^* and Z_2^* are two additive inverse to Z

$\Rightarrow Z_1^*=Z_2^* \Rightarrow Z+Z_1^*=0,\ Z^*+Z_2=0$

6

$$\left.\begin{array}{l} z + z_1^* = 0 \\ z + z_2^* = 0 \end{array}\right\} \quad z_1^* = z_2^* \text{ or}$$

$$z + z_1^* = z + z_2^* = 0 \quad \blacksquare$$

z^{-1} H.W بالتعريف

H.W

1) Show that if $z_2 \neq 0$ then
$$\frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \right.$$

2) $(\sqrt{2} - i) - i(1 - \sqrt{2}\, i) = -2i$

3) $(2, -3)(-2, 1) = (-1, 8)$

4) $(1 - i)^4 = -4$

5) Solve $z^2 + z + 1 = 0$

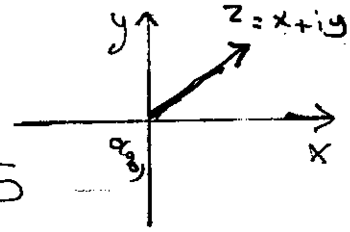
6) Show that $(1+z)^2 = 1 + 2z + z^2$

7) Show that $(z_1 + z_2)^n =$
$$z_1^n + \frac{n}{n!} z_1^{n-1} z_2 + \frac{n(n-1)}{2!} z_1^{n-2} z_2^2 + \cdots + z_2^n$$

3 Modulus and Conjugates

Defintion:- The modulus or absolute value of Complex number $z = x+iy$ is

$$|z| = \sqrt{x^2+y^2}$$



Example:- $z = 3-4i$

$$|z| = \sqrt{(3)^2+(-4)^2} = \sqrt{9+16} = 5$$

Remarkes:- For any Complex number:-

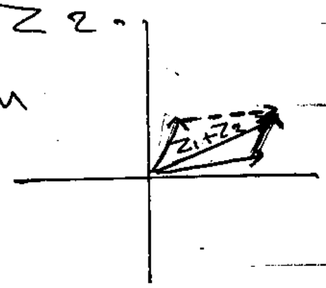
1- $z, |z| \in \mathbb{R}$ and $|z| \geq 0$

2- $|z| = 0$ iff $z = 0$

3- Geomatrically:- $|z|$ is the Length of the vector representing z.

4- $z_1 < z_2$ is meaningless unles bouth z_1 and z_2 realnumber.

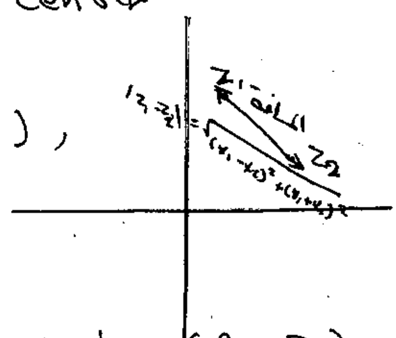
5- $|z_1| < |z_2|$ is means that the point z_1 is Closser to $(0,0)$ then the point z_2.



6- $|z_1 - z_2|$ gives the distance between The points z_1 and z_2.

$$|z_1 - z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

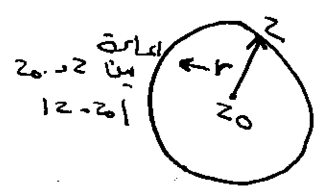
7- The equation of the circle with center z_0 and redius r is $|z - z_0| = r$



Example:- $|z - i| = 3$, $z_0 = i$, $z_0(0,1)$, $r = 3$

2- $|z| = 5$, $z_0 = (0,0)$, $r = 5$

3- $|z - 1 + i| = 3$ is circl with center $(2,-3)$ and redius $r = 3$



propostion:- Let z be a complex number then

$$1-\ Re(z) \leq |Re(z)| \leq |z|$$
$$2-\ Im(z) \leq |Im(z)| \leq |z|$$

proof:- $|z| = \sqrt{(Re(z))^2 + (Im(z))^2}$

$$|z|^2 = (Re(z))^2 + (Im(z))^2$$

1- $(Re(z))^2 \leq |z|^2$ بأخذ الجذر التربيعي للطرفين

$$|Re(z)| \leq |z|$$

$\therefore\ Re(z) \leq |Re(z)| \leq |z|$

2- H.W

Def:- The conjagate of the complex number $z = x+iy$ is $x - iy = \bar{z}$

Example:- $z = 2-3i$, $\bar{z} = 2+3i$

propesties of $|z|$ and $|\bar{z}|$:- Let z, z_1, z_2 be a complex number Then:-

1- $\bar{\bar{z}} = z$ proof $\overline{\overline{(x+iy)}} = x+iy$

2- $|\bar{z}| = |z|$

3- $\overline{z_1 \mp z_2} = \bar{z}_1 \mp \bar{z}_2$

4- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

5- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

6- $Re(z) = \dfrac{z+\bar{z}}{2}$

7- $Im(z) = \dfrac{z-\bar{z}}{2i}$

8- $z \cdot \bar{z} = |z|^2$

9- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

10- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

proof:- 8- $z \cdot \bar{z} = |z|^2$

$$z \cdot \bar{z} = (x+iy)\cdot(x-iy)$$
$$= x^2 - ixy + ixy + y^2$$
$$= x^2 + y^2 = |z|^2$$

Proof:- $z + \bar{z} = (x+iy) + (x-iy) = x + x + iy - iy = 2x = z + \bar{z}$

$$Re(z) = \frac{z+\bar{z}}{2}$$

Proof:- 5 - $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

sol:- $\frac{z_1}{z_2} \cdot \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1 \overline{z_2}}{|z_2|^2}$

$\overline{\left(\frac{z_1 \overline{z_2}}{|z_2|^2}\right)} = \frac{\overline{z_1} \cdot z_2}{\overline{z_2} \cdot z_2} = \frac{\overline{z_1}}{\overline{z_2}}$

Example:- $\frac{1}{3-4i} + \frac{3-2i}{6+8i}$

sol:-

$= \frac{1}{3-4i}\left(\frac{3+4i}{3+4i}\right) + \frac{3-3i}{6+8i}\left(\frac{6-8i}{6-8i}\right)$

$= \frac{3+4i}{9+16} + \frac{18-24i-12i-16}{36+64} = \frac{\cancel{3}\cancel{+}4i}{\cancel{25}} + \frac{1-8i}{50}$

= H.W.

Example:- show $\frac{1+2i}{3-4i} + \frac{2-i}{5i} = \frac{-2}{5}$

$\frac{1+2i}{3-4i}\left(\frac{3+4i}{3+4i}\right) + \frac{2-i}{5i}\left(\frac{-5i}{-5i}\right) =$

$= \frac{(1+2i)(3+4i)}{9+16} + \frac{-5i(2-i)}{25} = \frac{3+4i+6i+8(i)^2}{25} + \frac{-10i+5(i)^2}{25}$

$= \frac{-5+10i}{25} + \frac{-10i-5}{25} = \frac{-5+10i-10i-5}{25} = \frac{-2}{5}$

H.W. show ① $\frac{5}{(1-i)(2-i)(3-i)} = \frac{1}{2}i$

2 - $\frac{1}{\frac{1}{2}} = 2$

3 - $|z| = |3+2i|^5$

H.W. solve the equation

① $2z + |z| = 11 - 8i$

② $z^2 = 1 + 7i$

③ $(1-y)^{y'} = i$
$= -4$

Example:- Solve the equation $2z+i=\bar{z}-iz$

Sol. Let $z=x+iy$

$2(x+iy)+i=(x-iy)-i(x+iy)$

$2x+2iy+i=x-iy-ix-(i)^2y$

$2x+i(2y+1)=x+i(-y-x)+y$

$2x=x+y$ --- 1

$2y+1=-y-x$ --- 2

$2x-x=y \Rightarrow \boxed{x=y} \rightarrow$ نعوض في 2

$2y+1=-y-y \Rightarrow 2y+1 \Rightarrow \boxed{y=-\frac{1}{2}}$ نعوض في 1

$2x=x+(-\frac{1}{2}) \Rightarrow 2x=x-\frac{1}{2} \Rightarrow 2x-x=-\frac{1}{2}$ $\boxed{x=\frac{-1}{2}}$

$\therefore z=\frac{-1}{2}-\frac{1}{2}i$

Example:- Show that the hyperbola $y^2-x^2=1$ can be written as $z^2-\bar{z}^2=-2$ where $z=x+iy$

Sol:- $x=Re(z)=\frac{z+\bar{z}}{2}$

$y=Im(z)=\frac{z-\bar{z}}{2i}$

take:- $y^2-x^2=1 \Rightarrow \left(\frac{z-\bar{z}}{2i}\right)^2-\left(\frac{z+\bar{z}}{2}\right)^2=1$

$=\frac{z^2-2z\bar{z}+\bar{z}^2}{-4}-\frac{z^2+2z\bar{z}+\bar{z}^2}{4}=1$

$=\frac{-1}{4}z^2+\cancel{\frac{2}{4}z\bar{z}}-\frac{1}{4}\bar{z}^2-\frac{1}{4}z^2-\cancel{\frac{2}{4}z\bar{z}}-\frac{1}{4}\bar{z}^2=1$

$=\frac{-2}{4}z^2-\frac{2}{4}\bar{z}^2=1 \Rightarrow z^2-\bar{z}^2=-2$

4-Tringle Inequality:-

Proposition:- Let z_1 and z_2 be complex number. Then

1- $|z_1+z_2| \leq |z_1|+|z_2|$

2- $|z_1+z_2| \geq ||z_1|-|z_2|| \geq |z_1|-|z_2|$

Proof:- 1- $|z_1+z_2| \leq |z_1|+|z_2|$

$|z_1+z_2|^2=(z_1+z_2)\cdot(\overline{z_1+z_2})$ $\boxed{\text{By } |z|^2=z\cdot\bar{z}}$

$=(z_1+z_2)\cdot(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2}$

$= |z_1|^2 + |z_2|^2 + 2\,Re(z_1\bar{z}_2)$ By $[Re(z) \leq |z|$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$ $[$By $Re|z| \leq |Re|z|| \leq |z|$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1|\cdot|\bar{z}_2|$

$\leq \underbrace{|z_1|^2 + |z_2|^2 + 2|z_1|\cdot|z_2|}_{\text{مربع كامل}} \leq (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$ بأخذ الجذر التربيعي

$|z_1 + z_2| \leq |z_1| + |z_2|$

Proof:- 2- $|z_1 + z_2| \geq ||z_1| - |z_2||$

$-|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2|$

take $|z_1| = |z_1 + z_2| + |z_2|$

$|z_1| - |z_2| \leq |z_1 + z_2|$ ... ①

In the same way we can show that

$|z_2| - |z_1| \leq |z_1 + z_2|$ (بالضرب بـ −) نحصل

$|z_1| - |z_2| \geq -|z_1 + z_2|$ -- ②

$\therefore -|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2|$

Example:- Show that $|z_1 - z_2| \geq ||z_1| - |z_2||$

Sol:- $|z_1 - z_2| = |z_1 + (-z_2)| \geq ||z_1| - |-z_2||$

$= |z_1 - z_2| \geq ||z_1| - |z_2||$

Example:- If z lies on the circle $|z| = 2$ then show that $3 \leq |z^2 - 1| \leq 5$

Sol:- $|z^2 - 1| \leq ||z^2| + |-1|| = |z|^2 + 1 = 4 + 1 = 5$

$|z^2 - 1| \geq ||z|^2 - |1|| = 4 - 1 = 3$

Example:- Using the inequality $||z_1| - |z_2|| \leq |z_1 - z_2|$

Show that If z lies on the circle $|z| = 3$

The $|z^2 - 3z + 2| \geq 2$

Sol:- $|z^2-3z+2| = |(z-2)(z-1)| = |z-2||z-1|$

$|z-2| \geq ||z|-2| = 3-2 = 1$

$|z-1| \geq ||z|-1| = 3-1 = 2$

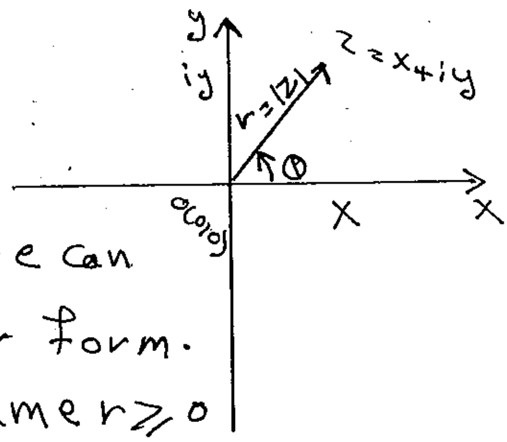
$||z^2-3z+1|| \geq 2$

Example: Show that $\left|\dfrac{z_1+z_2}{z_3+z_4}\right| \leq \dfrac{|z_1|+|z_2|}{||z_3|-|z_4||}$ H.W

5 - Polar Coordinates and Euler's formula

Let r and θ be the polar Coordinates of the point (x,y) that corresponds to the non zero complex number $z = x+iy$ since $x = r\cos\theta$, $y = r\sin\theta$ written in polar form as

$z = r(\cos\theta + i\sin\theta)$



Remarks:- 1- If $z = 0$ Then θ is not defined and we can not write $z = 0$ in polar form.

2- In complex analysis we assume $r \geq 0$ and $r = |z| = \sqrt{x^2+y^2}$

3- The real number θ repersent the angle measuredin inradians, that z makes with the postive realaxis.

4- θ has infinte number of values, both positive and negative, that differ by integer multiply 2π.

5- The values of θ can be determind by specifying the quadrant Containing $z = x+iy$ and by using one of the following three equations:-

1- $\tan\theta = \dfrac{y}{x}$, $\cos\theta = \dfrac{x}{|z|}$, $\sin\theta = \dfrac{y}{|z|}$

Example:- write $z = 1-i$ in polar form:-

Sol:- $x = 1$, $y = -1 \Rightarrow r = |z| = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = \frac{-\pi}{4}$

$\therefore z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

$z = (1-i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Definition:- Let z be a nonzero complex number

1- Each value of θ such that

$z = x+iy = r(\cos\theta + i\sin\theta)$

is called argument of z.

2- The set of an argument of z is denoted by $\arg z$.

3- The principal value of $\arg z$ is the unique θ such that $-\pi < \theta \leq \pi$ and is denoted by $\operatorname{Arg} z$.

Remark:- Note that $\arg z = \operatorname{Arg} z + 2n\pi$, $n \in \mathbb{Z}$

i.e $\arg z = \{\theta : z = r(\cos\theta + i\sin\theta)\}$

$z = r\{\cos(\arg z) + i\sin(\arg z)\}$

$$\operatorname{Arg} z = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{y}{x}\right), & x < 0, \ y \geq 0 \\ -\pi + \tan^{-1}\left(\frac{y}{x}\right), & x < 0, \ y < 0 \\ \frac{\pi}{2}, & x = 0, \ y > 0 \\ \frac{-\pi}{2}, & x = 0, \ y < 0 \end{cases}$$

Example:- ① Find $\operatorname{Arg}(5)$, $\operatorname{Arg}(-5)$, $\operatorname{Arg}(2i)$, $\operatorname{Arg}(-3i)$

Sol 1- $\operatorname{Arg}(5) \Rightarrow x = 5, y = 0$, $\operatorname{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right) = 0$

2- $\operatorname{Arg}(-5) \Rightarrow x = -5, y = 0$, $\operatorname{Arg}(-5) = \pi$

3- $\operatorname{Arg}(2i) \Rightarrow x = 0, y = 2 > 0 \therefore \operatorname{Arg}(2i) = \frac{\pi}{2}$

4- $\operatorname{Arg}(-3i) \Rightarrow x = 0, y = -3 < 0 \therefore \operatorname{Arg}(-3i) = \frac{-\pi}{2}$

13 Example:- Find the principal argument $z=\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$

Sol $z=\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\cdot\frac{1+i\sqrt{3}}{1+i\sqrt{3}}=\frac{-2+2i\sqrt{3}}{4}=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$

$x=\frac{-1}{2}$, $y=\frac{\sqrt{3}}{2}$ $\Rightarrow \operatorname{Arg} z=\pi+\tan^{-1}(\frac{y}{x})=\pi+\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)$

$\operatorname{Arg} z=\pi-\tan^{-1}(\sqrt{3})=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

Example:- Find $\arg(z)$ if $z=1+i$

Sol $\arg(z)=\operatorname{Arg} z+2n\pi$, $n\in\mathbb{Z}$

$x=1$, $y=1$ $\Rightarrow \tan^{-1}(1)=\frac{\pi}{4}=\operatorname{Arg}(z)$

$\arg(1+i)=\frac{\pi}{4}+2n\pi$

Euler's formula :— For any real number θ we will use the relation

$\cos\theta+i\sin\theta=e^{i\theta}$, $e=2.7$ ثابت

$\therefore z=re^{i\theta}$ or $z=r\exp(i\theta)$

Example:- ① write $z_1=-1$ and $z_2=1+i$ in polar form.

② Express $z=3e^{\frac{\pi}{3}i}$ in the form $x+iy$

Sol $|z|=r=|-1|=1$, $\operatorname{Arg}(-1)=\pi$

$\therefore \boxed{z=1e^{i\pi}}$

$z=1+i \Rightarrow r=\sqrt{1+1}=\sqrt{2}$, $\operatorname{Arg}(1+i)=\tan^{-1}(\frac{y}{x})$

$\operatorname{Arg}(z)=\tan^{-1}(1)=\frac{\pi}{4} \Rightarrow \therefore \boxed{z=\sqrt{2}\,e^{\frac{\pi}{4}i}}$

2- $z=3(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})=3(\frac{1}{2}+i\frac{\sqrt{3}}{2})=\frac{3}{2}+\frac{3\sqrt{3}}{2}i$

Proposition:- Let $z_1=r_1e^{i\theta_1}$ and $z_2=r_2e^{i\theta_2}$ Then $z_1=z_2$ if and only if $r_1=r_2$ and $\theta_1=\theta_2+2n\pi$, $n\in\mathbb{Z}$

Remark:- $z=z_0+Re^{i\theta}$, $0\leq\theta\leq 2\pi$ is a parametric representation of the circle $|z_0-z|=R$

Example: show that $|e^{i\theta}|=1$?

Sol:- $z = e^{i\theta} = \cos\theta + i\sin\theta$

$|e^{i\theta}| = |z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

6- proposition:- Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

Then: ① $z_1 \cdot z_2 = r_1 r_2 \, e^{i(\theta_1+\theta_2)}$

② $\frac{1}{z_1} = \frac{1}{r_1} e^{-i\theta}$

③ $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

1- proof:- $z_1 = r_1 e^{i\theta_1} \Rightarrow = r_1[\cos\theta_1 + i\sin\theta_1]$

$z_2 = r_2 e^{i\theta_2} = r_1[\cos\theta_2 + i\sin\theta_2]$

$z_1 \cdot z_2 = r_1[\cos\theta_1 + i\sin\theta_1] \cdot r_2[\cos\theta_2 + i\sin\theta_2]$

$= r_1 r_2 [\cos\theta_1 + i\sin\theta_1][\cos\theta_2 + i\sin\theta_2]$

$= r_1 r_2 \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$= r_1 r_2 \, e^{i(\theta_1+\theta_2)}$

~~2-proof:~~ 2- $\frac{1}{z_1} = \frac{1}{r_1} e^{i\theta_1}$?

$$\frac{1}{z_1} = \frac{1}{r_1 e^{\theta_1}} \cdot \frac{e^{-i\theta_1}}{e^{-i\theta_1}} = \frac{e^{-i\theta_1}}{r_1 e^{(\theta - -\theta_2)}} = \frac{1}{r_1} e^{-i\theta_1}$$

3- proof:- H.W

Example:- let $z_1 = -1 - i$, $z_2 = \sqrt{3} - i$ find $z_1 \cdot z_2$, $\frac{z_1}{z_2}$,

Sol 1- $z_1 \cdot z_2 \Rightarrow r_1 = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$, $\theta_1 = \mathrm{Arg}\, z_1 = \frac{-3\pi}{4}$

$z_1 = \sqrt{2}\, e^{\frac{-3\pi}{4} i}$

$r_2 = \sqrt{3+1} = \sqrt{4} = 2$, $\theta_2 = \mathrm{Arg}\, z_2 = \frac{-\pi}{6}$

$z_2 = 2\, e^{\frac{-\pi}{6} i}$

$z_1 \cdot z_2 = r_1 r_2 \, e^{i(\theta_1+\theta_2)} = 2\sqrt{2}\, e^{i\left(\frac{-3\pi}{4} - \frac{\pi}{6}\right)}$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} = \frac{\sqrt{2}}{2} e^{i\left(\frac{-3\pi}{4} + \frac{\pi}{6}\right)}$

$\frac{1}{z_1} = \frac{1}{r_1} e^{-i\theta_1} = \frac{1}{\sqrt{2}} e^{i\frac{3\pi}{4}}$

15

Proposition:- If $z = re^{i\theta}$ then $z^n = r^n e^{in\theta}$ for any $n \in \mathbb{Z}$

Proof:- $z^n = r^n e^{in\theta}$?? $n \in \mathbb{Z}$

by indaction for ① $n \geq 0$

$n = 0 \Rightarrow z^0 = r^0 e^{i0\theta} \Rightarrow 1 = 1$

It's true for $n = 0$.

② Assume it is true for k i.e $z^k = r^k e^{ik\theta}$

Then $z^{k+1} = z^k \cdot z = (r^k e^{ik\theta})(r e^{i\theta})$

$= r^{k+1} e^{i(k+1)\theta}$

There for $z^n = r^n e^{in\theta}$ is true for all $n \geq 0$

③ for $n < 0$

$\therefore z^n = (\frac{1}{z})^{-n}$

$z^n = (\frac{1}{z})^{-n} = (\frac{1}{r} e^{-i\theta})^{-n}$

$\Rightarrow n < 0 \Rightarrow -n > 0 \Rightarrow$

$z^n = (\frac{1}{n})^{-n} e^{-i(-n)}$

$= r^n e^{in\theta}$

Example:- write $(\sqrt{3}+i)^5$ in form $x+iy$

Sol:- $(\sqrt{3}+i)^5 \Rightarrow x+iy$

$\sqrt{3}+i = re^{i\theta}$

$r = \sqrt{(\sqrt{3})^2+(1)^2} = 2 \Rightarrow \theta = Arg(\sqrt{3}+i) = \frac{\pi}{6}$

$\Rightarrow \boxed{\sqrt{3}+i = 2e^{i\frac{\pi}{6}}}$

$(\sqrt{3}+i)^5 = (2e^{i\frac{\pi}{6}}) = 2^5 e^{i\frac{5\pi}{6}}$

$= 32\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$

$= -16(\sqrt{3}-i)$

Remarks:- (demoivre's formula):- Where $r = 1$, then above formula (proposition $z = re^{i\theta}$) become

$(e^{i\theta})^n = e^{in\theta}$ ⇒ This gives the usful demoivre's formula

$[\cos\theta + i\sin\theta]^n = \cos n\theta + i\sin(n\theta)$

16 Example:- Use demoivre's formula to derive the trigonometric identities $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ and $\sin^2\theta = 2\sin\theta\cos\theta$

Sol:- $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta$

$2i\sin\theta\cos\theta = \sin 2\theta$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

Proposition:- Let z_1 and z_2 be nonzero complex numbers then a- $\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2$

b- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Proof: Let $\arg z_1 = \operatorname{Arg} z_1 + 2n_1\pi$

$\arg z_2 = \operatorname{Arg} z_2 + 2n_2\pi$

$z_1 = r_1 e^{i\operatorname{Arg} z_1}, \quad z_2 = r_2 e^{i\operatorname{Arg} z_2}$

$z_1 z_2 = r_1 r_2 e^{i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2)}$

but

$z_1 z_2 = |z_1 z_2| e^{i\operatorname{Arg}(z_1 z_2)}$

$r_1 r_2 e^{i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2)} = r_1 r_2 e^{i\operatorname{Arg}(z_1 z_2)}$ [For شرط تساوي العدد مركب مكافئة

$\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi$

$\arg(z_1 z_2) = \operatorname{Arg}(z_1 z_2) + 2k\pi$

$= \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi + 2k\pi$

$= \arg z_1 - 2n_1\pi + \arg z_2 - 2n_2\pi + (k+n)2\pi$

$= \arg z_1 + \arg z_2 + 2\pi\underbrace{(-2n_1 - 2n_2 + k - n)}_{=0}$

$\therefore \arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2$

Example:- Let $z_1 = -1$ and $z_2 = i$ show that

$\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2$

and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

17

Sol:- $z_1 = -1$, $z_2 = i$

$z_1 z_2 = -i$, $\frac{z_1}{z_2} = \frac{-1}{i} = i$

$Arg(-1) = \pi$, $\arg z_1 = \pi + 2n_1\pi$, $n_1 \in Z$

$Arg(i) = \frac{\pi}{2}$, $\arg z_2 = \frac{\pi}{2} + 2n_2\pi$, $n_2 \in Z$

$Arg(z_1 z_2) = Arg(-i) = \frac{-\pi}{2}$, $\arg(z_1 z_2) = \frac{-\pi}{2} + 2n\pi$, $n \in Z$

$Arg(\frac{z_1}{z_2}) = Arg(i) = \frac{\pi}{2}$, $\arg(\frac{z_1}{z_2}) = \frac{\pi}{2} + 2m\pi$, $m \in Z$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$\frac{-\pi}{2} + 2\pi = \pi + \frac{\pi}{2}$ $(n_1 = n_2 = 0, \ n = 1$

$\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$

$\frac{\pi}{2} = \pi - \frac{\pi}{2}$ $(n_1 = n_2 = n = 0)$

Example:- Given that $z_1 z_2 \neq 0$, use the exponential forms of z_1 and z_2 to prove that

$Im(z_1 \bar{z}_2) = |z_1||z_2|$ iff $\arg z_1 - \arg z_2 = \frac{\pi}{2} + 2m\pi$, $m \in Z$

Proof:- $z_1 z_2 \neq 0$, $z_1 = z_2 \neq 0$.

$Im(z_1 \bar{z}_2) = |z_1||z_2| \Longleftrightarrow \arg z_1 - \arg z_2 = \frac{\pi}{2} + 2m\pi$

$z_1 \neq 0$, $z_1 = |z_1| e^{i Arg z_1}$

$z_2 \neq 0$, $z_2 = |z_2| e^{i Arg z_2}$

$\bar{z}_2 \neq 0$, $\bar{z}_2 = |z_2| e^{-i Arg z_2}$

$z_1 \bar{z}_2 = |z_1||z_2| e^{i(Arg z_1 - Arg z_2)}$

$Im(z_1 \bar{z}_2) = |z_1||z_2| [\sin(Arg z_1 - Arg z_2)]$

$Im(z_1 \bar{z}_2) = |z_1||z_2|$

$\Longleftrightarrow \sin(Arg z_1 - Arg z_2) = 1$

$\Leftrightarrow Arg z_1 - Arg z_2 = \frac{\pi}{2} + 2n\pi$, $n \in Z$

$\Leftrightarrow Arg z_1 = \arg z_1 - 2n_1\pi$, $n_1 \in Z$

$\Leftrightarrow Arg z_2 = \arg z_2 - 2n_2\pi$, $n_2 \in Z$

$\Leftrightarrow \arg z_1 - 2n_1\pi - [\arg z_2 - 2n_2\pi] = \frac{\pi}{2} + 2n\pi$

18 $\Longleftrightarrow \arg z_1 - \arg z_2 = \frac{\pi}{2} + 2\pi(\underbrace{n+n_1-n_2}_{m})$

$\therefore \arg z_1 - \arg z_2 = \frac{\pi}{2} + 2m\pi$

7- Root of complex numbers: In this section we use that fact $z^n = |z|^n e^{in\arg z}$, $n \in \mathbb{Z}$ to find the n-th root of any non zero complex number

Defintion:- Let z_0 be a non zero complex number. An n-th root of z_0 is a non zero complex number z such that $z^n = z_0$

How can we find all the n-th root of $z_0 \neq 0$?

since $z_0 \neq 0$, it has a polar form $z_0 = r_0 e^{i\theta_0}$

Similarly any n-th root z of z_0 nonzero and $z = r e^{i\theta}$

Therefore any n-th root of z_0 has the form

$z = \sqrt[n]{r_0}\, e^{\frac{i(\theta_0 + 2k\pi)}{2}}$, $k \in \mathbb{Z}$

Not that:- $|z| = \sqrt[n]{r_0}$ for any n-th root z of z_0

this means that the roots lie on the circle $|z| = \sqrt[n]{r_0}$.

Moreover they are equally spaced every $\frac{2\pi}{n}$ radians. Thus there are only n distinct roots, these distinct root can be obtained where $k = 0, \dots, n-1$

and denoted by C_k, $k_0 = 0$

$z_1 = \sqrt[n]{r_0}\, e^{i\theta_0}$

$C_1 = 1$

$z_2 = \sqrt[n]{r_0}\, e^{i\frac{\theta_0}{n} + \frac{2\pi}{n}}$

$C_2 = 2$

$z_3 = \sqrt[n]{r_0}\, e^{\frac{i\theta_0 + 4\pi}{n}}$

There for the n-th root of $z_0 = r_0 e^{i\theta_0}$ are given by $c_k = \sqrt[n]{r_0}\, e^{\frac{i(\theta_0+2k\pi)}{n}}$, $k=0,1,\dots,n-1$

Example:- Find the 3rd root of $z_0 = 1$ s.t $z = 1$,

$$(1)^{1/3} = ??$$

Sol:- $z_0 = 1$, $r_0 = |z_0| = 1$, $\theta_0(z_0) = 0$, $z_0 = e^{0} = 1$

The 3rd root of $z_0 = 1$ are

$$c_k = \sqrt[3]{1}\, e^{\frac{i(0+2k\pi)}{3}}, \quad k = 0,1,2$$

$$= e^{\frac{2k\pi}{3} i}$$

$$c_0 = e^{0} = 1, \quad c_1 = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$c_2 = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Remark:- $z^n = z_0$, $z = (z_0)^{\frac{1}{n}}$ if n- سالبة

$$n < 0, \quad z_0^{\frac{1}{n}} = \left(\frac{1}{z_0}\right)^{-\frac{1}{n}}.$$

Example:- $(2)^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}$

Remarks:- 1- For any complex number $z_0 \neq 0$

$z_0^{\frac{1}{n}}$ denotes the set of all n-th root of z_0 that is $z_0^{\frac{1}{n}} = \{z : z^n = z_0\}$

2- If r_0 is a positive real number, then $r_0^{\frac{1}{n}}$ denote the set of all n-th roots of $r_0^{\frac{1}{n}} = \{z : z^n = r_0\}$ but $\sqrt[n]{r_0}$ denote the unique positive real n-th root of r_0.

3- If $\theta_0 = \operatorname{Arg} z_0$, then $z_0 = \sqrt[n]{r_0}\, e^{\frac{i\theta_0}{n}}$ is called the principal root of z_0.

The n-th roots of unity:-

The n-th root of 1 are given by

$C_k = e^{\frac{2k\pi}{n} i}$, $k = 0, 1, \dots, n-1$

let $\omega_n = C_1 = e^{\frac{2\pi}{n} i}$ The

$$C_2 = e^{\frac{4\pi}{n} i} = \omega_n^2, \dots, C_{n-1} = e^{\frac{2(n-1)\pi i}{n}} = \omega_n^{n-1}$$

There fore the n-th root of unity are

$1, \omega_n, \omega_n^2, \omega_n^3, \dots, \omega_n^{n-1}$.

where $\omega_n = e^{\frac{2\pi}{n} i}$

<u>The n-th roots of Z_0:-</u> Let $\theta_0 = \text{Arg}\, Z_0$ then the n-th roots of $Z_0 = r_0 e^{i\theta_0}$ are given by

$$C_k = \sqrt[n]{r_0}\, e^{\frac{i(\theta_0 + 2k\pi)}{n}}, \quad k = 0, 1, \dots, n-1$$

They are cane wrtten as

$$C_k = \sqrt[n]{r_0}\, e^{i\theta_0} \cdot e^{\frac{2k\pi i}{n}} = C_0\, \omega_n^k, \quad k = 0, 1, \dots, n-1$$

Where $C_0 = \sqrt[n]{r_0}\, e^{i\theta_0}$ is the principal root of Z_0 and $\omega_n = e^{\frac{2k\pi}{n} i}$

Example:- Find the 2-th root of i, $k = 0, 1$?

Sol. $z^2 = i \Rightarrow z = i^{1/2}$, $n = 2$, $k = 0, 1$.

$r_0 = \sqrt{(1)^2} = 1$, $\text{Arg}(i) = \frac{\pi}{2}$

$C_k = C_0\, \omega_2^k$

$C_0 = 1 \cdot e^{\frac{\pi}{2} i} = \frac{1+i}{\sqrt{2}}$

$\omega_n = e^{\frac{2\pi}{2}} = e^{\pi i} = -1$

The roots are

$\frac{1+i}{\sqrt{2}}$, $\frac{-(1+i)}{\sqrt{2}}$.

Example:- 2- Find $(-4\sqrt{3}+4i)^{\frac{1}{3}}$

Sol:- $n=3$, $k=0,1,2$

$C_k = C_0 \omega_3$

$r_0 = \sqrt{|z_0|} = \sqrt{48+16} = 8$, $\mathrm{Arg}(z_0) = \pi + \tan^{-1}\left(\frac{-4\sqrt{3}}{4}\right)$

$C_0 = \sqrt[n]{|z_0|}\, e^{\mathrm{Arg}\, z_0\, i}$ $\qquad = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$= \sqrt[3]{8}\, e^{i\frac{5\pi}{18}} = 2e^{\frac{5\pi}{18}i}$

$\omega_3 = e^{\frac{2\pi}{3}i} = \frac{-1+\sqrt{3}}{2}$, $\omega_3^2 = \left(\frac{-1+\sqrt{3}}{2}\right)^2$

The root

C_0, $C_0\omega_3$, $C_0\omega_3^2$

$2e^{\frac{5\pi}{18}i}$, $2e^{\frac{5\pi}{18}i}\left(\frac{-1+\sqrt{3}}{2}\right)$, $2e^{\frac{5\pi}{18}i}\left(\frac{-1+\sqrt{3}}{2}\right)^2$

Example:3- $(2+2\sqrt{3}i)^{\frac{1}{4}}$ find all the root of z ?

Sol:- $n=4$, $k=0,1,2,3$

$z_0 = 2+2\sqrt{3}i$, $r_0 = \sqrt{4+12} = \sqrt{16} = 4$

$\mathrm{Arg}\, z_0 = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$C_k = C_0\omega_4^k$, $k=0,1,2,3$

$C_0 = \sqrt[4]{4}\, e^{\frac{\pi}{12}i} = \sqrt{2}\, e^{\frac{\pi}{12}i}$

$\omega_n = e^{\frac{2\pi}{4}i} = e^{\frac{\pi}{2}i} = i = c_1$

$\omega_4^2 = (i)^2 = -1$, $\omega_4^3 = (i)^3 = -i$

the roots, C_0, $\cancel{C_0}\omega_4$, $C_2\omega_4^2$, $C_3\omega_3^2$

$\sqrt{2}e^{\frac{\pi}{12}i}$, $i(\sqrt{2}\, e^{\frac{\pi}{12}i})$, $-\sqrt{2}\, e^{\frac{\pi}{12}i}$, $-i\sqrt{2}\, e^{\frac{\pi}{12}i}$

H-W:- 1- Find the 3-th root of i ?

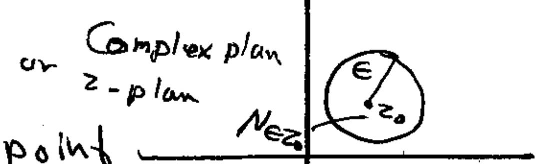
2- Find $(\sqrt{2}+i\sqrt{2})^{\frac{1}{3}}$?

3- Show that roots of $z^4+4=0$, are $1+i$, $i-1$, $1-i$, $-i+1$?

4- Find $(\sqrt{3}+i)^{\frac{2}{5}}$, 5- Find 3-rood of $27i$?

7-

Definition:- 1- An ϵ-Neighborhood of a point z_0 is the set $N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$ consisting of all points lying inside the circle $|z-z_0|<\epsilon$



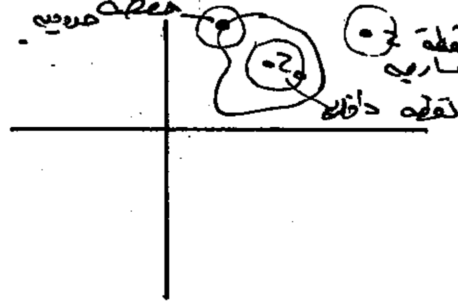
2- A deleted ϵ-Neighborhood of a point z_0 is an ϵ-Neighborhood of z_0 except z_0 is self. that is, it is the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$

3- A point z_0 is said to be an interior point of a set S of complex numbers if there is an ϵ-neighorhood $N_\epsilon(z_0)$ of z_0 such that $N_\epsilon(z_0) \subseteq S$

4- A point z_0 is an exterior point of a set S of Complex number if there is an ϵ-neighborhood of z_0 such that $N_\epsilon(z_0) \cap S = \emptyset$ → مجموعة خالية

5- A point z_0 is a boundary point of a set S if it is neither an interior nor an exterior point.

6- the set of all boundary points of a set S is Called the boundary of S.

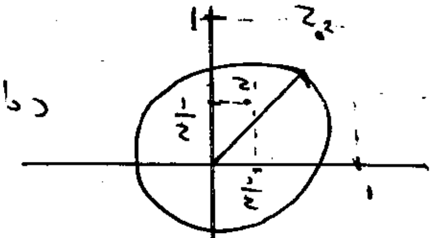


Example:- Let $S = \{ z : |z| < 1 \}$. then the boundary of S is the set $S = \{ z : |z| = 1 \}$

1- $z_1 = \frac{1+i}{2}$ is an interior point of S but $z_2 = 1+i$ is an exterior point.

Sol:- $z_1 = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2} i$ (داخلة)

$z_2 = 1+i$ (خارجة)



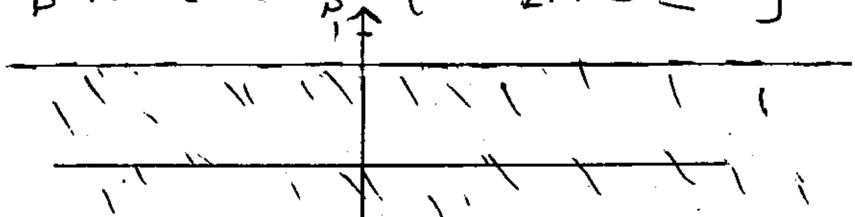
Definition 2:- Let S be a set of complex numbers then:-

1- S is open if it contains none of the boundary points

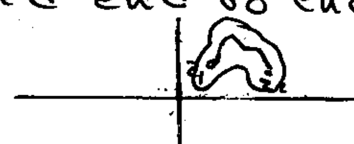
2- S is closed if it contains all its boundary points

3- The closure of S is the closed set consisting of all points in S together with the boundary of S

Example:- Let $S = \{ z : Im\, z < 1 \}$ Then S is open the closure of S is the set $S = \{ z : Im\, z \le 1 \}$



Definition 3:- 1- An open set S is said to be connected if each pair of points z_1 and z_2 in S can be joined by a polygonal line consisting of a finite number of line segments jointed end to end, that lies entirely in S



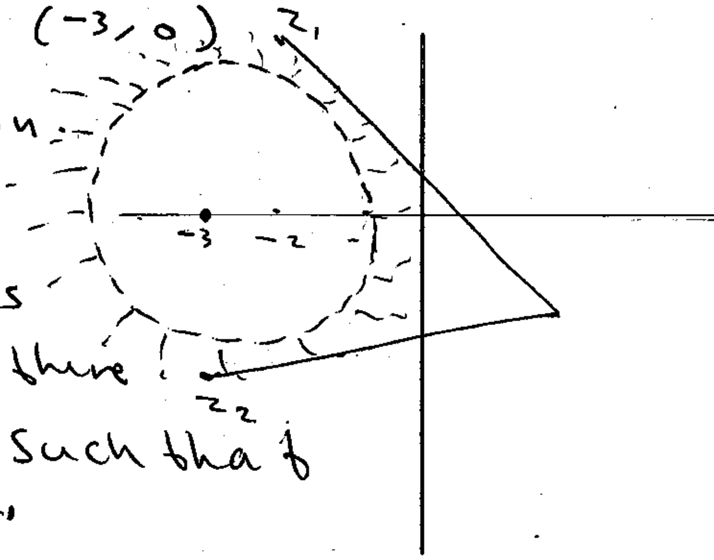
2- A set S is called a domain if it is open and connected.

3- A region is a domain together with some, none or all of its boundary.

Example:- sketch $S=\{z\in\mathbb{C} \mid |z+3|>2\}$ and determine whether it is a domain or not.

SoL:- $|z+3|>2$, $r=2$, $(-3,0)$

is domain and region.



Definition 4:- 1- A set S is said to be bounded if there is real number $R>0$ such that $|z|<R$ for all $z\in S$

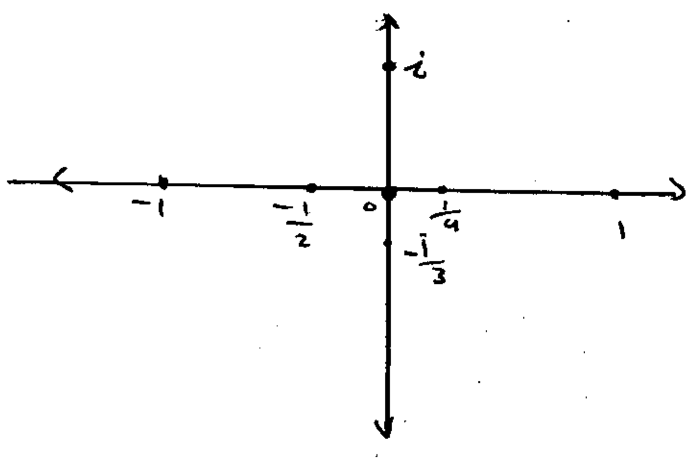
2- A point z_0 is called an accumulation point of a set S if each deleted neighborhood of z_0 contains at least on point of S.

Example:- Let $S=\{\frac{i^n}{n}\}$ $n=1,2,\ldots\}$ find the accumulation point of S

SoL:-

$S=\{i,\frac{-1}{2},\frac{-i}{3},\frac{1}{4},\frac{i}{5}\}$

Zero is accumulation point.



Chapter 2
Analytic Functions

In this chapter, we consider function of a complex vari-able and study continuity and differentiability of such function, the main goall of this chapter is to Introduce analytic functions.

9 – Function of complex Variable:–

Definition:– Let S be a set of complex numbers. A function f defined on S is rule that assinghs to each $z \in S$ a single complex number w. in such Case we write $w = f(z)$.

Remark:– 1- If f is a function defined on S, then S is the domain of f.

2- If the domain is not given explictly, then we agree that its the largest possible set

Example: 1- Let f be given by $f(z) = \frac{1}{z-1}$, the f is a function of complex variable with domain $S = \mathbb{C} - \{1\}$

2- Let f be a given by $f(z) = \frac{1}{z^2+1}$, then domain of f is $S = \mathbb{C} - \{i, -i\}$

H.w 1- $f(z) = \frac{z}{z+\bar{z}}$ 2- $f(z) = \frac{z}{1-|z|^2}$

3 – $f(z) = \frac{y}{x} + \frac{i}{1-y}$

Real and imaginary part of a function:-

Let $w=f(z)$, $z=x+iy$, be a function an Let $W=u+iv$ then we have.

$$u+iv=f(x+iy)$$

Both of the real numbers u and v dependes on x and y thus $f(z)=u(x,y)+iV(x,y)$

where u and v are real-valued function of two real varibles x and y.

Simliarly:- If $z=re^{i\theta}$, then $f(z)=u(r,\theta)+iv(r,\theta)$

Example:- Write $f(z)=z^2+2z$ in the forms

$$f(z)=u(x,y)+iv(x,y) \text{ and } f(z)=u(r,\theta)+iu(r,\theta)$$

Sol:- $z=x+iy$

$$f(z)=z^2+2z=(x+iy)^2+2(x+iy)=$$

$$=x^2-y^2+2xyi+2x+2yi=x^2-y^2+2x+i(2xy+2y)$$

$$=u(x,y)+iv(x,y)$$

$z=re^{i\theta}$

$$f(z)=r^2e^{2i\theta}+2re^{i\theta}=r^2(\cos 2\theta+i\sin 2\theta)+2r(\cos\theta+i\sin\theta)$$

$$=r^2\cos 2\theta+2r\cos\theta+i(r^2\sin 2\theta+2r\sin\theta)$$

$$=u(r,\theta)+iv(r,\theta).$$

Remark:- If $v(x,y)=0$ for all z, then $f(z)=u(x,y)$ is called a real-valued function of Complex variable.

Example:- $f(z)=|z|^2=x^2+y^2$ is a real-valued function of Complex variable.

Polynomials and rational functions:-

1- If n is a non negative integer, and if $a_0, a_1, \dots, a_n$ are Complex numbers such that $a_n \neq 0$, then the function

$$f(z) = a_0 + a_1 z + \dots + a_n z^n$$

is a polynomial of degree n. The domain of a polynomial function is the entire plane.

2- If $P(z)$ and $Q(z)$ are polynomials, then

$$f(z) = \frac{P(z)}{Q(z)}$$

is rational function.

The domain of a rational function $f(z) = \frac{P(z)}{Q(z)}$ is the set $\{z \in \mathbb{C} : Q(z) \neq 0\}$

Example:- $f(z) = i z^2 + (1+i) z^3 + 2z - 5 - 3i$ is a polynomial of degree 4 whose domain all $\mathbb{C}$.

Example:- $f(z) = \frac{2z^2 + 3iz + 4}{z^3 - (1+i) z^2 + iz}$ is rational function whose domain is $\mathbb{C} \setminus \{0, 1, i\}$.

Sol:- $z^3 - (i+1) z^2 + iz = 0 \Rightarrow z(z^2 - (i+1) z + i) = 0$

$z((z-i)(z-1)) = 0 \Rightarrow z = 0, z = i, z = 1.$

Example:- Let $f(z) = x^2 + y^2 + 2i(x+y)$. use $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$ to write $f(z)$ in terms of z.

Sol:- $f(z) = x^2 + y^2 + 2i(x+y)$

$$= \left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 + 2i\left(\frac{z+\bar{z}}{2} + \frac{z-\bar{z}}{2i}\right)$$

$$= \frac{1}{4}(z^2 + \bar{z}^2 + 2z\bar{z}) - \frac{1}{4}(z^2 + \bar{z}^2 - 2z\bar{z}) + i(z+\bar{z}) + z$$

$$= \frac{1}{2} z\bar{z} + z - \bar{z} + i(z + \bar{z}).$$

Definitions:- A function $w=f(z)$ is called a single value function if for each $z\in \mathbb{C}$, $f(z)$ is a unique value.

Examples:- $w=z^2+z+1$

Definitions:- A function $w=f(z)$ is called a multiple valued function if $\forall z\in\mathbb{C}$, $f(z)$ has more than one value.

Example:- $f(z)=z^2 \Rightarrow w^2=z \Rightarrow w=z^{\frac{1}{2}} \begin{cases} c_0 \\ c_0 w_n \end{cases}$

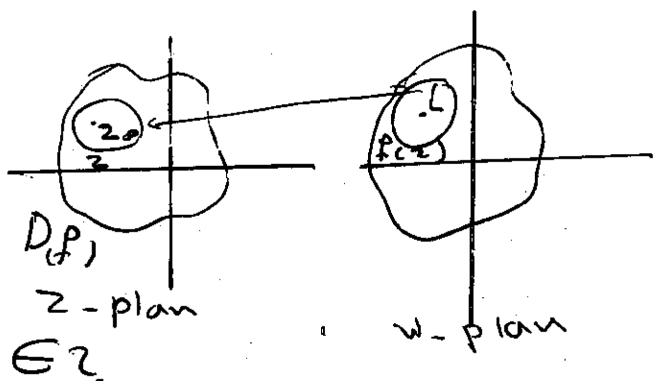
1D_ Limits

1-Definition (Limit of a function):- Let f be a function defined in some deleted neighborhood of a point $z_0\in\mathbb{C}$. we say that Limit of $f(z)$ as z approaches z_0 is a number L or $\lim\limits_{z\to z_0} f(z)=L$.

If for each $\epsilon>0$, there is $\delta>0$ such that $|f(z)-L|<\epsilon$ whenever $0<|z-z_0|<\delta$

Example:- Prove that

$\lim\limits_{z\to z_0} z = z_0$



sol:- $f(z)=z$, $L=z_0$

Given $\epsilon>0$, let $\delta=\epsilon$?

$0<|z-z_0|<\delta$, $|f(z)-L|<\epsilon$??

$|f(z)-L|=|z-z_0|<\delta=\epsilon \quad \therefore \delta=\epsilon$

Example Show that the $\lim_{z \to i} (3iz+2) = -1$

Sol $f(z) = 3iz+2$, $z_0 = -i$, $L = -1$

$\epsilon > 0$ be given and let $\delta =$

Now $0 < |z-i| < \delta \Rightarrow |f(z) - L| = |3iz+2+1| =$

$= |3(iz+1)|$

$= 3|(iz+1)|$

$= 3|i||z-i|$

$= 3\delta < \epsilon$

$\Rightarrow \delta = \frac{\epsilon}{3}$

$\therefore |3iz+2+1| < 3 \cdot \frac{\epsilon}{3} = \epsilon$

H.W 1 - prove that $\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$

2 - prove that $\lim_{z \to (2i-1)} (2z+3) = 4i+1$

3 - find the domain of ① $f(z) = \frac{2}{z+\bar{z}} - \frac{iz}{2x}$

② $f(z) = \frac{z}{1-|z|^2}$

4 Write $f(z)$ as $u(x,y) + iv(x,y)$

1 - $f(z) = (z-i)^2$, 2 - $f(z) = \frac{1}{z^2} + i$ 3 - $f(z) = \bar{z}^3$

5 - Rewrite the following complex function entirely in terms of complex variable z. its conjugate and constants - ① $f(z) = -2xy + i(x^2+y^2)$

② $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$

Sol:- 1 $2xy + i(x^2+y^2)$

$f(z) = -2\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) + iz\bar{z}$

$= \frac{i}{2}(z^2 - \bar{z}^2) + iz\bar{z}$.

3o We can extend the definition of limit to the case in wich z_0 is boundary point of the domain of f. In this case we require $|f(z)-L|<\epsilon$ only for z lies in bothe the domain of f and

$$0<|z-z_0|<\delta$$

3- If $\lim_{z\to z_0} f(z)=L$ exists, then it is unique.

4- The symbol $z \to z_0$ means that z is allowed to approach z_0 in any arbitrary manner. thas, if the limit approach different values from different direction, then the limit dose not exist.

Example:- Show that $\lim_{z\to 0} \frac{\bar{z}}{2z}$ dos not exist.

Sol:- ① $z=x \Rightarrow \lim_{z\to 0} \frac{\bar{z}}{2z} = \lim_{x\to 0} \frac{x}{2x} = \frac{1}{2}$ --- ①

② $z=iy \Rightarrow \lim_{z\to 0} \frac{\bar{z}}{2z} = \frac{-iy}{2iy} = \frac{-1}{2}$ --- ②

from ① and ② the $\lim_{z\to 0} \frac{\bar{z}}{2z}$ dose not exist.

12- Theorems on Limits:-

We can estabilsh a connection between Limits of f(z) and Limits of $u(x,y)$ and $v(x,y)$.

Theorem 1:- Suppose that $f(z)=u(x,y)+iv(x,y)$

$z_0=x_0+iy_0$, and $L=A+iB$ then

$\lim_{z\to z_0} f(z)=L$ if and only if $\lim_{(x,y)\to(x_0,y_0)} u(x,y)=A$

and

$\lim_{(x,y)\to(x_0,y_0)} v(x,y)=B$

31 proof:- $\lim_{z \to z_0} f(z) = L \overset{\text{المطلوب اثباته}}{\Rightarrow} \lim_{(x,y) \to (x_0,y_0)} u(x,y) = A$ and

$$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = B$$

Given $\epsilon > 0$ Let $\delta = \delta_1$

$\lim_{z \to z_0} f(z) = L$, $\exists\ \delta > 0$ such that $0 < |z - z_0| < \delta$

$\Rightarrow |f(z) - L| < \epsilon$

Now we have to prove $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$|u(x,y) - A| < \epsilon \Rightarrow 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$\Rightarrow 0 < |z - z_0| < \delta_1 = \delta$

and $|u(x) - A| = |Re[f(z) - L]| \leq |f(u) - L| < \epsilon$

$|v(x,y) - B| = Im[f(v) - B] \leq |f(v) - B| < \epsilon$

Example:- Find $\lim_{z \to 0} (z^2 + 2i)$?

sol:- $\lim_{z \to 0} (z^2 + 2i) \Rightarrow f(z) = z^2 + 2i$

$$= x^2 + y^2 + 2ixy + 2i$$

$\therefore u(x,y) = x^2 - y^2$, $v(x,y) = 2xy + 2$

$z \to 0 \Rightarrow (x,y) \Rightarrow (0,0)$

$$\lim_{(x,y) \to (0,0)} u(x,y) = \lim_{(x,y) \to (0,0)} (x^2 + y^2) = 0 = A$$

$$\lim_{(x,y) \to (0,0)} v(x,y) = \lim_{(x,y) \to (0,0)} 2xy + 2 = 2 = B$$

$\therefore \lim_{z \to 0} f(z) = A + iB = 0 + 2i = 2i$

Theorem:2 - Assume that $\lim_{z \to z_0} f(z) = L$ and

$\lim_{z \to z_0} g(z) = M$ then