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محاضرات بحوث العمليات

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Introduction to Linear Programming

A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2 + \ldots + a_nx_n$, where $a_1, a_2, \ldots, a_n$ are constants and $x_1, x_2 \ldots x_n$ are variables. The term Programming refers to the process of determining a particular program or plan of action.

So Linear Programming (LP) is one of the most important optimization (maximization / minimization) techniques developed in the field of Operations Research (OR).

The methods applied for solving a linear programming problem are basically simple problems; a solution can be obtained by a set of simultaneous equations. However a unique solution for a set of simultaneous equations in n-variables ($x_1, x_2 \ldots x_n$), at least one of them is non-zero, can be obtained if there are exactly n relations. When the number of relations is greater than or less than n, a unique solution does not exist but a number of trial solutions can be found.

In various practical situations, the problems are seen in which the number of relations is not equal to the number of the number of variables and many of the relations are in the form of inequalities ($\leq$ or $\geq$) to maximize or minimize a linear function of the variables subject to such conditions. Such problems are known as Linear Programming Problem (LPP).

Definition – The general LPP calls for optimizing (maximizing / minimizing) a linear function
of variables called the '**Objective function**' subject to a set of linear equations and / or
inequalities called the '**Constraints**' or '**Restrictions'**.

General form of LPP

We formulate a mathematical model for general problem of allocating resources to activities. In particular, this model is to select the values for $x_1, x_2 \ldots x_n$ so as to maximize or minimize

$Z = c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots + c_nx_n$

subject to restrictions

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \ (\leq \text{ or } \geq) \ b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \ (\leq \text{ or } \geq) \ b_2$

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$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n \ (\leq \text{ or } \geq) \ b_m$

and

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

Where

Z = value of overall measure of performance

x_j = level of activity (for $j = 1, 2, ..., n$)

c_j = increase in Z that would result from each unit increase in level of activity j

b_i = amount of resource i that is available for allocation to activities (for $i = 1,2, \ldots, m$)

a_{ij} = amount of resource i consumed by each unit of activity j

- The level of activities $x_1, x_2 \ldots\ldots\ldots x_n$ are called **decision variables**.
- The values of the c_j, b_i, a_{ij} (for $i=1, 2 \ldots m$ and $j=1, 2 \ldots n$)are the **input constants** for the model. They are called as **parameters** of the model.
- The function being maximized or minimized $Z = c_1x_1 + c_2x_2 + \ldots. + c_nx_n$ is called **objective function**.
- The restrictions are normally called as **constraints**. The constraint $a_{i1}x_1 + a_{i2}x_2 \ldots a_{in}x_n$ are sometimes called as **functional constraint** (L.H.S constraint). $x_j \geq 0$ restrictions are called **non-negativity constraint**.

Advantages of Linear Programming Techniques

1. It helps us in making the optimum utilization of productive resources.
2. The quality of decisions may also be improved by linear programming techniques.
3. Provides practically solutions.
4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique.

Formulation of LP Problems

Example 1

A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires 1minute of processing time on G and 2 minutes on H; type B requires 1 minute on G and 1 minute on H. The machine G is available for not more than 6 hours 40 minutes while machine H is available for 10 hours during any working day. Formulate the problem as a linear programming problem.

Solution

Let

x1 be the number of products of type A

x2 be the number of products of type B

After understanding the problem, the given information can be systematically arranged in the form of the following table.

	Type of products (minutes)		
Machine	Type A (x1 units)	Type B (x2 units)	Available time (mins)
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type A is Rs. 2 per product, 2 x1 will be the profit on selling x1 units of type A.

Similarly, 3x2 will be the profit on selling x2 units of type B. Therefore, total profit on selling x1units of A and x2 units of type B is given by

Maximize $Z = 2x_1 + 3x_2$ (**objective function**)

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by $x_1 + x_2$.
Similarly, the total number of minutes required on machine H is given by $2x_1 + 3x_2$.
But, machine G is not available for more than 6 hours 40 minutes (400 minutes). Therefore,
$x_1 + x_2 \leq 400$ (**first constraint**)
Also, the machine H is available for 10 hours (600 minutes) only, therefore,
$2x_1 + 3x_2 \leq 600$ (**second constraint**).
Since it is not possible to produce negative quantities
$x_1 \geq 0$ and $x_2 \geq 0$ (**non-negative restrictions**)
Hence
Maximize $Z = 2x_1 + 3x_2$
Subject to restrictions
$x_1 + x_2 \leq 400$
$2x_1 + 3x_2 \leq 600$
and non-negativity constraints
$x_1 \geq 0$, $x_2 \geq 0$

Example 2
A company produces two products A and B which possess raw materials 400 quintals and 450 labour hours. It is known that 1 unit of product A requires 5 quintals of raw materials and 10 man hours and yields a profit of Rs 45. Product B requires 20 quintals of raw materials, 15 man hours and yields a profit of Rs 80. Formulate the LPP.
Solution

Let x_1 be the number of units of product A
x_2 be the number of units of product B

	Product A	Product B	Availability
Raw materials	5	20	400
Man hours	10	15	450
Profit	Rs 45	Rs 80	

Hence
Maximize $Z = 45x1 + 80x2$
Subject to
$5x1 + 20\,x2 \leq 400$
$10x1 + 15x2 \leq 450$
$x1 \geq 0$, $x2 \geq 0$

Example 3
A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and below is given the required processing time in minutes for each machine on each product.

	Products		
Machine	A	B	C
X	4	3	5
Y	2	2	4

Machine X and Y have 2000 and 2500 machine minutes. The firm must manufacture 100 A's, 200 B's and 50 C's type, but not more than 150 A's.

Solution

Let
$x1$ be the number of units of product A
$x2$ be the number of units of product B
$x3$ be the number of units of product C

Max $Z = 3x1 + 2x2 + 4x3$
Subject to
$4x1 + 3x2 + 5x3 \leq 2000$
$2x1 + 2x2 + 4x3 \leq 2500$
$100 \leq x1 \leq 150$
$x2 \geq 200$
$x3 \geq 50$

Example 4
A company owns 2 oil mills A and B which have different production capacities for low, high and medium grade oil. The company enters into a contract to supply oil to a firm every week with 12, 8, 24 barrels of each grade respectively. It costs the company Rs 1000 and Rs 800 per day to run the mills A and B. On a day A produces 6, 2, 4 barrels of each grade and B produces 2, 2, 12 barrels of each grade. Formulate an LPP to determine number of days per week each mill will be operated in order to meet the contract economically.
Solution
Let
$x1$ be the no. of days a week the mill A has to work
$x2$ be the no. of days per week the mill B has to work

Minimize $Z = 1000x1 + 800\,x2$
Subject to
$6x1 + 2x2 \geq 12$
$2x1 + 2x2 \geq 8$
$4x1 + 12x2 \geq 24$
$x1 \geq 0\ ,\quad x2 \geq 0$

Example 5

A company has 3 operational departments weaving, processing and packing with the capacity to produce 3 different types of clothes that are suiting, shirting and woolen yielding with the profit of Rs. 2, Rs. 4 and Rs. 3 per meters respectively. 1m suiting requires 3mins in weaving 2 mins in processing and 1 min in packing. Similarly 1m of shirting requires 4 mins in weaving 1 min in processing and 3 mins in packing while 1m of woolen requires 3 mins in each department. In a week total run time of each department is 60, 40 and 80 hours for weaving, processing andpacking department respectively. Formulate a LPP to find the product to maximize the profit.

Solution

Let

x1 be the number of units of suiting

x2 be the number of units of shirting

x3 be the number of units of woolen

Maximize $Z = 2x1 + 4x2 + 3x3$

Subject to

$3x1 + 4x2 + 3x3 \leq 60$

$2x1 + 1x2 + 3x3 \leq 40$

$x1 + 3x2 + 3x3 \leq 80$

$x1 \geq 0, \quad x2 \geq 0, \quad x3 \geq 0$

Graphical Solution Procedure

The graphical solution procedure

1. Consider each inequality constraint as equation.
2. Plot each equation on the graph as each one will geometrically represent a straight line.
3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality constraint corresponding to that line is '≤' then the region below the line lying in the first quadrant is shaded. Similarly for '≥' the region above the line is shaded. The points lying in the common region will satisfy the constraints. This common region is called **feasible region.**
4. Choose the convenient value of Z and plot the objective function line.
5. Pull the objective function line until the extreme points of feasible region.

a. In the maximization case this line will stop far from the origin and passing
through at least one corner of the feasible region.

b. In the minimization case, this line will stop near to the origin and passing through
at least one corner of the feasible region.

6. Read the co-ordinates of the extreme points selected in step 5 and find the maximum or minimum value of Z.

Definitions

1. **Solution** – Any specification of the values for decision variable among (x1, x2… xn) is called a solution.
2. **Feasible solution** is a solution for which all constraints are satisfied.
3. **Infeasible solution** is a solution for which atleast one constraint is not satisfied.
4. **Feasible region** is a collection of all feasible solutions.
5. **Optimal solution** is a feasible solution that has the most favorable value of the objective function.

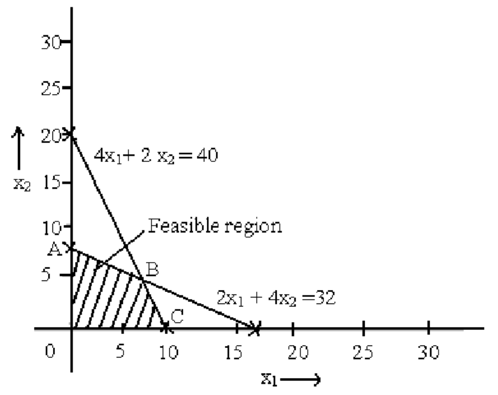
6. **Most favorable value** is the largest value if the objective function is to be maximized, whereas it is the smallest value if the objective function is to be minimized.
7. **Multiple optimal solution** – More than one solution with the same optimal value of the objective function.
8. **Unbounded solution** – If the value of the objective function can be increased or
decreased indefinitely such solutions are called unbounded solution.
9. **Feasible region** – The region containing all the solutions of an inequality
10. **Corner point feasible solution** is a solution that lies at the corner of the feasible region.

Example 1

Max $Z = 80x1 + 55x2$
Subject to
$4x1 + 2x2 \leq 40$
$2x1 + 4x2 \leq 32$
$x1 \geq 0$, $x2 \geq 0$

Solution
The first constraint $4x1 + 2x2 \leq 40$, written in a form of equation
$4x1 + 2x2 = 40$
Put $x1 = 0$, then $x2 = 20$
Put $x2 = 0$, then $x1 = 10$
The coordinates are (0, 20) and (10, 0)
The second constraint $2x1 + 4x2 \leq 32$, written in a form of equation
$2x1 + 4x2 = 32$
Put $x1 = 0$, then $x2 = 8$
Put $x2 = 0$, then $x1 = 16$
The coordinates are (0, 8) and (16, 0)

The graphical representation is



The corner points of feasible region are A, B and C. So the coordinates for the corner points are A (0, 8) B (8, 4) (Solve the two equations $4x1+ 2\ x2 = 40$ and $2x1 + 4x2 =32$ to get the coordinates) C (10, 0).

We know that Max $Z = 80x1 + 55x2$

At A (0, 8)

$Z = 80(0) + 55(8) = 440$

At B (8, 4)

$Z = 80(8) + 55(4) = 860$

At C (10, 0)

$Z = 80(10) + 55(0) = 800$

The maximum value is obtained at the point B. Therefore Max Z = 860 and x1 = 8, x2 = 4

Example 2

Minimize $Z = 10x1 + 4x2$

Subject to

$3x1 + 2x2 \geq 60$

$7x1 + 2x2 \geq 84$

$3x1 + 6x2 \geq 72$

$x1 \geq 0$, $x2 \geq 0$

Solution

The first constraint $3x1 + 2x2 \geq 60$, written in a form of equation

$3x1 + 2x2 = 60$

Put $x1 = 0$, then $x2 = 30$

Put $x2 = 0$, then $x1 = 20$

The coordinates are (0, 30) and (20, 0)

The second constraint $7x1 + 2x2 \geq 84$, written in a form of equation

$7x1 + 2x2 = 84$

Put $x1 = 0$, then $x2 = 42$

Put $x2 = 0$, then $x1 = 12$

The coordinates are (0, 42) and (12, 0)

The third constraint $3x1 + 6x2 \geq 72$, written in a form of equation

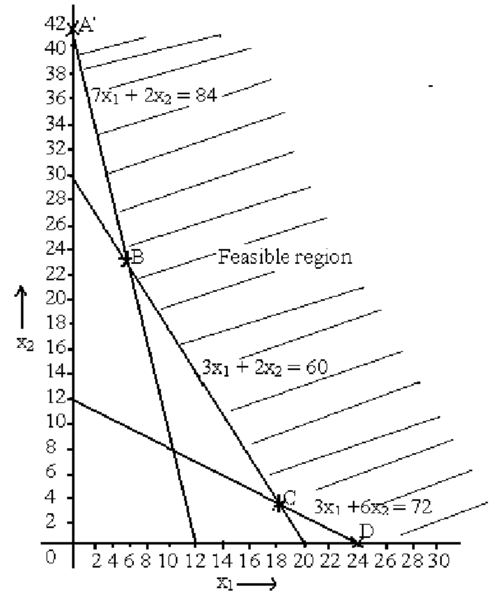
$3x1 + 6x2 = 72$

Put $x1 = 0$, then $x2 = 12$

Put $x2 = 0$, then $x1 = 24$

The coordinates are (0, 12) and (24, 0)

The graphical representation is



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points
are
A (0, 42)
B (6, 21) (Solve the two equations 7x1 + 2x2 = 84 and 3x1 + 2x2 = 60 to get the coordinates)
C (18, 3) Solve the two equations 3x1 +6x2 = 72 and 3x1 + 2x2 = 60 to get the coordinates)
D (24, 0)
We know that Min Z = 10x1 + 4x2
At A (0, 42)
Z = 10(0) + 4(42) = 168
At B (6, 21)
Z = 10(6) + 4(21) = 144
At C (18, 3)
Z = 10(18) + 4(3) = 192

At D (24, 0)
Z = 10(24) + 4(0) = 240
The minimum value is obtained at the point B. Therefore Min Z = 144 and $x_1 = 6$, $x_2 = 21$

Some Basic Definitions
Solution of LPP
Any set of variable ($x_1, x_2 \ldots x_n$) which satisfies the given constraint is called solution of LPP.
Basic solution
Is a solution obtained by setting any 'n' variable equal to zero and solving remaining 'm' variables. Such 'm' variables are called **Basic variables** and 'n' variables are called **Non-basic variables**.
Basic feasible solution
A basic solution that is feasible (all basic variables are non negative) is called basic feasible solution. There are two types of basic feasible solution.
1. Degenerate basic feasible solution
If any of the basic variable of a basic feasible solution is zero than it is said to be
degenerate basic feasible solution.
2. Non-degenerate basic feasible solution
It is a basic feasible solution which has exactly 'm' positive x_i, where i=1, 2, … m. In
other words all 'm' basic variables are positive and remaining 'n' variables are zero.
Optimum basic feasible solution
A basic feasible solution is said to be optimum if it optimizes (max / min) the objective function.

Introduction to Simplex Method

It was developed by G. Danztig . The simplex method provides an algorithm (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the fundamental theorem of linear programming.

The Simplex algorithm is an iterative procedure for solving LP problems in a finite number of steps. It consists of

- Having a trial basic feasible solution to constraint-equations
- Testing whether it is an optimal solution
- Improving the first trial solution by a set of rules and

repeating the process till an optimal solution is obtained

Advantages

- Simple to solve the problems
- The solution of LPP of more than two variables can be

obtained.

- The solution of LPP of more than two variables can be obtained.

1.5 Computational Procedure of Simplex Method

Consider an example

Maximize $Z = 3x_1 + 2x_2$

Subject to

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Step 1 – Write the given GLPP in the form of SLPP

Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$

Step 2 – Present the constraints in the matrix form

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step 3 – Construct the starting simplex table using the notations

		$C_j \rightarrow$	3	2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	4	1	1	1	0	
s_2	0	2	1	-1	0	1	
	$Z = C_B X_B$		Δ_j				

Step 4 – Calculation of Z and Δ_j and test the basic feasible solution for optimality by the rules given.

$Z = C_B X_B$

$= 0 * 4 + 0 * 2 = 0$

$\Delta_j = Z_j - C_j$

$= C_B X_j - C_j$

$\Delta_1 = C_B X_1 - C_j = 0 * 1 + 0 * 1 - 3 = -3$

$\Delta_2 = C_B X_2 - C_j = 0 * 1 + 0 * -1 - 2 = -2$

$\Delta_3 = C_B X_3 - C_j = 0 * 1 + 0 * 0 - 0 = 0$

$\Delta_4 = C_B X_4 - C_j = 0 * 0 + 0 * 1 - 0 = 0$

Procedure to test the basic feasible solution for optimality by the rules given

Rule 1 – If all $\Delta_j \geq 0$, the solution under the test will be **optimal**. Alternate optimal solution will exist if any non-basic Δ_j is also zero.

Rule 2 – If atleast one Δ_j is negative, the solution is not optimal and then proceeds to improve the solution in the next step.

Rule 3 – If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero, then the solution under test will be **unbounded**.

In this problem it is observed that Δ_1 and Δ_2 are negative. Hence proceed to improve this solution

Step 5 – To improve the basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined.

- **Incoming vector**
  The incoming vector X_k is always selected corresponding to the most negative value of Δ_j. It is indicated by (↑).

- **Outgoing vector**
  The outgoing vector is selected corresponding to the least positive value of minimum ratio. It is indicated by (→).

Step 6 – Mark the key element or pivot element by '□'.The element at the intersection of outgoing vector and incoming vector is the pivot element.

		$C_j \rightarrow$	3	2	0	0	
Basic Variables	C_B	X_B	X_1 (X_k)	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	4	1	1	1	0	4 / 1 = 4
s_2	0	2	[1]	-1	0	1	2 / 1 = 2 → outgoing
	$Z = C_B X_B = 0$		↑incoming $\Delta_1 = -3$	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	

- If the number in the marked position is other than unity, divide all the elements of that row by the key element.
- Then subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeroes in the remaining position of the column X_k.

Basic Variables	C_B	X_B	X_1	X_2 (X_k)	S_1	S_2	Min ratio X_B/X_k
s_1	0	2	(R_1=R_1 - R_2) 0	[2]	1	-1	2 / 2 = 1 → outgoing
x_1	3	2	1	-1	0	1	2 / -1 = -2 (neglect in case of negative)
	Z=0*2+3*2= 6	Δ_1=0	Δ_2= -5 ↑incoming	Δ_3=0	Δ_4=3		

Step 7 – Now repeat step 4 through step 6 until an optimal solution is obtained.

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B/X_k
x_2	2	1	(R_1=R_1 / 2) 0	1	1/2	-1/2	
x_1	3	3	(R_2=R_2 + R_1) 1	0	1/2	1/2	
	Z = 11		Δ_1=0	Δ_2=0	Δ_3=5/2	Δ_4=1/2	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 11, x_1 = 3 and x_2 = 1

Example 1

Maximize $Z = 80x_1 + 55x_2$

Subject to

$4x_1 + 2x_2 \leq 40$

$2x_1 + 4x_2 \leq 32$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 80x_1 + 55x_2 + 0s_1 + 0s_2$

Subject to

$4x_1 + 2x_2 + s_1 = 40$

$2x_1 + 4x_2 + s_2 = 32$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

	$C_j \rightarrow$		80	55	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	40	[4]	2	1	0	40 / 4 = 10⟶ outgoing
s_2	0	32	2	4	0	1	32 / 2 = 16
	$Z = C_B X_B = 0$		↑incoming $\Delta_1 = -80$	$\Delta_2 = -55$	$\Delta_3 = 0$	$\Delta_4 = 0$	
x_1	80	10	$(R_1 = R_1 / 4)$ 1	1/2	1/4	0	10/1/2 = 20
s_2	0	12	$(R_2 = R_2 - 2R_1)$ 0	[3]	-1/2	1	12/3 = 4⟶ outgoing
	Z = 800		$\Delta_1 = 0$	↑incoming $\Delta_2 = -15$	$\Delta_3 = 40$	$\Delta_4 = 0$	
x_1	80	8	$(R_1 = R_1 - 1/2R_2)$ 1	0	1/3	-1/6	
x_2	55	4	$(R_2 = R_2 / 3)$ 0	1	-1/6	1/3	
	Z = 860		$\Delta_1 = 0$	$\Delta_2 = 0$	$\Delta_3 = 35/2$	$\Delta_4 = 5$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 860, $x_1 = 8$ and $x_2 = 4$

Example 2

Maximize $Z = 5x_1 + 3x_2$

Subject to

$3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2$

Subject to

$3x_1 + 5x_2 + s_1 = 15$

$5x_1 + 2x_2 + s_2 = 10$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

		$C_j \rightarrow$	5	3	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	15	3	5	1	0	15 / 3 = 5
s_2	0	10	[5]	2	0	1	10 / 5 = 2 $\rightarrow$ outgoing
	$Z = C_B X_B = 0$		$\uparrow$incoming $\Delta_1 = -5$	$\Delta_2 = -3$	$\Delta_3 = 0$	$\Delta_4 = 0$	
s_1	0	9	$(R_1 = R_1 - 3R_2)$ 0	[19/5]	1	-3/5	9/19/5 = 45/19 $\rightarrow$
x_1	5	2	$(R_2 = R_2 / 5)$ 1	2/5	0	1/5	2/2/5 = 5
	$Z = 10$		$\Delta_1 = 0$	$\uparrow$ $\Delta_2 = -1$	$\Delta_3 = 0$	$\Delta_4 = 1$	
x_2	3	45/19	$(R_1 = R_1 / 19/5)$ 0	1	5/19	-3/19	
x_1	5	20/19	$(R_2 = R_2 - 2/5\ R_1)$ 1	0	-2/19	5/19	
	$Z = 235/19$		$\Delta_1 = 0$	$\Delta_2 = 0$	$\Delta_3 = 5/19$	$\Delta_4 = 16/19$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 235/19$, $x_1 = 20/19$ and $x_2 = 45/19$

Example 3

Maximize $Z = 5x_1 + 7x_2$

Subject to

$x_1 + x_2 \le 4$

$3x_1 - 8x_2 \le 24$

$10x_1 + 7x_2 \le 35$

and $x_1 \ge 0,\ x_2 \ge 0$

Solution

SLPP

Maximize $Z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$x_1 + x_2 + s_1 = 4$

$3x_1 - 8x_2 + s_2 = 24$

$10x_1 + 7x_2 + s_3 = 35$

$x_1 \ge 0,\ x_2 \ge 0,\ s_1 \ge 0,\ s_2 \ge 0,\ s_3 \ge 0$

		$C_j \rightarrow$	5	7	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	4	1	[1]	1	0	0	4 /1 = 4→outgoing
s_2	0	24	3	-8	0	1	0	–
s_3	0	35	10	7	0	0	1	35 / 7 = 5
	$Z = C_B X_B = 0$		-5	-7 ↑incoming	0	0	0	$\leftarrow \Delta_j$
x_2	7	4	1	1	1	0	0	
s_2	0	56	($R_2 = R_2 + 8R_1$) 11	0	8	1	0	
s_3	0	7	($R_3 = R_3 - 7R_1$) 3	0	-7	0	1	
	$Z = 28$		2	0	7	0	0	$\leftarrow \Delta_j$

Since all $\Delta_j \ge 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 28$, $x_1 = 0$ and $x_2 = 4$

Computational Procedure of Big – M Method (Charne's Penalty Method)

Step 1 – Express the problem in the standard form.
Step 2 – Add non-negative artificial variable to the left side of each of the equations
corresponding to the constraints of the type '≥' or '=' .
When artificial variables are added, it causes violation of the corresponding constraints. This
difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists).
On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by –M for maximization problems (+M for minimizing problem), where M > 0.
Step 3 – In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

Example 1

Max $Z = -2x_1 - x_2$

Subject to

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Max $Z = -2x_1 - x_2 + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$

		$C_j \rightarrow$	-2	-1	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	3	[3]	1	0	0	1	0	$3/3 = 1 \rightarrow$
a_2	-M	6	4	3	-1	0	0	1	$6/4 = 1.5$
s_2	0	4	1	2	0	1	0	0	$4/1 = 4$
	Z = -9M		↑ $2 - 7M$	$1 - 4M$	M	0	0	0	$\leftarrow \Delta_j$
x_1	-2	1	1	1/3	0	0	X	0	$1/1/3 = 3$
a_2	-M	2	0	[5/3]	-1	0	X	1	$6/5/3 = 1.2 \rightarrow$
s_2	0	3	0	5/3	0	1	X	0	$4/5/3 = 1.8$
	Z = -2 – 2M		0	↑ $\frac{(-5M+1)}{3}$	0	0	X	0	$\leftarrow \Delta_j$
x_1	-2	3/5	1	0	1/5	0	X	X	
x_2	-1	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	Z = -12/5		0	0	1/5	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = -12/5$, $x_1 = 3/5$, $x_2 = 6/5$

Example 2

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

and $x_1 \geq 0,\ x_2 \geq 0$

Solution

SLPP

Max $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - M\,a_1$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 3$

$x_2 + s_3 = 4$

$x_1,\ x_2,\ s_1,\ s_2,\ s_3,\ a_1 \geq 0$

		$C_j \rightarrow$	3	-1	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-M	2	[2]	1	-1	0	0	1	$2 / 2 = 1 \rightarrow$
s_2	0	3	1	3	0	1	0	0	$3 / 1 = 3$
s_3	0	4	0	1	0	0	1	0	-
	Z = -2M		↑ -2M-3	-M+1	M	0	0	0	$\leftarrow \Delta_j$
x_1	3	1	1	1/2	-1/2	0	0	X	-
s_2	0	2	0	5/2	[1/2]	1	0	X	$2/1/2 = 4 \rightarrow$
s_3	0	4	0	1	0	0	1	X	-
	Z = 3		0	5/2	↑ -3/2	0	0	X	$\leftarrow \Delta_j$
x_1	3	3	1	3	0	1/2	0	X	
s_1	0	4	0	5	1	2	0	X	
s_3	0	4	0	1	0	0	1	X	
	Z = 9		0	10	0	3/2	0	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 9,\ x_1 = 3,\ x_2 = 0$

Example 3

Min $Z = 2x_1 + 3x_2$

Subject to

$x_1 + x_2 \geq 5$

$x_1 + 2x_2 \geq 6$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Min Z = Max $Z' = -2x_1 - 3x_2 + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$x_1 + x_2 - s_1 + a_1 = 5$

$x_1 + 2x_2 - s_2 + a_2 = 6$

$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$

		$C_j \rightarrow$	-2	-3	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B/X_k
a_1	-M	5	1	1	-1	0	1	0	5 / 1 = 5
a_2	-M	6	1	[2]	0	-1	0	1	6 / 2 = 3 →
	Z' = -11M		-2M + 2	↑ -3M+3	M	M	0	0	← Δ_j
a_1	-M	2	[1/2]	0	-1	1/2	1	X	2/1/2 = 4 →
x_2	-3	3	1/2	1	0	-1/2	0	X	3/1/2 = 6
	Z' = -2M-9		↑ (-M+1) / 2	0	M	(-M+3)/2	0	X	← Δ_j
x_1	-2	4	1	0	-2	1	X	X	
x_2	-3	1	0	1	1	-1	X	X	
	Z' = -11		0	0	1	1	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Z' = -11 which implies Max Z = 11, $x_1 = 4$, $x_2 = 1$

Steps for Two-Phase Method

The process of eliminating artificial variables is performed in **phase-I** of the solution and **phase-II** is used to get an optimal solution. Since the solution of LPP is computed in two phases, it is called as **Two-Phase Simplex Method**.

Phase I – In this phase, the simplex method is applied to a specially constructed **auxiliary linear programming problem** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 – Assign a cost -1 to each artificial variable and a cost 0 to all other variables in the objective function.

Step 2 – Construct the Auxiliary LPP in which the new objective function Z* is to be
maximized subject to the given set of constraints.

Step 3 – Solve the auxiliary problem by simplex method until either of the following
three possibilities do arise

i. Max Z* < 0 and atleast one artificial vector appear in the optimum basis at
a positive level ($\Delta j \geq 0$). In this case, given problem does not possess any feasible solution.

ii. Max Z* = 0 and at least one artificial vector appears in the optimum basis
at a zero level. In this case proceed to phase-II.

iii. Max Z* = 0 and no one artificial vector appears in the optimum basis. In
this case also proceed to phase-II.

Phase II – Now assign the actual cost to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. Simplex method is applied to the modified simplex table obtained at the end of phase-I, until an optimum basic feasible solution has been attained. The artificial variables which are non-basic at the end of phase-I are removed.

Example 1

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 2$

$x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	0	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-1	2	[2]	1	-1	0	0	1	1→
s_2	0	2	1	3	0	1	0	0	2
s_3	0	4	0	1	0	0	1	0	-
		$Z^* = -2$	↑ -2	-1	1	0	0	0	←Δ_j
x_1	0	1	1	1/2	-1/2	0	0	X	
s_2	0	1	0	5/2	1/2	1	0	X	
s_3	0	4	0	1	0	0	1	X	
		$Z^* = 0$	0	0	0	0	0	X	←Δ_j

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
	$C_j \rightarrow$		3	-1	0	0	0	
x_1	3	1	1	1/2	-1/2	0	0	-
s_2	0	1	0	5/2	1/2	1	0	$2 \rightarrow$
s_3	0	4	0	1	0	0	1	-
	Z = 3		0	5/2	-3/2 ↑	0	0	$\leftarrow \Delta_j$
x_1	3	2	1	3	0	1	0	
s_1	0	2	0	5	1	2	0	
s_3	0	4	0	1	0	0	1	
	Z = 6		0	10	0	3	0	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 6, $x_1 = 2$, $x_2 = 0$

Example 2

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1 - 1a_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

$\text{Max } Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1 - 1a_2$

Subject to

$$3x_1 + 2x_2 - s_1 + a_1 = 3$$
$$x_1 + 4x_2 - s_2 + a_2 = 4$$
$$x_1 + x_2 + s_3 = 5$$
$$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$$

Phase I

		$C_j \rightarrow$	0	0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	A_2	Min ratio X_B / X_k
a_1	-1	3	3	2	-1	0	0	1	0	3/2
a_2	-1	4	1	[4]	0	-1	0	0	1	1→
s_3	0	5	1	1	0	0	1	0	0	5
	$Z^* = -7$		-4	↑ -6	1	1	0	0	0	←Δ_j
a_1	-1	1	[5/2]	0	-1	1/2	0	1	X	2/5→
x_2	0	1	1/4	1	0	-1/4	0	0	X	4
s_3	0	4	3/4	0	0	1/4	1	0	X	16/3
	$Z^* = -1$		↑ -5/2	0	1	-1/2	0	0	X	←Δ_j
x_1	0	2/5	1	0	-2/5	1/5	0	X	X	
x_2	0	9/10	0	1	1/10	-3/10	0	X	X	
s_3	0	37/10	0	0	3/10	1/10	1	X	X	
	$Z^* = 0$		0	0	0	0	0	X	X	←Δ_j

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

		$C_j \rightarrow$	5	8	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
x_1	5	2/5	1	0	-2/5	[1/5]	0	2→
x_2	8	9/10	0	1	1/10	-3/10	0	-
s_3	0	37/10	0	0	3/10	1/10	1	37
	$Z = 46/5$		0	0	-6/5	↑ -7/5	0	←Δ_j
s_2	0	2	5	0	-2	1	0	-
x_2	8	3/2	3/2	1	-1/2	0	0	-
s_3	0	7/2	-1/2	0	[1/2]	0	1	7→
	$Z = 12$		7	0	↑ -4	0	0	←Δ_j
s_2	0	16	3	0	0	1	2	
x_2	8	5	1	1	0	0	1/2	
s_1	0	7	-1	0	1	0	2	

Phase II

	$C_j \rightarrow$		5	8	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B/X_k
x_1	5	2/5	1	0	-2/5	[1/5]	0	$2 \rightarrow$
x_2	8	9/10	0	1	1/10	-3/10	0	-
s_3	0	37/10	0	0	3/10	1/10	1	37
	Z = 46/5		0	0	-6/5	-7/5 ↑	0	$\leftarrow \Delta_j$
s_2	0	2	5	0	-2	1	0	-
x_2	8	3/2	3/2	1	-1/2	0	0	-
s_3	0	7/2	-1/2	0	[1/2]	0	1	$7 \rightarrow$
	Z = 12		7	0	-4 ↑	0	0	$\leftarrow \Delta_j$
s_2	0	16	3	0	0	1	2	
x_2	8	5	1	1	0	0	1/2	
s_1	0	7	-1	0	1	0	2	
	Z = 40		3	0	0	0	4	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 40, $x_1 = 0$, $x_2 = 5$

Example 3

Max $Z = -4x_1 - 3x_2 - 9x_3$

Subject to

$2x_1 + 4x_2 + 6x_3 \geq 15$

$6x_1 + x_2 + 6x_3 \geq 12$

and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

Solution

Standard LPP

Max $Z = -4x_1 - 3x_2 - 9x_3$

Subject to

$$2x_1 + 4x_2 + 6x_3 - s_1 + a_1 = 15$$
$$6x_1 + x_2 + 6x_3 - s_2 + a_2 = 12$$
$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 - 0x_3 + 0s_1 + 0s_2 - 1a_1 - 1a_2$

Subject to

$$2x_1 + 4x_2 + 6x_3 - s_1 + a_1 = 15$$
$$6x_1 + x_2 + 6x_3 - s_2 + a_2 = 12$$
$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

Phase I

	$C_j \rightarrow$		0	0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-1	15	2	4	6	-1	0	1	0	15/6
a_2	-1	12	6	1	[6]	0	-1	0	1	2→
	$Z^* = -27$		-8	-5	↑ -12	1	1	0	0	←Δ_j
a_1	-1	3	-4	[3]	0	-1	1	1	X	1→
x_3	0	2	1	1/6	1	0	-1/6	0	X	12
	$Z^* = -3$		4	↑ -3	0	1	-1	0	X	←Δ_j
x_2	0	1	-4/3	1	0	-1/3	1/3	X	X	
x_3	0	11/6	22/18	0	1	1/18	-4/18	X	X	
	$Z^* = 0$		0	0	0	0	0	X	X	

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

	$C_j \rightarrow$		-4	-3	-9	0	0	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	Min ratio X_B / X_k
x_2	-3	1	-4/3	1	0	-1/3	1/3	-
x_3	-9	11/6	[22/18]	0	1	1/18	-4/18	3/2→
	$Z = -39/2$		↑ -3	0	0	1/2	1	←Δ_j
x_2	-3	3	0	1	12/11	-3/11	1/11	
x_1	-4	3/2	1	0	18/22	1/22	-4/22	
	$Z = -15$		0	0	27/11	7/11	5/11	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = -15$, $x_1 = 3/2$, $x_2 = 3$, $x_3 = 0$

Example 4

Example 4

Min $Z = 4x_1 + x_2$

Subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$

and $x_1 \geq 0,\ x_2 \geq 0$

Solution

Standard LPP

Min Z = Max $Z' = -4x_1 - x_2$

Subject to

$$3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - s_1 + a_2 = 6$$
$$x_1 + 2x_2 + s_2 = 4$$
$$x_1,\ x_2,\ s_1,\ s_2,\ a_1,\ a_2 \geq 0$$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 + 0s_1 + 0s_2 - 1a_1 - 1a_2$

Subject to

$$3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - s_1 + a_2 = 6$$
$$x_1 + 2x_2 + s_2 = 4$$
$$x_1,\ x_2,\ s_1,\ s_2,\ a_1,\ a_2 \geq 0$$

Phase I

	$C_j \rightarrow$		0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-1	3	[3]	1	0	0	1	0	1→
a_2	-1	6	4	3	-1	0	0	1	6/4
s_2	0	4	1	2	0	1	0	0	4
	$Z^* = -9$		↑ -7	-4	1	0	0	0	
x_1	0	1	1	1/3	0	0	X	0	3
a_2	-1	2	0	[5/3]	-1	0	X	1	6/5→
s_2	0	3	0	5/3	0	1	X	0	9/5
	$Z^* = -2$		0	↑ -5/3	1	0	X	0	
x_1	0	3/5	1	0	1/5	0	X	X	
x_2	0	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	$Z^* = 0$		0	0	0	0	X	X	

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase I

	$C_j \rightarrow$		0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-1	3	[3]	1	0	0	1	0	1→
a_2	-1	6	4	3	-1	0	0	1	6/4
s_2	0	4	1	2	0	1	0	0	4
	$Z^* = -9$		↑ -7	-4	1	0	0	0	
x_1	0	1	1	1/3	0	0	X	0	3
a_2	-1	2	0	[5/3]	-1	0	X	1	6/5→
s_2	0	3	0	5/3	0	1	X	0	9/5
	$Z^* = -2$		0	↑ -5/3	1	0	X	0	
x_1	0	3/5	1	0	1/5	0	X	X	
x_2	0	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	$Z^* = 0$		0	0	0	0	X	X	

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

	$C_j \rightarrow$		-4	-1	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
x_1	-4	3/5	1	0	1/5	0	3
x_2	-1	6/5	0	1	-3/5	0	-
s_2	0	1	0	0	[1]	1	1→
	$Z' = -18/5$		0	0	↑ -1/5	0	$\leftarrow \Delta_j$
x_1	-4	2/5	1	0	0	-1/5	
x_2	-1	9/5	0	1	0	3/5	
s_1	0	1	0	0	1	1	
	$Z' = -17/5$		0	0	0	1/5	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z' = -17/5$
Min $Z = 17/5$, $x_1 = 2/5$, $x_2 = 9/5$