وزارة التعليم العالي و البحث العلمي
جامعة ديالى
كلية التربية المقداد
٢٠٢٠ - ٢٠٢١

الاستاذة : م. م أيناس حسن عبد كاظم

المادة : **التحليل الدالي**

القسم : **الرياضيات**

المرحلة : **الرابعة**

عدد طلاب المرحلة : ٩٩

قائمة حضور طلبـــة المرحلـــة الرابعـــة صباحي قســـم الرياضيـــات للعام الدراسي

2020-2021 / الشهر: 2 المادة: تحليل دالي الاستاذ:ماجستير ايناس حسن

ت	التاريخ / اسم الطالب					الملاحظات
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.3	احمد كريم عبد علوان	ح	ح	غ	ح	
.4	اسماء جليل حسن حمد	ح	ح	ح	ح	
.5	اسيل رافد عباس داود	غ	ح	ح	ح	
.6	امامة ابراهيم حسين حمد	ح	ح	ح	غ	
.7	امير حسن عبدالهادي حميد	ح	ح	ح	ح	
.8	اياد عبد علوان حسون	غ	غ	ح	ح	
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.14	حسن عبد السادة فزيع جابر	غ	ح	ح	ح	
.15	حسن عشب زعال	غ	غ	ح	ح	
.16	حسين عبدالله شبلاوي عبد	غ	غ	ح	غ	
.17	حقي محمد علي محمود	غ	ح	ح	غ	
.18	حيدر حسين ابراهيم	غ	غ	ح	ح	
.19	محمد رحيم ناصر عليوي	غ	ح	ح	ح	
.20	خليل علي حسين	ح	ح	ح	ح	
.21	رقية تركي عبعوب مطلك	غ	غ	ح	ح	
.22	رقية عدوان حكمان عابد	ح	ح	ح	ح	
.23	رقية نوح عباس	ح	ح	ح	ح	
.24	رندة طه ياسين حسين	غ	ح	ح	ح	
.25	رونق عدنان مبارك كاظم	غ	غ	ح	ح	
.26	رياض ابراهيم حاتم رحمان	غ	غ	ح	غ	
.27	ريسان علي رشيد كصير	غ	ح	ح	غ	
.28	زهراء جمال علي داود	غ	غ	ح	ح	
.29	زينب سمير قحطان نصيف	غ	ح	ح	ح	
.30	سحر حميد ابراهيم فارس	ح	ح	ح	ح	
.31	سكينة حسين علي سعدون	غ	غ	ح	ح	
.32	سيف عبد الكريم مري كاظم	ح	ح	ح	ح	
.33	شذى سهيل نجم عبعوب	ح	ح	ح	ح	
.34	شهد شيرزاد نامق سمين	غ	ح	ح	ح	
.35	طاهر صالح عبد الكاظم داود	غ	غ	ح	ح	
.36	علي خليل سلطان	غ	غ	ح	غ	
.37	علي هادي عبيد منشود	غ	ح	ح	غ	
.38	غزوان فيصل خليل عبد	غ	غ	ح	ح	
.39	فاطمة اسماعيل حسون غفور	غ	ح	ح	ح	
.40	قيصر ضياء فاضل علو	ح	ح	ح	ح	
.41	لمياء كريم عبود حسين	غ	غ	ح	ح	
.42	محمد متعب سلمان شاكر	ح	ح	ح	ح	
.43	محمد مسير هاشم محمد	ح	ح	ح	ح	
.44	محمد مقداد كريم يحيى	غ	ح	ح	ح	
.45	محمود عدنان علي	غ	غ	ح	ح	
.46	مريم حمزة عارف منصور	غ	غ	ح	غ	

قائمة حضور طلبـــة المرحلـــة الرابعـــة صباحي قســـم الرياضيـــات للعام الدراسي 2020-2021 / الشهر: 2 المادة: تحليل دالي الاستاذ:ماجستير ايناس حسن

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.48	هدى سلمان خادم صالح	ح	ح	ح	ح	
.49	واثق مهدي صالح عبد	ح	ح	ح	غ	
.50	وفاء عبد الكريم سلطان سدره	ح	ح	غ	ح	
.51	ياسر اياد اسعد حسن	ح	ح	ح	ح	
.52	يمام مجيد حميد	غ	ح	ح	ح	
.53	ازهر شوكت حمادي سلمان	ح	ح	ح	غ	
.54	اسماء هادي نجم عبد الوهاب	ح	ح	ح	غ	
.55	بيداء حسين حميد حسين	غ	ح	ح	ح	
.56	حسين سليمان فليح حسن	ح	ح	ح	ح	
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.59	حيدر احمد جواد	ح	ح	ح	ح	
.60	حيدر مكصد عبد الامير	غ	ح	ح	ح	
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.63	زهراء طالب حسن حميد	غ	ح	ح	ح	
.64	زهراء عدنان عثمان	ح	ح	ح	ح	
.65	زينب حقي اسماعيل	ح	ح	ح	غ	
.66	سجاد نعيم عباس	غ	ح	ح	ح	
.67	سعد كاظم راشد	ح	ح	ح	ح	
.68	شهد عدنان محمود عباس	ح	ح	ح	غ	
.69	صابرين شهيد عبد الرحمن حميد	ح	ح	غ	ح	
.70	عزيز خليل منصور عباس	ح	ح	ح	ح	
.71	علي باسم علي احمد	غ	ح	ح	ح	
.72	علي حسين علي محمد	ح	ح	ح	غ	
.73	علي زامل مديد مدير	ح	ح	ح	غ	
.74	علي عبد الحكيم قادرسليمان	غ	ح	ح	ح	
.75	علي يوسف خرموش علي	ح	ح	ح	ح	
.76	عهود ثائر جابر وداعه	ح	ح	ح	غ	
.77	فاطمة حسن كاظم يحيى	ح	ح	غ	ح	
.78	قاسم خالد عنون مبارك	غ	ح	ح	ح	
.79	كاظم حسين علي كاظم	غ	غ	ح	ح	
.80	كاظم عقيل كريم	غ	غ	ح	غ	
.81	كرار حسن جواد كاظم	غ	ح	ح	غ	
.82	كرار ستار خلف	غ	غ	ح	ح	
.83	كرار فليح حسن مسعود	غ	ح	ح	ح	
.84	محمد حسن عجمي نعمة	ح	ح	ح	ح	
.85	محمد مزهر عمير مصلح	غ	غ	ح	ح	
.86	محمود احمد حسين محمد	ح	ح	ح	ح	
.87	مرتضى مجيد فنجان	ح	ح	ح	ح	
.88	مروة خالد تحسين علي	ح	ح	ح	غ	
.89	مريم علي حسن دكدك	ح	ح	غ	ح	
.90	مريم مشعان طه خميس	ح	ح	ح	غ	
.91	مصطفى رياض حسن شناوة	غ	غ	ح	غ	

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ت	التاريخ / اسم الطالب					الملاحظات
.92	مصطفى فؤاد حسن نعمان	غ	ح	ح	ح	
.93	منتظر حسين حاتم حسين	ح	ح	ح	ح	
.94	نهى رشيد مجيد حميد	ح	ح	ح	غ	
.95	نوار صالح سلمان خلف	ح	ح	غ	ح	
.96	نور الهدى عدنان اسماعيل	ح	ح	ح	ح	
.97	همام جلال جميل كربول	غ	ح	ح	ح	
.98	هيه علي كامل اسماعيل	ح	ح	ح	غ	
.99	وديان حافظ رشيد سعد	ح	ح	ح	غ	

جامعة ديالى
كلية التربية المقداد
قسم الرياضيات – المرحلة الرابعة

٢٠٢٠ - ٢٠٢١

محاضرات التحليل الدالي

أعداد استاذة المادة : م. م ايناس حسن عبد كاظم

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Chapter 1

Linear Space

A linear space (also called vector space), denoted by L or V, is a collection of objects called **vectors**, which may be added together and multiplied by numbers, called **scalars** which are taken from a field F. Before defining .linear space, we first define an arbitrary field

. *Field*1.1Definition

.Let F be a non-empty set and + and . be two binary operations on F

The ordered triple $(F, +, .)$ is called **field** if and only if

$F, +)$ is a commutative group)(1)

$F - \{e\}, .)$ is a commutative group, where e is the identity with respect)(2)

.(+) to

(is distributed over (+) (from left and right (.)(3)

.1.2Example

Let (+) and (.) are **ordinary addition and multiplications**. Then

- Each of $(\mathbb{R}, +, .), (C, +, .)$, and $(Q, +, .)$ are examples of fields
-) does not hold) and $(Z, +, .)$ is 1(1.1$Z, -, .)$ is not field (Definition) not field (Definition 1.1)2 () does not hold

. *Linear Space*1.3Definition

Let $(F,+,.)$ be a field whose elements are called **scalars**. Let L is a non empty set whose elements are called **vectors**.Then L is a **linear space** (or a **vector space**) over the field F, if

addition: There is a binary operation $+$ on L called **addition** (not **(١)** .usual addition) such that $(L,+)$ is a commutative group

$\forall\alpha \in F.\ \ x \in L,\forall$**scalar multiplication:** $\alpha.x \in L$**(٢)**

)The scalar multiplication and addition satisfy3(

$\forall\alpha \in F\, x, y \in L,\forall \quad \alpha.(x + y) = \alpha.x + \alpha.y(i)$

$\forall\alpha,\beta \in F\ x \in L,\forall$(ii)$(\ \alpha + \beta).x = \alpha.x + \beta.x$

$1\ \forall\alpha,\beta \in F$ (iv)$\forall x \in L,$(iii)$(\ \alpha.\beta).x = \alpha.(\beta.x)$

is the unity $F1\ \ x \in L$ and $\forall.x = x$

.1.4Remark

If L is a linear space over F, we say that $L(F)$ is a linear space. We also .can say L is a linear space

Examples of Linear Space1.1

.1.5Example

-The set of real numbers R, with **ordinary** addition and **ordinary** multi ,plication, is a linear space over $(F,+,.) = (R,+,.)$. Indeed

R,+) is an abelian group) (١)

$x \in R, \alpha \in R \forall \alpha.x \in R$ (٢)

(!All other conditions are satisfied (Check (٣)

.This linear space $(R,+,.)$ is called **real** linear space

.1.6Example

The set of complex numbers C, with **ordinary** addition and **ordinary** ,multiplication, is a linear space over $(F,+,.) = (C,+,.)$. Indeed

C,+) is an abelian group) (١)

$x \in C, \alpha \in C \forall \alpha.x \in C$ (٢)

(!All other conditions are satisfied (Check (٣)

.This linear space $(C,+,.)$ is called **complex** linear space

.1.7Example

$\ni$, ..., x_n1, ..., x_n) : x_1Let $(R,+,.)$ be the field of real numbers. Let $R^n = \{(x$, ..., y_n) of R^n, $_1$, ..., x_n) and $Y = (y_1$R}. For any two elements $X = (x$ define ordinary addition

$$.(, ..., x_n + y_{n1} + y_1 X + Y = (x$$

Also, define scalar multiplication in R^n over R by

$$.\forall \alpha \in R,\ \forall X \in R^n, ..., \alpha.x_n)_1 \alpha.X = (\alpha.x$$

.Show that R^n is a linear space over R

Solution: Let us check linear space conditions

We show that $(R^n, +)$ is a commutative group (1)

, ..., x_n1+ y_1 , ..., $y_n \in R^n$). Since x_1, ..., x_n), Y = (y_1a)Let X = (x)

+ $y_n \in$ R, then $X + Y \in R^n$. Hence, R^n is closed with respect

.to ordinary addition

, ..., z_n) $\in R^n$1, ..., y_n), Z = (z_1, ..., x_n), Y = ($y_1 X$ = (xb)For all)

[(, ..., z_n1, ..., y_n) + (z_1, ..., x_n) + [($y_1 X$ + ($Y + Z$) = (x

(, ..., y_n + z_n1+ z_1 , ..., x_n) + ($y_1 x$) =

(), ..., x_n + (y_n + z_n1+ z_1 + (y_1 x =

, ..., (x_n + y_n) + z_n1) + z_1+ y_1 x) =

., ..., z_n) = ($X + Y$)+Z_1, ..., x_n+ y_n)+(z_1+ $y_1 x$) =

, ..., y_n) $\in R^n$1, ..., x_n), Y = ($y_1 X$ = (xc)For all)

, ..., y_n+x_n) = 1+x_1, ..., x_n+y_n) = (y_1+y_1, ..., y_n) = (x_1, ..., x_n)+($y_1 X$+ Y = (x
Y

) $\in R^n$ such that 0, ..., 0, ..., x_n) $\in R^n$ we have (1X = (xFor all (d)

) is the 0, ..., 0, ..., x_n). Thus, (1) = (x0, ...0, ..., x_n) + (1(x

.additive identity

, ..., $-x_n$) $\in R^n$ such 1, ..., x_n) $\in R^n$ then $-X$= ($-x_1 X$ = (xIf (e)

that

.). Thus, $-X$ is the additive inverse of X0, ..., 0X + ($-X$) = (

.From (a)-(e) we get $(R^n, +)$ is a commutative group

$, \ldots, \alpha x_n \in$ R, then$_1, \ldots, x_n) \in$ R^n and $\alpha \in$ R. Since αx_1Let $X = (x$ (٢)

$., \ldots, \alpha.x_n) \in$ R$^n{}_1 \alpha.X = (\alpha.x$

.Hence, R^n is closed with respect to scalar multiplication

The scalar multiplication and addition satisfy (٣)

$, \ldots, y_n) \in$ R^n and $\alpha \in$ R, then$_1, \ldots, x_n), (y_1 X = (x$If (i)

$$(, \ldots, x_n + y_{n1} + y_1 \; \alpha.(X + Y) = \alpha.(x$$

$$((), \ldots, \alpha.(x_n + y_{n1} + y_1 \; \alpha.(x) =$$

$$(, \ldots, \alpha.x_n + \alpha.y_{n1} + \alpha.y_1 \; \alpha.x) =$$

$$(, \ldots, \alpha.y_{n1}, \ldots, \alpha.x_n) + (\alpha.y_1 \alpha.x) =$$

$$, \ldots, y_n) = \alpha.X + \alpha.Y_1, \ldots, x_n) + \alpha.(y_1 \alpha.(x =$$

$, \ldots, x_n) \in$ R^n and $\alpha, \beta \in$ R, then$_1 X = (x$If (ii)

$$, \ldots, (\alpha + \beta).x_{n1} \alpha + \beta).X = (\alpha + \beta).x)$$

$$, \ldots, \alpha.x_n + \beta.x_{n1} + \beta.x_1 \; \alpha.x =$$

$$(, \ldots, \beta.x_{n1}, \ldots, \alpha.x_n) + (\beta.x_1 \alpha.x) =$$

$$, \ldots, x_n) = \alpha.X + \beta.Y_1, \ldots, x_n) + \beta.(x_1 \alpha.(x =$$

$, \ldots, x_n) \in$ R^n and $\alpha, \beta \in$ R, then $_1 X = (x$If (iii)

$$, \ldots, (\alpha.\beta)x_{n1} \; (\alpha.\beta).X = (\alpha.\beta).x$$

$$, \ldots, x_n) = \alpha.(\beta.X) \; _1, \ldots, \beta.x_n) = \alpha. \; \beta.(x_1 \alpha.(\beta.x =$$

is the unity of R, then1 $, \ldots, x_n) \in$ R^n and $_1 X = (x$(iv)If

$$, \ldots, x_n) = X_1.x_n) = (x1, \ldots, _1.x1 X = (.1$$

.Hence R^n is a linear (vector)space over R

.1.8Example

, ..., x_n) : $_1$Let (C, +, .) be the field of complex numbers. Let C^n = {(x

, ...,$_1$, ..., x_n) and Y = (y_1, ..., $x_n \in C$}. For any two elements X = ($x_1 x$

y_n) of C^n, define

.(, ..., $x_n + y_{n1} + y_1$ $X + Y$ = (x

Define scalar multiplication in C^n over C by

.$\forall \alpha \in C$, $\forall X \in C^n$, ..., $\alpha.x_n)_1 \alpha.X$ = ($\alpha.x$

Show that C^n is a vector space over C. (**Verify that**)

.1.9Example

, y) : x, y >0), (0Let (R, +, .) be the field of real numbers. Let M = {(x,

, y) of M , define 0) and Y = (0}. For any two elements X = (x, 0

.(ordinary addition $X + Y$ = (x, y

) and0Also, define scalar multiplication in M over R by $\alpha.X$ = ($\alpha.x$,

?, $\alpha.y$) $\forall \alpha \in$ R, $\forall X, Y \in M$. Is M a linear space over R0$\alpha.Y$ = (

.**Solution:** Let us check if (M, +) is a commutative group

) $\notin M$. Thus, M is 1, 1) = (1, 0) + (0, 1) $\in$ M but (1, 0), (0, 1Since (

) = 0, 1.(1not closed under addition, then (M, +) is not group. Also, −

.) $\notin M$. Thus, M is not closed under scalar multiplication0, 1(−

.1.10Example

Let C^b(R) = {f : f : R → R ; f is bounded and continuous} set of all bounded and continuous functions defined on R. For any f, $g \in C^b$(R) and for any $\alpha \in$ R, define

$.\alpha f)(x) = \alpha.f(x)\ \forall x \in R, \forall \alpha \in R)andf+g)(x) = f(x)+g(x)\ \forall x \in R)$

Show that $C^b(R)$ is a linear space over R.

.Now, let us check linear space conditions

We show $(C^b(R),+)$ is a commutative group (١)

Let $f,\ g \in C^b(R))$ such that $f,\ g$ are continuous and bounded (a)

func- tions. We want to prove $f + g \in C^b(R)$. (i.e., $f + g$ is

(continuous and bounded

Since f, g are continuous, the sum $(f + g)$ is a continuous func-

(I)tion

$\in R_+$ such that$_2$ $, M_1$Also, since f, g are bounded functions, $\exists M$

. Hence, for all $x \in R_2$and $|g(x)| \le M_1\ f(x)| \le M|$

$._2+ M_1\ f+g)(x)| = |f(x)+g(x)| \le |f(x)| + |g(x)| \le M)|$

. Thus, $f + g$ is bounded function$_2+ M_1\ f+g)(x)| \le M)|$ **(II)**

.(By **(I)** and **(II)**, $f + g \in C^b(R$

$f,g,h \in C^b(R)$ and for all $x \in R$b)For all)

$[(f + (g+h)](x) = f(x) + [(g+h)(x]$

$(f(x) + g(x)] + h(x] =$

$.(f+g)(x) + h(x) = [(f+g)+h](x) =$

$(f,g \in C^b(R$For all (c)

$.(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$

$.0(x) = 0$by $\hat{0} : R \to 0\ f \in C^b(R)$, define $\hat{F}$or all (d)

$\ni \hat{0}$is continuous and bounded function. Thus, $\hat{0}$It is clear that $\hat{}$

$C^b(R)$ and

.(= $f(x0\,(x) = f(x) + 0)(x) = f(x) + \hat{0}f + \hat{}$)

,+ $f(x) = f(x)$ Thus0 $(x) + f(x) = 0 + f)(x) = \hat{0}$ Similarly, ($\hat{}$

$+ f = f0 = \hat{0} f + \hat{}$

.is called the additive identity $\hat{0}$

∋ e)For any $f \in C^b(\mathrm{R})$, define $-f : \mathrm{R} \to \mathrm{R}$ by $(-f)(x) = -[f(x)]\ \forall x)$

.R

.Since f is continuous, then $-f$ is continuous

.Moreover, $\forall x \in \mathrm{R}, |-f(x)| = |f(x)| \le M$. Then, $-f$ is bounded

Thus, $-f \in C^b(\mathrm{R})$ and

$f+(-f)](x) = f(x)+(-f)(x) = f(x)+(-f(x)) = f(x)-f(x)$]

$.0 = \hat{0} =$

= (Similarly, $[(-f) + f](x) = (-f)(x) + f(x) = (-f(x)) + f(x$

$0 = \hat{0}\ f(x) + f(x) = -$

.From (a)-(e) we get $(C^b(\mathrm{R}), +)$ is a commutative group

(٢) Let $f \in C^b(\mathrm{R})$ and $\alpha \in \mathrm{R}$. We want to prove $\alpha f \in C^b(\mathrm{R})$. (i.e., αf is

(continuous and bounded

.Since f is continuous, then αf is a continuous function

Also, since f is bounded functions, $\exists M \in \mathrm{R}_+$ such that $|f(x)| \le M$

. Hence, for all $x \in \mathrm{R}$

$.\alpha f)(x)| = |\alpha . f(x)| = |\alpha|\,|f(x)| \le |\alpha| M)|$

Therefore, $\alpha f \in C^b(\mathrm{R})$ ($C^b(\mathrm{R})$ is Thus, αf is bounded function.

.(closed with respect to scalar multiplication

(٣) The scalar multiplication and addition satisfy

$f, g \in C^b(R)$ and $\alpha \in R$, theni)If)

$[(\alpha(f+g))(x) = \alpha.(f+g)(x) = \alpha.[(f(x)+g(x)$

$(\alpha.f(x)+\alpha.g(x =$

$\alpha f)(x)+(\alpha g)(x) = (\alpha f +) =$

$f \in C^b(R)$ and $\alpha, \beta \in R$, then $\alpha g)(x)$ (ii)If

$(\alpha+\beta)f](x) = (\alpha+\beta).f(x)]$

$(\alpha.f(x)+\beta.f(x =$

$\alpha f)(x)+(\beta f)(x) = (\alpha f+\beta f) =$

$f \in C^b(R)$ and $\alpha, \beta \in R$, then $)(x)$ (iii)If

$.(\alpha.\beta)f](x) = (\alpha.\beta).f(x) = \alpha.(\beta.f(x)) = \alpha.[(\beta f)(x)] = [\alpha(\beta f)](x)]$

.(Hence, $(\alpha.\beta)f = \alpha(\beta f$

is the unity of R, then 1 $f \in C^b(R)$ and iv)If)

$.(.f(x) = f(x1f)(x) = 1($

.Hence, $C^b(R)$ is a linear (vector)space over R

.1.11Exercise

Let $C^b[a, b] = \{f : f : [a, b] \to R \;\; f$ is bounded and continuous$\}$ set (١) of all bounded continuous functions defined on $[a, b]$. Show that $C^b[a, b]$ is a linear space over R where $f + g$ and αf are defined in the .1.10same way as in Example

and $F = (R, +, .)$. Define the following **two operations**:2 Let $L =$ R (٢)

$.^2) \in R_2, y_1), (y_2, x_1) \;\; \forall(x_2 + y_2, x_1 + y_1) = (x_2, y_1) + (y_2, x_1)$ **(x1(**

$.\forall \alpha \in R,^2) \in R_2, x_1 \forall(x)_2, x_1) = (\alpha.x_2, x_1 \alpha.(x$ **(2)**

?Show that L is not a linear space over R

Let L be the set of all real valued sequences $\langle x_n \rangle$. Define usual addition(٣)

and multiplication of a sequence as follows: for any $\langle x_n \rangle, \langle y_n \rangle \in L$

and each $\alpha \in$ R

and $\alpha.\langle x_n \rangle = \langle \alpha.x_n \rangle$. Show that $\langle x_n \rangle$ is a $x_n \rangle + \langle y_n \rangle = \langle x_n + y_n \rangle \rangle$

.linear space over R

: $(_3, x_2, x_1$Let $N = \{(x$Let (R,+,.) be the field of real numbers.(٤)

:}. Define the following **two operations**$0 >{}_3, x_2, x_1 x$

=), $Y_3, x_2, x_1 \forall X = (x)._3 + y_3, x_2 + y_2, x_1 + y_1\ X + Y = (x$ **(1)**

$) \in N_3, y_2, y_1 y)$

$.\forall \alpha \in$ R, $\forall X \in N)_3, \alpha.x_2, \alpha.x_1 \alpha.X = (\alpha.x$ **(2)**

?Is N linear space over R

. ***Properties of Linear Space*****1.12Theorem**

$_L$ is a zero vector of L. Then**0**Let $L(F)$ be a linear space and

$_L\ \forall \alpha \in F.\ \mathbf{0}_L = \mathbf{0}\alpha.$ (1)

$._L \forall x \in L\mathbf{0}.x = 0$)2 (

$\forall \alpha \in F.\ \ \alpha.(-x) = -(\alpha.x)\ \ \forall x \in L,$ (3)

$.\forall \alpha \in F)(\ -\alpha).x = -(\alpha.x)\ \ \forall x \in L,$4(

$\forall \alpha \in F.\ \ \alpha.(x - y) = \alpha.x - \alpha.y\ \ \forall x, y \in L,$ (5)

$._L\mathbf{0}$or $x = 0$ then $\alpha = 0\ \alpha.x =$)If6(

Linear Subspace1.2

.1.13Definition

Let L be a linear space over a field F and let φ /= $H \subseteq L$. Then H is called

. a **linear subspace** of L if H itself is a linear space over F

.1.14Theorem

Let H be a non empty subset of a linear space $L(F)$. H is called a subspace

. of L if and only if $\alpha x + \beta y \in H$ for all $x, y \in H$ and for all $\alpha, \beta \in F$

.1.15Exercise

^{3}be a linear space over R. Which of the following subsets of R^3 RLet(١)

.^{3}are subspaces of R

$\{\in R_3 , x_2 : x^3) \in R_3, x_2, x2= \{(_1$ i) $H)$

$= {}_3 + x_2 + x_1 : x^3) \in R_3, x_2, x_1 = \{(x_2$ ii) $H)$

$= {}_2\, x2 + {}_1 : x^3) \in R_3, x_2, x_1 = \{(x_3 \}$ (iii) $H0$

$\{1$

] be a linear space over R. Which of the following subsets 1, 1C[−Let (٢)

.[1, 1] are subspaces of C[−1, 1of C[−

$\{0) = 0] : f(1, 1 = \{f \in C[-_4$ i) $H)$

$\{[1, 1, \forall x \in [-0] : f(x) \leq 1, 1 = \{f \in C[-_5$ ii) $H)$

$\{(1) = f(1] : f(-1, 1 = \{f \in C[-_6$ iii) $H)$

then$_1$) $\in H_2, y_1, y2), (_3, x_2, x2$**(i)):** Take (**1Solution (**

$_1) \notin H_3 + y_3 , x_2 + y_2 , x4) = (_2, y_1, y2) + (_3, x_2 x$ ،2)

is $_1$, H1.13Then, the closure condition is not satisfied. From Definition

,and $\alpha, \beta \in R_1$) $\in H_2, y_1, y2), (_3, x_2, x2$**(i)):** Let (**1Another Solution (**

then

$_1) \notin H_3+y_3, x_2+y_2(\alpha+\beta), x2) = (_3+y_3, x_2+y_2\beta, x2\alpha+2) = (_2, y_1, y2)+\beta(_3, x_2, x2\alpha($

.1if and only if $\alpha + \beta = 2$ $(\alpha + \beta) = 2$because

.^{3}is not a subspace of R_1 , H1.14Thus, from Theorem

.and $\alpha, \beta \in R_6$ **(iii)):** Let $f, g \in H$**2Solution (**

$] \Rightarrow \alpha f$ and βg are continuous on1, 1$\Rightarrow f, g$ are continuous on $[-_6$ $f, g \in H$

.[1, 1Thus, $\alpha f + \beta g$ is continuous on [− .[1 ،1−] **(I)**

$(1) + (\beta g)(-1) = (\alpha f)(-1 \alpha f + \beta g)(-)$

$(1) + \beta.g(-1 \alpha.f(- =$

.$(1) = (\alpha f + \beta g)(1) + \beta.g(1 \alpha.f(=$ **(II)**

.[1, 1is a subspace of $C[-_6$. Thus, H_6From **(I)** and **(II)**, $\alpha f + \beta g \in H$

Linear Transformation Mapping .٣

.1.16Definition

Let $L(F)$ and $L'(F)$ be two linear spaces over the same field F. A mapping $T : L \to L'$ is called a **Linear Operator** or **Linear Transformation** if .$\forall \alpha, \beta \in F \forall x, y \in L, T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

.1.17Example

.$\in R_3$, $x_2, x_1)$ $\forall x_2, x_1) = (x_3, x_2, x_1$defined by $T(x^2 \to R^3$ Let $T : R$

.T is a linear transformationShow that(١)

). Compute1, 5, −0) = $(_3, y_2, y_1), Y = (y3, -1, 2) = (_3, x_2, x_1 X = (x$If(2)

.(X) and $T(X + Y2T($

∋ and α, β^3) ∈ R3, $y_2, y_1, Y = (y^3)$ ∈ R3, x_2, x_1**):** Let $X = (x$**1Solution (**

R. Then

$[(3, y_2, y_1) + \beta(y_3, x_2, x_1 T(\alpha X + \beta Y) = T[\alpha(x$

$(3+ \beta y_3 , \alpha x_2+ \beta y_2 , \alpha x_1+ \beta y_1 T(\alpha x =$

$(2+ \beta y_2 , \alpha x_1+ \beta y_1 \alpha x) =$

$(2, \beta y_1) + (\beta y_2, \alpha x_1 \alpha x) =$

$(2, y_1) + \beta(y_2, x_1 \alpha(x =$

$.() = \alpha T(X) + \beta T(Y_3, y_2, y_1) + \beta T(y_3, x_2, x_1 \alpha T(x =$

$.(2,4) = (6, -2, 4X) = T(2$**):** $T($**2Solution (**

$.(4, -2) = (2, -4, -2 T(X + Y) = T($

.1.18Exercise

is a linear space over $F=$ R with usual addition and multipli- ^{2} Let R (١)

is a $^2 \to$ R^2 cation. Show that each of the following mappings T: R

linear transformation

$2+ x_1 x2) = 2, x_1(x_1$i) $T)$

$(2, x0) = (2, x_1(x_2$ii) $T)$

) where $a \in$ R$_2, ax_1) = (ax_2, x_1(x_3$iii) $T)$

Let C^b(R) be the set of all bounded continuous functions defined (٢)

on R such that C^b(R) is a linear space over R with usual addition

and multiplication. Let T : C^b(R) → C^b(R) such that $T(f(x))$ =

.$f(x)$. Show that T is a linear transformation mapping$^2 x$

.1.19Theorem

Let $T : L(F) \to L'(F)$ be a linear transformation. Then

$_{L'}$ is the zero 0_L is the zero vector of L and $0_{L'}$ where $0_L) = 0\,T($ *(i)*

$'$vector of L

$(\,T(-x) = -T(x$ *(ii)*

$(\,T(x-y) = T(x) - T(y$ *(iii)*

.1.20Theorem

$: L \to L'$ linear $_2$, T_1Let L, L' be linear spaces over same field F. Let T

$)(x) = {}_2+\ T_1 : L \to L'$ as $(T_2+\ T_1$ transformations. Define the function T

$(x)\ \forall x \in L_2(x) +\ T_1\ \ T$

$=\ ()(x_1\!: L \to L'$ is defined as $(\alpha T_1$ If $\alpha \in F$, then the function αT

$(x)\ \ \forall x \in L$. Then$_1\,\alpha.T$

is a linear transformation. $_2+\ \ T_1\ T$Show that(i)

.is a linear transformation$_1$ αT(ii)Show that

Proof. (i) Let $\alpha, \beta \in F$ and $x, y \in L$. Then

$(\alpha x + {}_2(\alpha x + \beta y) +\ T_1)(\alpha x + \beta y) =\ T_2+\ T_1\ T)$ (+ Definition of)

$(\beta y$

$(x) +\ {}_2(y) +\ \alpha T_1(x) +\ \beta T_1 \alpha T =$ $_2$, T_1since $T)$

$((y_2 \beta T$

(.linear trans

$(((y_2(y) +\ T_1(x)) + \beta(T_2(x) +\ T_1 \alpha(T =$

$.()(y_2+\ T_1\ )(x) + \beta(T_2+\ T_1\ \alpha(T =$

.is a linear transformation$_2$ + T_1 Thus, T

$\in F$ and $x, y \in L$. Then$_2$, $\beta_1$ii) Let β)

($y_2 x + \beta_1(\beta_1 y) = \alpha . T_2 x + \beta_1)(\beta_1 \alpha T)$ -Definition of scalar multiplica)

(tion

$((y_1 . T_2(x) + \beta_1 . T_1 \alpha . \ \beta =$ (.linear trans$_1$ since T)

$((y_1 . T_2(x) + \alpha . \beta_1 . T_1 \alpha . \beta =$

$()(y_1 .(\alpha T_2)(x) + \beta_1 .(\alpha T_1 \beta =$

.is a linear transformation$_1$ Thus, αT □

.1.21Definition

Let L be a linear space. A linear transformation $T : L \to F$ is said to be **Linear functional**. (Note that F can be regarded as a linear space over .(F

.1.22Example

= Let L F^n , ..., $x_n \in F$} be a linear space over$_1$, ..., x_n) : $x_1 x$)} =

.the field F + ... + $_1 x_1, ..., x_n) = \alpha_1$Let $T : F^n \to F$ defined by $T(x$

Prove that T is a linear, ..., $\alpha_n \in F$.$_1$, ..., $x_n) \in F^n$ and $\alpha_1 x)\forall \alpha_n x_n$

.transformation

, ..., $y_n) \in F^n$ and $\alpha, \beta \in F$. Then$_1$, ..., $x_n), y = (y_1$**Solution:** Let $x = (x$

$[(, ..., y_{n1}, ..., x_n) + \beta(y_1 T(\alpha x + \beta y) = T[\alpha(x$

$(, ..., \alpha x_n + \beta y_{n1} + \beta y_1 \ T(\alpha x =$

$() + ... + \alpha_n(\alpha x_n + \beta y_{n1} + \beta y_1 \ (\alpha x_1 \alpha =$

$(+ ... + \alpha_n y_{n1} \ y_1 + ... + \alpha_n x_n) + \beta(\alpha_1 \ x_1 \alpha(\alpha =$

$.(, ..., y_{n1}, ..., x_n) + \beta T(y_1 \alpha T(x =$

.(Thus, T is a linear transformation (i.e., linear functional

Chapter 2

Normed Linear Space

.2.1Definition

Let $L(F)$ be a linear space over a field F. A mapping $\| \| : L \to \mathrm{R}$ is called

norm if the following conditions hold

((Positivity $x \in L.\forall 0 \|x\| \geq$ (١)

.0if and only if $x = 0$ $\|x\| =$ (٢)

((Triangle Inequality $x, y \in L.\forall \|x + y\| \leq \|x\| + \|y\|$(٣)

$.x \in L,\ \forall \alpha \in F \forall \|\alpha x\| = |\alpha|\,\|x\|$ (4)

$L, \| \|$) is called **normed linear space**.)

.2.2 Remark

.From now on, the field F is either R or C

.2.3Theorem

Let $(L, \| \|)$ be a normed linear space. Then, for each $x, y \in L$

$.0_{L}\| = \mathbf{0}\|$ (1)

$.\|x\| = \|-x\|$ (2)

$.\ \|x - y\| = \|y - x\|$ (3)

((Reverse Triangle Inequality$\|x\| - \|y\|$ $| \le \|x - y\|$. | (٤)

)Every 6(Reverse Triangle Inequality) $(\|x\| - \|y\|$ $| \le \|x + y\|$. |(٥)

subspace of a normed space is itself normed space with respect

.to the same norm

((1(1.12see Theorem)$_{\mathbf{L}}\|0\mathbf{0}_{\mathbf{L}}\| = \|\mathbf{0})$ $\|1$ *Proof.* (

$.0_{\mathbf{L}}\| = \mathbf{0}\|$ $0 =$

$.x \in L\forall| \|x\| = \|x\|1\|-x\| = |-$ (2)

.((2by part ()$\|x - y\| = \|-(y-x)\| = \|y - x\|$ (3)

We must prove $-\|x - y\| \le \|x\| - \|y\| \le \|x - y\|$ (٤)

)). 3(2.1by Definition)$\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$

(I) Hence, $\|x\| - \|y\| \le \|x - y\|$

)). 3(2.1by Definition)Similarly, $\|y\| = \|y - x + x\| \le \|y - x\| + \|x\|$

(II) Hence, $\|y\| - \|x\| \le \|x - y\|$

$.\forall x, y \in L$Hence, by (I) and (II), we get $\|x - y\| \ge |\|x\| - \|y\||$

We must prove $-\|x + y\| \le \|x\| - \|y\| \le \|x + y\|$ (٥)

)). 3(2.1by Definition)$\|x\| = \|x + y - y\| \le \|x + y\| + \|-y\|$

(III) Hence, $\|x\| - \|y\| \le \|x + y\|$

.((3(2.1by Definition)Similarly, $\|y\| = \|y + x - x\| \le \|y + x\| + \|-x\|$

Hence, $\|y\| - \|x\| \le \|x + y\|$

(IV) $\|x\| - \|y\| \ge -\|x + y\|$

Hence, by (III) and (IV), we get $-\|x + y\| \le \|x\| - \|y\| \le \|x + y\|$

$.x, y \in L\forall$

□

Examples of Normed Linear Space2.1

.2.4Example

Let $L = \mathbb{R}$ be a linear space over R with $\| \ \| : L \to \mathbb{R}$ such that $\|x\| =$

$.|x|$. Show that $(\mathbb{R}, \| \ \|)$ is a normed space

Solution: We show that

$.0 x \in \mathbb{R}$; hence $\|x\| \geq \forall 0 \|x\| = |x| \geq$ (1)

$.0 \Longleftrightarrow \ x = 0 \ \Longleftrightarrow \ |x| = 0$ Let $x \in \mathbb{R}$, $\|x\| =$ (2)

$,x \in \mathbb{R}, \forall \alpha \in \mathbb{R} \forall$ (3)

$.\|\alpha x\| = |\alpha x| = |\alpha|\,|x| = |\alpha|\,\|x\|$

$.x, y \in \mathbb{R} \forall \|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\|$ (4)

.2.5Example

Let $L = C$ be a complex linear space over C with $\| \ \| : C \to \mathbb{R}$ such that

$.z = a + ib$. Show that $(C, \| \ \|)$ is a normed space $\forall$ $\|z\| = |z| = \sqrt{a^2 + b^2}$

Solution: We show that

$.0 z = a + ib \in C$; hence $\|z\| \geq \forall 0 \geq$ $\|z\| = |z| = \sqrt{a^2 + b^2}$ (1)

Let $z = a + ib \in C$ (٢)

$.0 i = 0 + 0 \Longleftrightarrow \ z = 0 \ \Longleftrightarrow \ a = b = 0 =$ $\|z\| = |z| = \sqrt{a^2 + b^2}$

Let $z, w \in C$ (٣)

$\|z + w\|^2 = (z + w)(\overline{z + w})$ where $\overline{z + w}$ =conjugate of $z + w$

$(z + w)(\bar{z} + \bar{w}) = (z\bar{z} + w\bar{w} + w\bar{z} + \bar{w}z) =$

$z\bar{z} + w\bar{w} + w\bar{z} + \overline{\bar{w}z}$

$= z\bar{z} + w\bar{w} + 2 Re\, w\bar{z}$

$\geq \|z\|^2 + \|w\|^2 + 2\|z\|\,\|w\| = (\|z\| + \|w\|)^2$

, hence, $\|z + w\| \leq \|z\| + {}^2 \leq (\|z\| + \|w\|)^2$ Thus, $\|z + w\|$
$.\|w\|$

,Let $z \in C, \alpha \in C$ (4)

$|(\|\alpha z\| = |\alpha z| = |\alpha(a + ib$
$= |= |\alpha||z^2 + b^2 \sqrt{\ } a^2) = \sqrt{\ } \alpha^2 + b^2 (a^2 = \sqrt{\ } \alpha^2 + (\alpha b)^2 \alpha a)) \sqrt{\ } =$

$.\alpha| \|z\||$

$-i$, then1 $i, w = 3 + 2$ **:** Let $z =$ **<u>2.5As an application to Example</u>**
$. 13 = \sqrt{\ } 2^2 + 3^2 i\| = \sqrt{\ } 2 + 3\, i - i)\| = \|3) + (1 + 2\|z + w\| = \|($
$.13 \sqrt{\ } 5 = 325 = \sqrt{\ } 15^2 + 10^2 i\| = \sqrt{\ } 15 + 10\, z\| = \|5\, \|$
$.13 \sqrt{\ } 5 = 3^2 + 2^2 \sqrt{\ } 5\|z\| = 5$

.2.6Example

Show that the linear space $C^b(\mathrm{R})$ is a normed space under the norm

$.(\forall f \in C^b(\mathrm{R}\|f\| = \sup\{|f(x)| : x \in \mathrm{R}\},$

.0. Hence, $\|f\| \geq 0 \forall x \in \mathrm{R}$. Then, $\|f\| = \sup|f(x)| \geq 0$ Since $|f(x)| \geq$ (1)

$0 \Longleftrightarrow \sup\{|f(x)| : x \in \mathrm{R}\} = 0 \;\; \|f\| =$ (2)

$\forall x \in \mathrm{R} 0 \;\; f(x)| = | \Rightarrow \Leftarrow$

((zero mapping$0 \; \forall x \in \mathrm{R} \;\; \Longleftrightarrow \;\; f = \hat{0} \;\; f(x) = \;\; \Rightarrow \Leftarrow$

Let $f, g \in C^b(\mathrm{R})$. Then (٣)

$\{\|f + g\| = \sup\{|f(x) + g(x)| : x \in \mathrm{R}$

$\{\sup\{|f(x)| + |g(x)| : x \in \mathrm{R} \geq$

$.\sup\{|f(x)| : x \in \mathrm{R}\} + \sup\{|g(x)| : x \in \mathrm{R}\} = \|f\| + \|g\| \geq$

.Hence, $\|f + g\| \leq \|f\| + \|g\|$

Let $f \in C^b(\mathrm{R}), \alpha \in \mathrm{R}$. Then (٤)

$\{\|\alpha f\| = \sup\{|(\alpha f)(x)| : x \in \mathrm{R}$

$\{\sup\{|\alpha||f(x)| : x \in \mathrm{R} =$

below where $A = |f2.7\ \alpha| \sup\{|f(x)| : x \in \mathrm{R}\}$ (By Theorem $| =$

$(|(x)|$ and $\beta = |\alpha$

$.\alpha| \|f\|| =$

.2.7Theorem

, then βA is bounded above and 0If A is a bounded above set and $\beta >$

$.(\sup(\beta A) = \beta \sup(A$

$= ($**:**Let $f, g \in C^b(\mathrm{R})$ such that $f(x$**<u>2.6As an application to Example</u>**

,. Hence1$cos(x) + 2sin(x)$ and $g(x) =$

$.(\forall x \in \mathrm{R},$1since $|sin(x)| \leq)1\|f\| = \sup\{|sin(x)| : x \in \mathrm{R}\} =$

$.\{| : x \in \mathrm{R}1cos(x) + 2\|f\| = \sup\{|$

.3. So $\|g\| = 3 = 1\ |cos(x)| + 2| \leq 1cos(x) + 2$But $|$

.2.8Example

] is 1, 0] of all real valued continuous functions on [1, 0The linear space C^b[

(**.**. (**H.W**2.6a normed space under the norm defined in Example

.2.9Example

] is 1, 0] of all real valued continuous functions on [1, 0The linear space C[

a normed space with the norm defined as

$.[1, 0|\int_0^1 f(x)|\ dx\ \ \forall f \in C[\|f\| =$

,. Thus0$f(x)|\ dx \geq |]$, then1, 0$\forall x \in [,0)$ Since$\int_0^1 |f(x)| \geq 1$**solution:** (

$.0\|f\| \geq$

$\Rightarrow \Leftarrow 0\|f\| =\ \ (2)\ \int_0^1\ f(x)|\ dx = |$

$[1,0 \forall x \in [0 \ \ f(x)| = | \ \Rightarrow\Leftarrow$

$[1,0 \forall x \in [0 \ \ f(x) = \ \ \Rightarrow\Leftarrow$

.((zero mapping0 $f =$ ̂ $\Rightarrow\Leftarrow$

]. Then1, 0Let $f, g \in C$[(3)

$|f(x) + g(x)|\, dx \|f + g\| = \int_0^1$

$f(x)| + |g(x)|)\, dx|) \geq \int_0^1$

$\int_0^1 |g(x)|\ dx = \int_0^1 \|f\| + \|f(x)|\ dx + | = g\|$

], $\alpha \in$ R . Then1, 0Let $f \in C$[(4)

$|(|f(x \int_0^1 \quad |\alpha|\,|f(x)| \int_0^1 dx = |\alpha||\alpha f(x)|\ dx = \int_0^1 \|\alpha f\| = \ .dx = |\alpha|\,\|f\|$

= (] such that $f(x$1, 0: Let $f \in C$[**<u>2.9As an application to Example</u>**

.. Find $\|f\|$, $\|g\|$ and $\|f + g\|^2$and $g(x) = -x^3\ x$

$= |f(x)|\ dx\|f\| = \int_0^1 \cdot^3\cdot \quad = \int_0^1 x_4^1\ dx dx = {}^3_0 x$

$= |g(x)|\ dx\|g\| = \int_0^1 \cdot\ ^2\cdot \quad = \int_0^1 x_0\ dx^1 dx = x^2_3$

$= \|f + g\| \int_0^1 = f + g)(x)|\ dx)| \quad \int_0^1 .x^2 x\ \underline{x}^3\ dx$

$\int_1 \cdot 12 = {}^3_4\ {}^2_3\ x \ -x \tfrac{1}{)} dx \tfrac{1}{=} \tfrac{1}{)} =$

.2.10Example

Consider the linear space F^n over F ($F =$ R or C). Define $\|\ \| : F^n \to$ R

$, ..., x_n) \in F^n$. Then $(F^n, \|\ \|)$ is a $_1| , ..., |x_n|\} \ \forall X = (x_1$by $\|X\| = \max\{|x$

.normed space

$., ..., n1,\ \forall i = 0, ..., x_n) \in F^n,\ |x_i| \geq {}_1)$ For any $X = (x1$**solution:** (

$.0$, then $\|X\| \geq 0| , ..., |x_n|\} \geq {}_1$Then $\max\{|x$

$, ..., x_n) \in F^n{}_1$, where $X = (x0\|X\| =$ (2)

$\Rightarrow \max\{|x_1|, \ldots, |x_n|\} = 0$

$\Rightarrow |x_1| = \ldots = |x_n| = 0 \Leftrightarrow x_1 = \ldots = x_n = 0$

$\Rightarrow X = (x_1, \ldots, x_n) = (0, \ldots, 0) = \mathbf{0}_{F^n}$

(3) Let $X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in F^n$

$\|X + Y\| = \max\{|x_1 + y_1|, \ldots, |x_n + y_n|\}$

$\leq \max\{|x_1| + |y_1|, \ldots, |x_n| + |y_n|\}$

$\leq \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\} = \|X\| + \|Y\|$

(4) Let $X = (x_1, \ldots, x_n) \in F^n$ and $\alpha \in F$

$\|\alpha X\| = \max\{|\alpha x_1|, \ldots, |\alpha x_n|\}$

$= \max\{|\alpha|\,|x_1|, \ldots, |\alpha|\,|x_n|\} = |\alpha| \max\{|x_1|, \ldots, |x_n|\} = |\alpha|\,\|X\|$

2.10 As an application to Example 3 **:** Consider the linear space $\mathbb{R}^3$ over $\mathbb{R}$. Let $X = (x_1, x_2, x_3) = (1, -2, 5), Y = (y_1, y_2, y_3) = (0, -7, 3)$. Then

(1) $\|X\| = \max\{|1|, |-2|, |5|\} = 5$ and

$\|Y\| = \max\{|0|, |-7|, |3|\} = 7$.

$\|X + 2Y\| = \max\{|1|, |-12|, |11|\} = 12$

(2) Find $\|2X\|$, $\|X - Y\|$, $\|2X + 3Y\|$

(3) Show that

$\max\{|x_1|+|y_1|, |x_2|+|y_2|, |x_3|+|y_3|\} \leq \max\{|x_1|, |x_2|, |x_3|\} + \max\{|y_1|, |y_2|, |y_3|\}$.

.2.11Exercise

→ R such ^{2} be a linear space over $F =$ C. Define ‖ ‖ : C^2 Let $L =$ C(١)

. Show that ‖0and $a, b >$ ^{2}) ∈ C_2, x_1|, ∀X = (x_2| + b|$x_1$that ‖X‖ = a|x

(.. (**H.W**^{2}‖ is a norm on C

(. Let ‖X‖ = min{|x| , |y|}, ∀X = (x, y^2Consider the linear space R (٢)
∋

.^{2}. Show that ‖ ‖ is not a norm on R^{2}R

^{2}) ∈ R3, −0**solution:** Let X = (

0} = 3,0|} = min{3|,|−0‖X‖ = min{|

) of the definition of the 2. Condition (0, but ‖X‖ = $_{2R}$**0**Since X /=

.^{2}norm is not valid. Hence, ‖ ‖ is not a norm on R
، |. Let ‖X‖ = |x| +|y^2Consider the^{2}linear space R (٣)∀X = (x, y) ∈ R

.(4Show that ‖ ‖ does not satisfies condition (

2), α = 3,1**solution:** Let X = (

20) = 3^2+ 1^2 (2(|x| ^{2} + |y| ^{2}) = 2α| ‖X‖ = |

40= 6^2 + 2^2)‖ = 6,2)‖ = ‖(3,1(2‖αX‖ = ‖

.40/= ‖αX‖ = 20 Thus, |α| ‖X‖ =

$x, y \in L$. ∀Let (L, ‖ ‖) be a normed space. Let ‖$x + y$‖ = ‖x‖ + ‖y‖(٤)

. ‖y‖2 ‖x‖ + 3 y‖ = 2x + 3Show that ‖

y‖2x + 3‖y‖ and ‖2 ‖x‖ + 3 y‖ ≥ 2x + 3**solution:** We must show ‖
≥

‖y‖2 ‖x‖ + 3

(x + y) −y‖3y −y‖ = ‖3x + 3y‖ = ‖2x + 3‖

((4(2.3(By Theorem (x + y)‖ −‖y‖ |3‖ | ≤

((4(By axiom (‖x + y‖) −‖y‖ |)3| =

2 ‖x‖ + 3 ‖y‖ | = 2 ‖x‖ + 3| =

(1) ‖y‖2 ‖x‖ + 3 y‖ ≥ 2x + 3‖y‖ Thus, ‖

(2( ))3-4By axioms ()‖y‖2 ‖x‖ + 3 y‖ ≤ 2x + 3On the other hand, ‖

. ‖y‖2 ‖x‖ + 3 y‖ = 2x + 3), ‖2) and (1From (

Some Important Inequalities

To give more examples about normed space, it is important to present .some inequalities

is a real number andIf $l_p = \{\langle x_n\rangle : x_n$ $\sum_{i=1}^{\infty}$ $_i{}^p < \infty\}$ be a set of x || = ,...,⟩ ∈ l_p, y_n2, x_1)). Let x_n = ⟨x2(1.11sequence space (see Excercise ,...,⟩ ∈ l_q. Then2, $y_1 y$⟩

Holder's Inequality(1)

$$\sum_{i=1}^{\infty} \geq |x_i y_i| \quad \sum_{i=1}^{\infty} |x_i|^p \,{}^{\frac{1}{p}} \quad \sum_{i=1}^{\infty} {}_i{}^q | y|^{\frac{1}{q}}$$

.1= $\frac{1}{q}$ + $\frac{1}{p}$ and 1 , q > 1where p >

Cauchy Schwarz's Inequality (2)

$$\sum_{i=1}^{\infty} \geq |_i x y| \quad \sum_{i=1}^{\infty} {}_i{}^2 \,{}^{\frac{1}{2}}| \, |y \quad \sum_{i=1}^{\infty} {}_i{}^2 x^{\frac{1}{2}}|$$

-Note that Cauchy Schwarz' s inequality is a special case of Holder' s inequal .2ity where p = q =

Minkowski's Inequality (3)

1If p ≥

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{\frac{1}{p}}$$

2.12. Example

Let $L = \mathbb{R}^2$ be a linear space over $\mathbb{R}$. If $X = (-5, 0)$, $Y = (2, 1) \in \mathbb{R}^2$ and $p = 3$.

(1) Verify Cauchy Shwarz inequality.

(2) Verify Minkowski's inequality.

2.13. Remark

The three inequalities above hold for finite sum.

Now we can give the following examples

2.14. Example

Show that the linear space $\mathbb{R}^n$ over $\mathbb{R}$ (or C^n over C) is a normed space with $\|X\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$, $\forall X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ or C^n. The space $(\mathbb{R}^n, \|X\|)$ is called **Euclidian space** and $(C^n, \|X\|)$ is called **Unitary space**.

Solution: Let $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ (or C^n) and $\alpha \in \mathbb{R}$ (or C^n).

(1) Since $|x_i| \geq 0$, $\forall i = 1, \ldots, n$. Then, $\left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \geq 0$; that is $\|X\| \geq 0$.

(2) $\|X\| = 0 \Leftrightarrow \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i|^2 = 0$

$\Leftrightarrow |x_i|^2 = 0$, $\forall i = 1, \ldots, n$

$\Leftrightarrow x_i = 0$, $\forall i = 1, \ldots, n$

$\Rightarrow\Leftarrow \; X = (x_1, \dots, x_n) = 0_{R^n}$

$$(3)\quad \|X + Y\| = \|(x_1 + y_1, \dots, x_n + y_n)\|$$

$$= \left(\sum_{i=1}^{n} |x_i + y_i|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2}} \quad \text{(Minkoski's Inequality)}$$

$$= \|X\| + \|Y\|$$

$$(4)\quad \|\alpha X\| = \|(\alpha x_1, \dots, \alpha x_n)\| = \left(\sum_{i=1}^{n} |\alpha x_i|^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_{i=1}^{n} |\alpha|^2 |x_i|^2\right)^{\frac{1}{2}}$$

$$= |\alpha| \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = |\alpha| \, \|X\|$$

Example 2.14: As an application

(1) Let $(R^3, \| \ \|)$ be an Euclaidian space and $X = (x_1, x_2, x_3) = (4, 2, -1)$.

Then, find $\|X\|$.

(2) Let $(C^2, \| \ \|)$ be a Unitary space and $X = (x_1, x_2) = (2+i, i-1)$.

Then, find $\|X\|$.

Product of Normed Spaces2.2

.2.15Definition

Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed linear spaces over a field F. Let $L \times L' = \{(X, Y) : X \in L, Y \in L'\}$ be the Cartesian product of L and L'. Define $+$ on $L \times L'$ by

$) \in L_2, Y_2) + (X_1, Y_1 \forall(\underbrace{X}_{\text{sum on L}}), _2 + \underbrace{Y_1}_{\text{sum on } L'}, Y_2 + X_1) = (X_2, Y_2) + (X_1, Y_1 X)$
$.' \times L$

Define a scalar multiplication

$. \forall(X, Y) \in L \times L', \forall \alpha \in F \alpha(X, Y) = (\alpha X, \alpha Y),$

.2.16Proposition

(.Show that $(X \times Y, +, \times)$ is a linear space over F. (**H. W**

.2.17Remark

The product linear space defined above can be made a normed space by .different ways as we show in the following example

.2.18Example

Define $\| \| : L \times L' \to \mathbb{R}$ such that

$= \|X\|_L + \|Y\|_{L1} \|(X, Y)\|$ (1)

$\{ = \max\{\|X\|_L, \|Y\|_{L2} \|(X, Y)\|$ (2)

.) are normed spaces$_2$), $(L \times L', \| \|_1$Show that $(L \times L', \| \|$

,) is a normed space$_1$To show $(L \times L', \| \|$ (١)

$\forall X \in L, \forall Y \in L'$, then0 and $\|Y\|_{L'} \geq 0$ Since $\|X\|_L \geq$ (i)

$.0 \geq {}_1 \|X\|_L + \|Y\|_{L'} = \|(X, Y)\|$

$0 \Leftarrow\Rightarrow \|X\|_L + \|Y\|_{L'} = 0 = {}_1$ ii) $\|(X, Y)\|)$

$0\|X\|_L = \|Y\|_{L'} = \ \Rightarrow\Leftarrow$

$(L, \| \|_L), (L', \| \|_{L'})$ are normed spaces)) $0X = Y = \ \Rightarrow\Leftarrow$

$(0, 0X, Y) = () \ \Rightarrow\Leftarrow$

$') \in L \times L_2, Y_2), (X_1, Y_1$ For each $(X$ (iii)

${}_1)\|_2 + Y_1, Y_2 + X_1 = \|(X_1)\|_2, Y_2) + (X_1, Y_1\|(X$

$\cdot\|_{L2} + Y_1 \|_L + \|Y_2 + X_1 \|X =$

$\cdot\|_{L2}\|_{L'} + \|Y_1\|_L + \|Y_2\|_L + \|X_1\|X \geq$

$(\cdot\|_{L2}\|_L + \|Y_2\|_{L'}) + (\|X_1\|_L + \|Y_1\|X) =$

${}_1)\|_2, Y_2 + \|(X_1)\|_1, Y_1\|(X =$

For each $(x, y) \in X \times Y$ and for each $\alpha \in F$ (iv)

$= \|\alpha X\|_L + \|\alpha Y\|_{L1} = \|(\alpha X, \alpha Y)\|_1 \|\alpha(X, Y)\|$

$\alpha| \|X\|_L + |\alpha| \|Y\|_{L'} = |\alpha|(\|X\|_L + \|Y\|_{L'}) = |\alpha| \|(X, Y| =$
${}_1)\|$

$'$is a norm on $L \times L_2$ Now, we show that $\| \|$ (٢)

$\forall X \in L, \forall Y \in L', 0$ and $\|Y\|_{L'} \geq 0$ Since $\|X\|_L \geq$ (i)

$.0 \geq {}_2$ then $\max\{\|X\|_L, \|Y\|_{L'}\} = \|(X, Y)\|$

$0 \Leftarrow\Rightarrow \max\{\|X\|_L, \|Y\|_{L'}\} = 0 = {}_2$ ii) $\|(X, Y)\|)$

$0\|X\|_L = \|Y\|_{L'} = \ \Rightarrow\Leftarrow$

$(L, \| \|_L), (L', \| \|_{L'})$ are normed spaces)) $0X = Y = \ \Rightarrow\Leftarrow$

$(0, 0X, Y) = () \ \Rightarrow\Leftarrow$

$') \in L \times L_2, Y_2), (X_1, Y_1$ For each $(X$ (iii)

$_2)\|_2 + \ Y_1, Y_2 + \ X_1 = \|(X_2)\|_2, Y_2) + (X_1, Y_1\|(X$

$\{ \cdot\|_{L2} + \ Y_1\|_L, \| Y_2 + \ X_1 \max\{\|X =$

$\{ \cdot\|_{L2}\|_{L'} + \| Y_1\|_L, \| Y_2\|_L + \|X_1 \max\{\|X \geq$

$\{ \cdot\|_{L2}\|_L, \| Y_2\|_{L'}\} + \max\{\|X_1\|_L, \| Y_1 \max\{\|X \geq$

$_2)\|_2, Y_2 + \|(X_2)\|_1, Y_1\|(X =$

For each $(X, Y) \in L \times L'$ and for each $\alpha \in F$ (iv)

$\{ \cdot = \max\{\|\alpha X\|_L, \|\alpha Y\|_{L2} = \|(\alpha X, \alpha Y)\|_2 \ \|\alpha(X, Y)\|$

$\{ \cdot \max\{|\alpha|\|X\|_L, |\alpha|\|Y\|_L =$

$_2\alpha| \max\{\|X\|_L, \|Y\|_{L'}\} = |\alpha|\|(X, Y)\|| =$

$)_2, \| \ \|_R{}^2$**:** Let $L = (R, |\ |)$ and $L' = (R$ **<u>2.18As an application to Example</u>**

$\ni (2, -1 \in L = R$ and $Y = (3$ is the Euclaidian norm. If $X = {}_2$ where $\| \ \|_R$

$_2$and $\|(X, Y)\|_1$. Find $\|(X, Y)\|^2 L' = R$

$_2)\|_R 2, -1\|_R + \|(3 = \|_1))\|2, -1, (3 = \|(_1$ **Solution:** $\|(X, Y)\|$

$+ |3| = \quad {}^2 \ |\sum_{i=1}^{2} x_i|^2 \ {}^{\frac{1}{2}}$

$-| + |1|^2 + 3 = \ {}^2 5^{\frac{1}{2}} \sqrt{\ } + 3 = \overline{|2}$

$(.$(**H.W**$_2$ Find $\|(X, Y)\|$

Normed space and Metric space2.3

.2.19Definition

Let X be a non empty set and $d : X \times X \to$ R be a mapping. Then d is

called metric if

$x, y \in X \forall 0 d(x, y) \geq$ (1)

$x, y \in X \forall \Longleftrightarrow \; x = y 0 \;\; d(x, y) =$ (2)

$\forall x, y \in X \; d(x, y) = d(y, x)$ (3)

$\forall x, y, z \in X . \; d(x, y) \leq d(x, z) + d(z, y)$ (4)

(X, d) is called **metric space**

.2.20Theorem

Let $(L, \| \|)$ be a normed linear space. Let $d : L \times L \to$ R defined by $d(x, y) = \|x - y\| \quad \forall x, y \in X$. Prove that (L, d) is a metric space. (i.e., every normed space is a metric space). The metric d is called .metric **induced** by the norm

.*Proof.* To prove (L, d) is a metric space

= (Hence, $d(x, y x, y \in L . \forall 0$i) By definition of norm, $\|x - y\| \geq$)

$0 \|x - y\| \geq$

(ii) $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$

$\Longleftrightarrow \; x = y 0 \; \Longleftrightarrow \; x - y = 0 \; \Longleftrightarrow \; \|x - y\| = 0$ iii) $d(x, y) =$)

iv) $d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) +$)

$(d(y, z$ □

.2.21Lemma

(Let d be a metric induced by a normed space $(L, \| \ \|)$ (i.e., $d(x, y$
=

:$\|x - y\|$). Then d satisfies the following

.$\forall x, y, a \in L \; d(x + a, y + a) = d(x, y)$ *(i)*

.$\forall x, y \in L, \; \forall a \in F \; d(\alpha x, \alpha y) = |\alpha| \, d(x, y)$ *(ii)*

$\forall x, y, a \in L)$ $d(x + a, y + a) = \|x + a - (y + a)\| = d(x, y)$ 1 *Proof.* (
□
.$(d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = \|\alpha(x - y)\| = |\alpha| \, \|x - y\| = |\alpha| \, d(x, y$ (2)

.2.22Remark

Not every metric space is a normed space as we show in the next example

.2.23Example

Let d be the discrete metric on a space X. Then d can' t be obtained from a norm on X (i.e., $(X, \| \ \|)$, where

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & x \neq y \end{cases}$$

Solution: Suppose d induced by a norm on X. Then, by previous

,Lemma

.$d(\alpha x, \alpha y) = |\alpha| \, d(x, y)$ $\forall x, y \in X$ and $\forall \alpha \in F$

Then $\alpha x \neq \alpha y$ such that $d(x, y)$ Let $x, y \in X$ such that $x \neq y$.

(1) 1, $d(\alpha x, \alpha y) = 1$=

(2(But $|\alpha| \, d(x, y) = |\alpha|$

. Thus, d can $1 \neq |\alpha| = |\alpha| \, d(x, y)$ for any $\alpha \neq 1$ Hence, $d(\alpha x, \alpha y) =$

.not be induced by a normed space

2.24. Example

Let $d(x, y) = |x| + |y| \ \forall x, y \in \mathbb{R}$. Then, d is a metric on $\mathbb{R}$ (check!). However, d is not induced by a normed space. To show this, let $x = 1$, $y = 3$, $a = 2 \in \mathbb{R}$.

$$d(x, y) = d(1, 3) = |1| + |3| = 4$$

On the other hand, $d(x + a, y + a) = d(3, 5) = |3| + |5| = 8$

Thus, $d(x, y) \neq d(x + a, y + a)$. By Lemma 2.21, d is not induced by a norm.

2.4 Generalizations of Some Concepts from Metric Space

In what follow, we give generalizations of some known concepts from metric space such as open (closed) ball, open (closed) set, interior set, closure of a set, convergent sequence, Cauchy sequence, and bounded sequence.

2.25. Definition

Let $(L, \| \ \|)$ be a normed linear space. Let $x_0 \in L$, $r \in \mathbb{R}$, $r > 0$. Then the set

$$B_r(x_0) = \{x \in L : \|x - x_0\| < r\}$$

is called an **open ball** with center x_0 and radius r. Similarly,

$$\bar{B}_r(x_0) = \{x \in L : \|x - x_0\| \leq r\}$$

is called an **closed ball** with center x_0 and radius r.

.2.26Definition

Let $(L, \| \|)$ be a normed space and $A \subseteq L$. Then A is said to be

- .such that $B_r(x) \subseteq A$0 **open set** if $\forall x \in A, \exists r >$
- **closed set** if $A^c = L \setminus A$ is open set

.2.27Remark

Let $(L, \| \|)$ be a normed space. Then

.L, φ are closed and open (١)

The union of any family of open sets is open (٢)

)The union of finite family of closed sets is closed 3(

)The intersection of finite family of open sets is open4(

.The intersection of any family of closed sets is closed(5)

.2.28Theorem

.Any finite subset of a normed space is closed

.*Proof.* Let L be a normed space and $A \subseteq L$

((1(2.27by Remark)If $A = \varphi$, then A is closed

If $A = \{x\}$ to prove A is closed (i.e., to prove $L \setminus A$ is

open) Let $y \in L \setminus A = L \setminus \{x\}$ so that $y \neq x$. Put $\|x -$,Thus .0 <

$x \notin B_r(y)$ and hence $B_r(y) \subseteq A^c = L \setminus \{x\}$. Thus, A^c is open and thus A

.is closed

If $A = \{x_1, \ldots, x_n$,(3(2.27$\{x_i\}$. By Remark then $A = \bigcup_{i=1}^{n}$ $n \in Z$, $n >$ ،
{
A is closed

□