(ii) $T_2(x_1, x_2) = (x_2, x_1)$

Let $X \in \mathbb{R}^2$: $X = (x_1, x_2)$

$y \in \mathbb{R}^2$: $y = (y_1, y_2)$

$$\alpha X = \alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$

$$\beta y = \beta(y_1, y_2) = (\beta y_1, \beta y_2)$$

$$\alpha X + \beta y = (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2) = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

$$T_2(\alpha X + \beta y) = T_2(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

$$= (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1)$$

$$= (\alpha x_2, \alpha x_1) + (\beta y_2, \beta y_1)$$

$$= \alpha(x_2, x_1) + \beta(y_2, y_1)$$

$$= \alpha T(x_1, x_2) + \beta T(y_1, y_2)$$

(iii) $T_3(x_1, x_2) = (0, x_2)$ H.W

[illegible]

$L = \mathbb{R}$, or $\mathbb{R}^n$

عدد { حقيقي - كلي

تحويل خطي $T : L \longrightarrow F$ linear functional

(L: linear space; F: حقل $\mathbb{R}$, عدد $\mathbb{C}$)

تعريف $L : F^n = \{ (x_1, \dots, x_n) : x_1, x_2, \dots, x_n \in F \}$ ($\mathbb{R}^n$, $\mathbb{C}^n$)

$$T : F^n \longrightarrow F \qquad L = F^n$$

$$T(x) = T((x_1, x_2, \dots, x_n)) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_n x_n$$

$$= \sum_{i=1}^{n} \alpha_i x_i \quad \forall x = (x_1, \dots, x_n) \in F^n$$

$$x_1, \dots, x_n \in F$$

T is linear transform ?

$$T(\alpha x + \beta y) = T[\alpha(x) + \beta(y)] = \alpha T(x) + \beta T(y)$$

$x \in F^n$: $x = (x_1, x_2, \dots, x_n)$: $\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

$y \in F^n$: $y = (y_1, y_2, \dots, y_n)$: $\beta y = (\beta y_1, \beta y_2, \dots, \beta y_n)$

$$\alpha x + \beta y = (\alpha x_1, \alpha x_2, \dots \alpha x_n) + (\beta y_1, \beta y_2, \dots, \beta y_n)$$

$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n)$$

نأخذ T للطرفين

$$T(\alpha x + \beta y) = T((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n))$$

$$= \sum_{i=1}^{n} \alpha_i (\alpha x_i + \beta y_i)$$

$$= \alpha_1(\alpha x_1 + \beta y_1) + \alpha_2(\alpha x_2 + \beta y_2) + \dots + \alpha_n(\alpha x_n + \beta y_n)$$

$$= \alpha_1 \alpha x_1 + \alpha_1 \beta y_1 + \alpha_2 \alpha x_2 + \alpha_2 \beta y_2 + \dots + \alpha_n \alpha x_n + \alpha_n \beta y_n$$

$$= \alpha_1 \alpha x_1 + \alpha_2 \alpha x_2 + \dots + \alpha_n \alpha x_n + \alpha_1 \beta y_1 + \alpha_2 \beta y \text{ (عناصر)}$$

$$= \alpha(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n) + \beta(\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n)$$

$$= \alpha \sum \alpha_i x_i + \beta \sum \alpha_i y_i$$

$$= \alpha T(x) + \beta T(y)$$

Theorem 2.3 $L : \| \ \| ; L \to \mathbb{R}$

المحاضرة ~~الرابعة~~ الخامسة

٤c

(1) $\|0_L\| = 0 \quad \forall x \in L$

let $x = 0_L = (0, 0, 0, \dots, 0) \in L$, $\alpha = 0 \in R$

(1) $\|x\| = \|\alpha x\| = \|0 \cdot x\| = |0| \, \|x\|$
$= 0$

(2) $\|x\| = \|-x\| \quad \forall x \in L$

$\|-x\| = |-1| \, \|x\| = 1 \cdot \|x\| = \|x\|$

(3) $\|x - y\| = \|y - x\|$

$\|y - x\| = \|-1(-y + x)\|$
$= \|-1(x - y)\|$
$= |-1| \, \|x - y\|$
$= 1 \cdot \|x - y\|$
$= \|x - y\|$

تعريف
$|x| < 1$
$-1 < x < 1$

$\|x - y\| = \|-1(-x + y)\|$
$= \|-1(y - x)\|$
$= |-1| \, \|y - x\|$
$= 1 \cdot \|y - x\|$
$= \|y - x\|$

$\big| \, \|x\| - \|y\| \, \big| \leq \|x - y\|$

$-(\|x - y\|) \leq \|x\| - \|y\| \leq \|x - y\|$

$\|x\| = \|x + y - y\| \leq \|x - y\| + \|y\|$

$\|x\| - \|y\| \leq \|x - y\|$

$\|y\| = \|y + x - x\|$ ~~$\neq \|y - x\|$~~

$\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\|$

$\|y\| \leq \|x - y\| + \|x\|$

$\|y\| - \|x\| \leq \|x - y\|$

$-(\|x\| - \|y\|) \leq \|x - y\| \quad * -1 \Rightarrow \|x\| - \|y\| \geq -\|x - y\|$

(5) $\left| \|x\| - \|y\| \right| \leq \|x+y\|$ $\quad \|-y\| = |-1| \|y\| = \|y\|$

$-\|x+y\| \leq \|x\| - \|y\| \leq \|x+y\|$

$\|x\| = \|x+y-y\| = \|x+y-y\| \leq \|x+y\| + \|y\|$

$\|x\| - \|y\| \leq \|x+y\|$ ✓

$\|y\| = \|y+x-x\| = \|x+y-x\| \leq \|x+y\| + \|x\|$

$\|y\| - \|x\| \leq \|x+y\|$

$-1\left(\|x\| - \|y\|\right) \leq \|x+y\| \quad * -1$

$\|x\| - \|y\| \geq -\|x+y\|$

$L: \| \ \| \longrightarrow \mathbb{R}$

$E: \| \ \| \longrightarrow \mathbb{R}$

$E \subseteq L$, E normed space

(E)

2-5: $Z = 2+3i$

$w = 1-i$

$\|Z+w\|$: ?

$Z+w = 2+3i+1-i$

$= 3+2i = a+ib$

$\|3+2i\| = \sqrt{a^2+b^2}$

$= \sqrt{(3)^2+(2)^2}$

$= \sqrt{9+4} = \sqrt{13}$

$\|5Z\| = 5 \cdot \|Z\| = 5 \cdot \sqrt{13}$

Example 2.6:

$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\} \quad \forall f \, C^b(\mathbb{R})$

$C^b(\mathbb{R}) = L : \|f\| \longrightarrow \mathbb{R}$

$\|f\| \geq 0$?

$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}$

$|f(x)| \geq 0 \Rightarrow \|f\| \geq 0$

$\|f\| = 0 \Rightarrow f = \underline{0}$

$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}$

$|f(x)| = 0 \Longleftrightarrow f(x) = 0$

$f = \underline{0}$

$\|f+g\| \leq \|f\| + \|g\|$

$\|f+g\| = \sup\{|f(x)+g(x)| : x \in R\}$

$\leq \sup\{|f(x)|\} + \sup\{g(x) : x \in R\}$

$= \|f\| + \|g\|$

$\|\alpha f\| = |\alpha| \|f\|$?

$\|\alpha f\| = \sup\{|\alpha f(x)| : x \in \mathbb{R}\}$

$= \sup\{|\alpha||f(x)| : x \in R$

$= |\alpha| \sup\{|f(x)| : x \in R\}$

$= |\alpha| \|f\|$

EX: 2.7 : $f(x) = \sin(x)$, $g(x) = 2\cos x + 1$ } $f, g : \|\ \| \longrightarrow \mathbb{R}$

$\|f(x)\| = \sup\{|f(x)| : x \in R\}$
$= \sup\{|\sin x| : x \in R = 1$

$|\sin x| \leq 1$

$\|g(x)\| = \sup\{|g(x)| : x \in R\}$
$= \sup\{|2\cos x + 1| : x \in R = 3$

$|2\cos x + 1| \leq |2\cos x| + |1|$
$\leq 2|\cos x| + 1$
$\leq 2 \cdot 1 + 1$
≤ 3

$\|f+g\| \leq \|f\| + \|g\|$
$= 1 + 3$
$= 4$

$C^b(\mathbb{R}) = \|f\| \to \mathbb{R}$

$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}$

$f(x) = \sin(x)$

هل تحقق شروط normed sp.

① $\|f\| \geq 0$?

$\|f\| = \sup\{|\sin(x)| : x \in \mathbb{R}\} = 1 \geq 0$

$|\sin(x)| \leq \boxed{1}$

$\|f\| = 0$?

$\|f\| = \sup\{|\sin(x)| : x \in \mathbb{R}\}$

$|\sin(x)| = 0$

$\sin(x) = 0 \quad , x = 0$

$\|f\| = 0$

$\|\alpha f\| = |\alpha| \|f\|$?

$\|\alpha f\| = \sup\{|\alpha \sin x| : x \in \mathbb{R}\}$

$|\alpha \sin x| = |\alpha| |\sin x|$

$\leq |\alpha| . 1$

$= |\alpha| . 1$

$g(x) = 2\cos(x) + 1$ H.w:

Ex: 2.8 H.w

EX. 2.9:

$\|f\| = \int_0^1 |f(x)|\,dx$

$\|f\| \geq 0$?

$\|f\| = \int_0^1 |f(x)|\,dx$

ناخذ تكامل للطرفين $|f(x)| \geq 0$

$\int_0^1 |f(x)|\,dx \geq \int 0\,dx$

$\|f\| \geq 0$

$\|f\| = 0$

$\|f\| = \int_0^1 |f(x)|\,dx$

$|f(x)| = 0 \Longleftrightarrow f(x) = 0$

$f = \underline{0}$

$\|f+g\| \leq \|f\| + \|g\|$

$\|f+g\| = \int_0^1 |f(x)+g(x)|\,dx$

ناخذ تكامل للطرفين $|f(x)+g(x)| \leq |f(x)| + |g(x)|$

$\int_0^1 |f(x)+g(x)|\,dx \leq \int_0^1 \left(|f(x)| + |g(x)| \right) dx$

$= \int_0^1 |f(x)|\,dx + \int_0^1 |g(x)|\,dx$

$\leq \|f\| + \|g\|$

$$\|\alpha f\| = |\alpha| \|f\|$$

$$\|\alpha f\| = \int_0^1 |\alpha f(x)| dx$$

$$|\alpha f(x)| = |\alpha||f(x)|$$

$$\int_0^1 |\alpha f(x)| dx = \int_0^1 |\alpha| |f(x)| dx$$

$$= |\alpha| \int_0^1 |f(x)| dx$$

$$= |\alpha| \|f\|$$

محاضرة تحليل دالي رقم O

Normed space and Metric space :-

Def // Let X be a non empty set and

$X \times X \longrightarrow \mathbb{R}$ be a mapping then is called metric if

1) $d(x,y) \geq 0 \quad \forall\, x,y \in X$

2) $d(x,y) = 0 \iff x = y \quad \forall\, x,y \in X$

3) $d(x,y) = d(y,x) \quad \forall\, x,y \in X$

4) $d(x,y) \leq d(x,z) + d(z,y)$,

$\forall\, x,y,z \in X$

then (X,d) is called metric space.

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التوقيع :

م. م. هاجر كامل راضي

Theorem: 2.20 // Let $(L, \|\ \|)$ be a normed space.
Let $d: L \times L \longrightarrow \mathbb{R}$ define by

$d(x,y) = \|x-y\| \ \forall x,y \in X$ Prove that (L,d) is a metric space (i.e every normed space is metric space).

Then metric d is called metric induced by the norm. (ناتج)

~~Solution~~ Proof // By using the definition of norm we get.

1) $\|x-y\| \geq 0 \ \forall x,y \in L$, then $d(x,y) = \|x-y\| \geq 0$

2) $d(x,y) = 0 \Leftrightarrow \|x-y\| = 0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$
$\forall x,y \in L$

3) $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

4) $d(x,y) = \|x-y\| = \|x-z+z-y\|$ (نطرح z ونضيف z)

$\leq \|x-z\| + \|z-y\|$

$= d(x,z) + d(z,y)$

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محاضرة تحليل دالي

Lemma 2-21

Let d b a metric induced by a normd space $(L, \| \|)$ (i.e $d(x,y) = \|x-y\|$). Then satisfies the following

1) $d(x+a, y+a) = d(x,y) \quad \forall x,y,a \in L$

2) $d(\alpha x, \alpha y) = |\alpha| \, d(x,y) \quad \forall x,y \in L, \forall \alpha \in F$

Solution //

1) $d(x+a, y+a) = \|(x+a) - (y+a)\|$

$$= \|x + a - y - a\| = \|x-y\| = d(x,y)$$

2) $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = \|\alpha (x-y)\|$

$$= |\alpha| \|x-y\| = |\alpha| \, d(x,y)$$

Remark: Not every metric space is normd space

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Example // 2.23

Let d be a discrete metric space X. Then

Then d can't be obtained from on X

(i.e $(X, \| \|)$) where

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

lemma 2.21 باستخدام لخاصية الثانية من

$$d(\alpha x, \alpha y) = |\alpha| \, d(x,y)$$

$$x \neq y \Rightarrow \alpha x \neq \alpha y$$

$$d(\alpha x, \alpha y) = 1$$

$$d(x,y) = 1$$

$$d(\alpha x, \alpha y) = |\alpha| \, d(x,y)$$

$$1 = |\alpha| \cdot 1$$

$$|\alpha| \neq 1 \qquad \alpha \neq \pm 1$$

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(4)

$x \neq y$

$2 \neq 3$

$2 \cdot 4 \neq 3 \cdot 4$

$8 \neq 12$

$\alpha = 4$

$|4| = 4$

$\alpha = 5$

$2 \neq 3$

$2 \cdot 5 \neq 3 \cdot 5$

$10 \quad 15$

$|5| = 1$

مثال تطبيقي عن مجاميع الجبر الديكارتي

Example 2.18 // Let $L = (\mathbb{R}, \| \|_{\mathbb{R}})$ and $L' = (\mathbb{R}^2, \| \|_{\mathbb{R}})$

where $\| \|_{\mathbb{R}^2}$ is the Euclaidian norm.

If $x = 3 \in L = \mathbb{R}$ and $Y = (1,-2) \in L' = \mathbb{R}^2$.

Find $\|(X,Y)\|_1$ and $\|(x,Y)\|_2$

(Ex 2.18) ملاحظ // حسب المثال السابقة

تعريف $\|(X,Y)\|_1$ و $\|(X,Y)\|_2$ هو كالاتي

1) $\|(X,Y)\|_1 = \|X\|_1 + \|Y\|_{1'}$

2) $\|(X,Y)\|_2 = \max\{\|X\|_L , \|Y\|_{L'}\}$

solution //

① $\|(X,Y)\|_1 = \|3\| + \|(1,-2)\|$

$\|X\|_1 = \sqrt{3^2} = 3$ تعريف Norm هو

$\|(1,-2)\|_{L'} = \sqrt{1^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}$

$\therefore \|(X,Y)\|_1 = \|3\| + \|(1,-2)\| = \boxed{3+\sqrt{5}}$

② $\|(X,Y)\|_2 = \max\{\|3\|, \|(1,-2)\|\}$

$= \max\{3, \sqrt{5}\} = \boxed{3}$

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محاضرات التحليل الدالي

أعداد أستاذة المادة

م. م ايناس حسن عبد كاظم

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٢٠٢٠ - ٢٠٢١

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Chapter 1

Linear Space

A linear space (also called vector space), denoted by L or V, is a collection of objects called **vectors**, which may be added together and multiplied by numbers, called **scalars** which are taken from a field F. Before defining .linear space, we first define an arbitrary field

. *Field*1.1Definition

.Let F be a non-empty set and + and . be two binary operations on F

The ordered triple $(F, +, .)$ is called **field** if and only if

$F, +)$ is a commutative group)(1)

$F-\{e\}, .)$ is a commutative group, where e is the identity with respect)(2)

.(+) to

(is distributed over (+) (from left and right (.)(3)

.1.2Example

Let (+) and (.) are **ordinary addition and multiplications**. Then

- Each of $(\mathbb{R}, +, .)$, $(C, +, .)$, and $(Q, +, .)$ are examples of fields
-) does not hold) and $(Z, +, .)$ is 1(1.1Z, $-$,.) is not field (Definition) () does not hold2(1.1not field (Definition

. ***Linear Space*1.3Definition**

Let $(F,+,.)$ be a field whose elements are called **scalars**. Let L is a non empty set whose elements are called **vectors**.Then L is a **linear space** (or a **vector space**) over the field F, if

addition: There is a binary operation $+$ on L called **addition** (not **(١)**
.usual addition) such that $(L,+)$ is a commutative group

$\forall\alpha \in F.\ \ x \in L,\forall$**scalar multiplication:** $\alpha.x \in L$**(٢)**

)The scalar multiplication and addition satisfy3(

$\forall\alpha \in F\, x, y \in L,\forall \quad \alpha.(x+y) = \alpha.x + \alpha.y(i)$

$\forall\alpha,\beta \in F\ \ x \in L,\forall$(ii)$(\ \alpha+\beta).x = \alpha.x + \beta.x$

$1\ \ \forall\alpha,\beta \in F$ (iv)$\forall x \in L,$(iii)$(\ \alpha.\beta).x = \alpha.(\beta.x)$

is the unity F1 $x \in L$ and $\forall.x = x$

.1.4Remark

If L is a linear space over F, we say that $L(F)$ is a linear space. We also .can say L is a linear space

Examples of Linear Space1.1

.1.5Example

-The set of real numbers R, with **ordinary** addition and **ordinary** multi

,plication, is a linear space over $(F,+,.) = (R,+,.)$. Indeed

R,+) is an abelian group) (١)

$x \in R, \alpha \in R \forall \alpha . x \in R$ (٢)

(!All other conditions are satisfied (Check (٣)

.This linear space (R,+,.) is called **real** linear space

.1.6Example

The set of complex numbers C, with **ordinary** addition and **ordinary**

,multiplication, is a linear space over $(F,+,.) = (C,+,.)$. Indeed

C,+) is an abelian group) (١)

$x \in C, \alpha \in C \forall \alpha . x \in C$ (٢)

(!All other conditions are satisfied (Check (٣)

.This linear space $(C,+,.)$ is called **complex** linear space

.1.7Example

$\ni$, ..., x_n1, ..., x_n) : x_1Let (R, +, .) be the field of real numbers. Let R^n = {(x

, ..., y_n) of R^n, $_1$, ..., x_n) and $Y = (y_1$R}. For any two elements X = (x

define ordinary addition

.(, ..., $x_n + y_n$1+ y_1 $X + Y = (x$

Also, define scalar multiplication in R^n over R by

.$\forall \alpha \in R, \ \forall X \in R^n$, ..., $\alpha . x_n)_1 \alpha . X = (\alpha . x$

.Show that R^n is a linear space over R

Solution: Let us check linear space conditions

We show that $(R^n, +)$ is a commutative group (١)

$, ..., x_{n1} + y_1 , ..., y_n \in R^n)$. Since $x_1, ..., x_n)$, $Y = (y_1$a)Let $X = (x)$

$+ \quad y_n \in$ R, then $X + Y \in R^n$. Hence, R^n is closed with respect

.to ordinary addition

$, ..., z_n) \in R^n{}_1, ..., y_n), Z = (z_1, ..., x_n), Y = (y_1 X = (x$b)For all)

$$[(, ..., z_{n1}, ..., y_n) + (z_1, ..., x_n) + [(y_1 X + (Y + Z) = (x$$

$$(, ..., y_n + z_{n1} + z_1 , ..., x_n) + (y_1 x) =$$

$$(), ..., x_n + (y_n + z_{n1} + z_1 + (y_1 x \; =$$

$$, ..., (x_n + y_n) + z_{n1}) + z_1 + y_1 x) \; =$$

$$., ..., z_n) = (X + Y) + Z_1, ..., x_n + y_n) + (z_1 + y_1 x) =$$

$, ..., y_n) \in R^n{}_1, ..., x_n), Y = (y_1 X = (x$c)For all)

$$, ..., y_n + x_n) = {}_1 + x_1, ..., x_n + y_n) = (y_1 + y_1, ..., y_n) = (x_1, ..., x_n) + (y_1 X + Y = (x \; Y$$

$) \in R^n$ such that $0, ..., 0, ..., x_n) \in R^n$ we have $({}_1 X = (x$For all (d)

$)$ is the $0, ..., 0, ..., x_n)$. Thus, $({}_1) = (x0, ...0, ..., x_n) + ({}_1(x$

.additive identity

$, ..., -x_n) \in R^n$ such $_1, ..., x_n) \in R^n$ then $-X = (-x_1 X = (x$If (e)

that

.). Thus, $-X$ is the additive inverse of $X0, ..., 0X + (-X) = ($

.From (a)-(e) we get $(R^n, +)$ is a commutative group

$, \ldots, \alpha x_n \in$ R, then$_1, \ldots, x_n) \in$ R^n and $\alpha \in$ R. Since αx_1Let $X = (x$ (٢)

$., \ldots, \alpha.x_n) \in$ R$^n{}_1\alpha.X = (\alpha.x$

.Hence, R^n is closed with respect to scalar multiplication

The scalar multiplication and addition satisfy (٣)

$, \ldots, y_n) \in$ R^n and $\alpha \in$ R, then$_1, \ldots, x_n),(y_1X = (x$If (i)

$(, \ldots, x_n + y_{n1} + y_1 \ \alpha.(X + Y) = \alpha.(x$

$((), \ldots, \alpha.(x_n + y_{n1} + y_1 \ \alpha.(x) =$

$(, \ldots, \alpha.x_n + \alpha.y_{n1} + \alpha.y_1 \ \alpha.x) =$

$(, \ldots, \alpha.y_{n1}, \ldots, \alpha.x_n) + (\alpha.y_1\alpha.x) =$

$, \ldots, y_n) = \alpha.X + \alpha.Y_1, \ldots, x_n) + \alpha.(y_1\alpha.(x =$

$, \ldots, x_n) \in$ R^n and $\alpha, \beta \in$ R, then$_1X = (x$If (ii)

$, \ldots, (\alpha + \beta).x_{n1}\alpha + \beta).X = (\alpha + \beta).x)$

$, \ldots, \alpha.x_n + \beta.x_{n1} + \beta.x_1 \ \alpha.x =$

$(, \ldots, \beta.x_{n1}, \ldots, \alpha.x_n) + (\beta.x_1\alpha.x) =$

$, \ldots, x_n) = \alpha.X + \beta.Y_1, \ldots, x_n) + \beta.(x_1\alpha.(x =$

$, \ldots, x_n) \in$ R^n and $\alpha, \beta \in$ R, then $_1X = (x$If (iii)

$, \ldots, (\alpha.\beta)x_{n1} \ (\alpha.\beta).X = (\alpha.\beta).x$

$, \ldots, x_n) = \alpha.(\beta.X)\ _1, \ldots, \beta.x_n) = \alpha.\ \beta.(x_1\alpha.(\beta.x =$

is the unity of R, then1 $, \ldots, x_n) \in$ R^n and $_1X = (x$(iv)If

$, \ldots, x_n) = X_1.x_n) = (x1, \ldots, _1.x1X = (.1$

.Hence R^n is a linear (vector)space over R

.1.8Example

$, ..., x_n)$: $_1$Let $(C, +, .)$ be the field of complex numbers. Let $C^n =$ $\{(x$

$, ...,_1, ..., x_n)$ and $Y = (y_1, ..., x_n \in C\}$. For any two elements $X = (x_1 x$

$y_n)$ of C^n, define

$.(, ..., x_n + y_{n1} + y_1\ X + Y = (x$

Define scalar multiplication in C^n over C by

$.\forall \alpha \in C,\ \forall X \in C^n, ..., \alpha.x_n)_1 \alpha.X = (\alpha.x$

Show that C^n is a vector space over C. (**Verify that**)

.1.9Example

$, y) : x, y > 0), (0$Let (R, +, .) be the field of real numbers. Let $M = \{(x,$

$, y)$ of M , define 0) and $Y = (0\}$. For any two elements $X = (x, 0$

.(ordinary addition $X + Y = (x, y$

) and0Also, define scalar multiplication in M over R by $\alpha.X = (\alpha.x,$

?, $\alpha.y)$ $\forall \alpha \in$ R, $\forall X, Y \in M$. Is M a linear space over R0$\alpha.Y = ($

,**Solution:** Let us check if $(M, +)$ is a commutative group

) $\notin M$. Thus, M is 1, 1) = (1, 0) + (0, 1) $\in$ M but (1, 0), (0, 1Since (

) = 0, 1.(1not closed under addition, then $(M, +)$ is not group. Also, $-$

.) $\notin M$. Thus, M is not closed under scalar multiplication0, 1($-$

.1.10Example

Let $C^b(R) = \{f : f : R \to R$; f is bounded and continuous$\}$ set of all bounded and continuous functions defined on R. For any $f, g \in C^b(R)$ and for any $\alpha \in$ R, define

$.\alpha f)(x) = \alpha . f(x) \ \ \forall x \in R, \forall \alpha \in R)andf + g)(x) = f(x) + g(x) \ \ \forall x \in R)$

Show that $C^b(R)$ is a linear space over R.

.Now, let us check linear space conditions

We show $(C^b(R), +)$ is a commutative group (1)

Let $f, g \in C^b(R))$ such that f, g are continuous and bounded (a)

func- tions. We want to prove $f + g \in C^b(R)$. (i.e., $f + g$ is

(continuous and bounded

Since f, g are continuous, the sum $(f + g)$ is a continuous func-

(I)tion

$\in R_+$ such that$_2$, M_1Also, since f, g are bounded functions, $\exists M$

. Hence, for all $x \in R_2$and $|g(x)| \le M_1 \ f(x)| \le M|$

$._2 + M_1 \ f + g)(x)| = |f(x) + g(x)| \le |f(x)| + |g(x)| \le M)|$

. Thus, $f + g$ is bounded function$_2 + M_1 \ f + g)(x)| \le M)|$ **(II)**

.(By **(I)** and **(II)**, $f + g \in C^b(R$

$f, g, h \in C^b(R)$ and for all $x \in R$b)For all)

$[(f + (g + h)](x) = f(x) + [(g + h)(x]$

$(f(x) + g(x)] + h(x] =$

$.(f + g)(x) + h(x) = [(f + g) + h](x) =$

$(f, g \in C^b(R$For all (c)

$.(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$

$.0(x) = 0$by $\hat{0} : R \to 0 \ f \in C^b(R)$, define $\hat{F}$or all (d)

$\ni 0$is continuous and bounded function. Thus, $\hat{0}$It is clear that $\hat{}$

$C^b(R)$ and

$.(= f(x0 (x) = f(x) + 0)(x) = f(x) + \hat{0}f + \hat{})$

$,+ f(x) = f(x)$ Thus$0 (x) + f(x) = 0+ f)(x) = \hat{0}$ Similarly, $(\hat{}$

$+ f = f0 = \hat{0} f + \hat{}$

.is called the additive identity $\hat{0}$

$\ni$ e)For any $f \in C^b(\mathrm{R})$, define $-f: \mathrm{R} \to \mathrm{R}$ by $(-f)(x) = -[f(x)] \ \forall x)$

.R

.Since f is continuous, then $-f$ is continuous

.Moreover, $\forall x \in \mathrm{R}$, $|-f(x)| = |f(x)| \leq M$. Then, $-f$ is bounded

Thus, $-f \in C^b(\mathrm{R})$ and

$f+(-f)](x) = f(x)+(-f)(x) = f(x)+(-f(x)) = f(x)-f(x)]$

$.0= \hat{0} =$

$=$ (Similarly, $[(-f) + f](x) = (-f)(x) + f(x) = (-f(x)) + f(x$

$0= \hat{0} f(x) + f(x) = -$

.From (a)-(e) we get $(C^b(\mathrm{R}), +)$ is a commutative group

Let $f \in C^b(\mathrm{R})$ and $\alpha \in \mathrm{R}$. We want to prove $\alpha f \in C^b(\mathrm{R})$. (i.e., αf is (٢)

(continuous and bounded

.Since f is continuous, then αf is a continuous function

Also, since f is bounded functions, $\exists M \in \mathrm{R}_+$ such that $|f(x)| \leq M$

. Hence, for all $x \in \mathrm{R}$

$.\alpha f)(x)| = |\alpha . f(x)| = |\alpha| |f(x)| \leq |\alpha| M)|$

Therefore, $\alpha f \in C^b(\mathrm{R})$ ($C^b(\mathrm{R})$ is Thus, αf is bounded function.

.(closed with respect to scalar multiplication

The scalar multiplication and addition satisfy (٣)

$f, g \in C^b(\mathrm{R})$ and $\alpha \in \mathrm{R}$, theni)If)

$[(\alpha(f+g))(x) = \alpha.(f+g)(x) = \alpha.[(f(x)+g(x)$

$(\alpha.f(x) + \alpha.g(x =$

$\alpha f)(x) + (\alpha g)(x) = (\alpha f +) =$

$f \in C^b(\mathrm{R})$ and $\alpha, \beta \in \mathrm{R}$, then $\alpha g)(x)$ (ii)If

$(\alpha+\beta)f](x) = (\alpha+\beta).f(x)]$

$(\alpha.f(x) + \beta.f(x =$

$\alpha f)(x) + (\beta f)(x) = (\alpha f + \beta f) =$

$f \in C^b(\mathrm{R})$ and $\alpha, \beta \in \mathrm{R}$, then $)(x)$ (iii)If

$.(\alpha.\beta)f](x) = (\alpha.\beta).f(x) = \alpha.(\beta.f(x)) = \alpha.[(\beta f)(x)] = [\alpha(\beta f)](x)]$

.(Hence, $(\alpha.\beta)f = \alpha(\beta f$

is the unity of R, then 1 $f \in C^b(\mathrm{R})$ and iv)If)

$.(.f(x) = f(x1f)(x) = 1($

.Hence, $C^b(\mathrm{R})$ is a linear (vector)space over R

.1.11Exercise

Let $C^b[a, b] = \{f : f : [a, b] \to \mathrm{R}$ f is bounded and continuous$\}$ set (١) of all bounded continuous functions defined on $[a, b]$. Show that $C^b[a, b]$ is a linear space over R where $f + g$ and αf are defined in the .1.10same way as in Example

and $F = (\mathrm{R}, +, .)$. Define the following **two operations**:2 Let $L = \mathrm{R}$ (٢)

$.^2) \in \mathrm{R}_2, y_1), (y_2, x_1)$ $\forall(x_2 + y_2, x_1 + y_1) = (x_2, y_1) + (y_2, x_1)$ **)** $(x$**1(**

$.\forall \alpha \in \mathrm{R},^2) \in \mathrm{R}_2, x_1 \forall (x)_2, x_1) = (\alpha.x_2, x_1 \alpha.(x$ **(2)**

?Show that L is not a linear space over R

Let L be the set of all real valued sequences $\langle x_n \rangle$. Define usual addition(٣) and multiplication of a sequence as follows: for any $\langle x_n \rangle, \langle y_n \rangle \in L$ and each $\alpha \in$ R

and $\alpha.\langle x_n \rangle = \langle \alpha.x_n \rangle$. Show that $\langle x_n \rangle$ is a $x_n \rangle + \langle y_n \rangle = \langle x_n + y_n \rangle\rangle$

.linear space over R

: $(_3, x_2, x_1$Let $N = \{(x$Let (R,+,.) be the field of real numbers.(٤)

:}. Define the following **two operations**$0 > {}_3, x_2, x_1 x$

$= \;), Y_3, x_2, x_1 \forall X = (x).{}_3+ y_3, x_2+ y_2, x_1+ y_1 \; X + Y = (x$ **(1)**

$) \in N_3, y_2, y_1 y)$

$.\forall \alpha \in$ R, $\forall X \in N)_3, \alpha.x_2, \alpha.x_1 \alpha.X = (\alpha.x$ **(2)**

?Is N linear space over R

. *Properties of Linear Space*1.12Theorem

$_L$ is a zero vector of L. Then**0**Let $L(F)$ be a linear space and

$_L \; \forall \alpha \in F.\; \mathbf{0}_L = \mathbf{0}\alpha.$ (1)

$._L \forall x \in L\mathbf{0}.x = 0$)2 (

$\forall \alpha \in F,\; \alpha.(-x) = -(\alpha.x) \;\; \forall x \in L,$ (3)

$.\forall \alpha \in F)(-\alpha).x = -(\alpha.x) \;\; \forall x \in L,$4(

$\forall \alpha \in F,\; \alpha.(x - y) = \alpha.x - \alpha.y \;\; \forall x, y \in L,$ (5)

$._L\mathbf{0}$or $x = 0$ then $\alpha = 0$ $\alpha.x =$)If6(

Linear Subspace1.2

.1.13Definition

Let L be a linear space over a field F and let φ /= $H \subseteq L$. Then H is called

.a **linear subspace** of L if H itself is a linear space over F

.1.14Theorem

Let H be a non empty subset of a linear space $L(F)$. H is called a subspace

.of L if and only if $\alpha x + \beta y \in H$ for all $x, y \in H$ and for all $\alpha, \beta \in F$

.1.15Exercise

^{3}be a linear space over R. Which of the following subsets of R^3 RLet(١)

.^{3}are subspaces of R

{∈ R3 , x_2: x^3) ∈ R3, x_2, x2= {($_1$ i) H)

= $_3$ + x_2 + x_1 : x^3) ∈ R3, x_2, x_1= {(x_2 ii) H)

= $_2$ x2+ $_1$: x^3) ∈ R3, x_2, x_1= {(x_3 } (iii) H0

{1

] be a linear space over R. Which of the following subsets 1, 1C[−Let (٢)

.[1, 1] are subspaces of C[−1, 1of C[−

{0) = 0] : f(1, 1= {f ∈ C[−$_4$ i) H)

{[1, 1, ∀x ∈ [−0] : f(x) ≤ 1, 1= {f ∈ C[−$_5$ ii) H)

{(1) = f(1] : f(−1, 1= {f ∈ C[−$_6$ iii) H)

then$_1$) ∈ H_2, y_1, y2), ($_3$, x_2, x2**(i)):** Take (**1Solution (**

$_1$) ∉H_3+ y_3 , x_2+ y_2 , x4) = ($_2$, y_1, y2) + ($_3$, x_2x $_4$2)

is $_1$, H1.13Then, the closure condition is not satisfied. From Definition

,and $\alpha, \beta \in$ R$_1$) $\in H_2, y_1, y2), (_3, x_2, x2$**(i)):** Let (**1Another Solution (**

then

$_1) \notin H_3 + y_3, x_2 + y_2(\alpha+\beta), x2) = (_3 + y_3, x_2 + y_2\beta, x2\alpha + 2) = (_2, y_1, y2) + \beta(_3, x_2, x2\alpha($

.1if and only if $\alpha + \beta = 2$ $(\alpha + \beta) =$ 2because

.^{3}is not a subspace of R$_1$, H1.14Thus, from Theorem

.and $\alpha, \beta \in$ R$_6$ **(iii)):** Let $f, g \in H$**2Solution (**

] $\Rightarrow$ αf and βg are continuous on1, 1$\Rightarrow$ f, g are continuous on [$-_6$ $f, g \in H$

.[1, 1Thus, $\alpha f + \beta g$ is continuous on [$-$.[1 4$-$] **(I)**

(1) + $(\beta g)(-1) = (\alpha f)(-1$ $\alpha f + \beta g)(-)$

(1) + $\beta.g(-1$ $\alpha.f(- =$

.(1) = $(\alpha f + \beta g)(1) + \beta.g(1$ $\alpha.f(=$ **(II)**

.[1, 1is a subspace of C[$-_6$. Thus, H_6From **(I)** and **(II)**, $\alpha f + \beta g \in H$

Linear Transformation Mapping .٣

.1.16Definition

Let $L(F)$ and $L'(F)$ be two linear spaces over the same field F. A mapping $T : L \to L'$ is called a **Linear Operator** or **Linear Transformation** if .$\forall \alpha, \beta \in F \forall x, y \in L, T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

.1.17Example

.$\in$ R$_3$, $x_2, x_1)$ $\forall x_2, x_1) = (x_3, x_2, x_1$defined by $T(x^2 \to$ R^3 Let T : R

.T is a linear transformationShow that(١)

). Compute1, 5, $-0) = (_3, y_2, y_1), Y = (y3, -1, 2) = (_3, x_2, x_1X = (x$If(2)

.(X) and $T(X + Y2$ $T($

∋ and α, β^{5}) $\in$ R3, $y_2, y_1, Y = (y^3) \in$ R3, x_2, x_1**):** Let X = (x**1Solution (**

R. Then

$[(3, y_2, y_1) + \beta(y_3, x_2, x_1\, T(\alpha X + \beta Y) = T[\alpha(x$

$(3+ \beta y_3\, , \alpha x_2+ \beta y_2\, , \alpha x_1+ \beta y_1\, T(\alpha x =$

$(2+ \beta y_2\, , \alpha x_1+ \beta y_1\, \alpha x) =$

$(2, \beta y_1) + (\beta y_2, \alpha x_1 \alpha x) =$

$(2, y_1) + \beta(y_2, x_1 \alpha(x =$

$.( ) = \alpha T(X) + \beta T(Y_3, y_2, y_1) + \beta T(y_3, x_2, x_1 \alpha T(x =$

$.(2,4) = (6, -2, 4X) = T(2)$**:** $T($**2Solution (**

$.(4, -2) = (2, -4, -2\, T(X + Y) = T($

.1.18Exercise

is a linear space over $F=$ R with usual addition and multipli-^{2} Let R (1)

is a $^2 \rightarrow$ R^2 cation. Show that each of the following mappings T : R

linear transformation

$2+ x_1\, x2) = 2, x_1(x_1$i) $T)$

$(2, x0) = (2, x_1(x_2$ii) $T)$

) where $a \in$ R$_2$, $ax_1) = (ax_2, x_1(x_3$iii) $T)$

Let C^b(R) be the set of all bounded continuous functions defined (2)

on R such that C^b(R) is a linear space over R with usual addition

and multiplication. Let T : C^b(R) $\rightarrow$ C^b(R) such that $T(f(x))$ =

.$f(x)$. Show that T is a linear transformation mapping$^2 x$

1.19 Theorem.

Let $T : L(F) \to L'(F)$ be a linear transformation. Then

(i) $T(0_L) = 0_{L'}$ where 0_L is the zero vector of L and $0_{L'}$ is the zero vector of L'

(ii) $T(-x) = -T(x)$

(iii) $T(x-y) = T(x) - T(y)$

1.20 Theorem.

Let L, L' be linear spaces over same field F. Let $T_1, T_2 : L \to L'$ linear transformations. Define the function $T_1 + T_2 : L \to L'$ as $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ $\forall x \in L$. If $\alpha \in F$, then the function $\alpha T_1 : L \to L'$ is defined as $(\alpha T_1)(x) = \alpha . T_1(x)$ $\forall x \in L$. Then

(i) Show that $T_1 + T_2$ is a linear transformation.

(ii) Show that αT_1 is a linear transformation.

Proof. (i) Let $\alpha, \beta \in F$ and $x, y \in L$. Then

$(T_1 + T_2)(\alpha x + \beta y) = T_1(\alpha x + \beta y) + T_2(\alpha x + \beta y)$ (Definition of $T_1 + T_2$)

$= \alpha T_1(x) + \beta T_1(y) + \alpha T_2(x) + \beta T_2(y)$ (since T_1, T_2 linear trans.)

$= \alpha(T_1(x) + T_2(x)) + \beta(T_1(y) + T_2(y))$

$= \alpha(T_1 + T_2)(x) + \beta(T_1 + T_2)(y).$

Thus, $T_1 + T_2$ is a linear transformation.

$\in F$ and $x, y \in L$. Then$_2$, $\beta_1$ii) Let β)

($y_2 x + \beta_1(\beta_1 y) = \alpha . T_2 x + \beta_1)(\beta_1 \alpha T)$ -Definition of scalar multiplica)

(tion

$((y_1 . T_2(x) + \beta_1 . T_1 \alpha . \beta =$ (.linear trans$_1$ since T)

$((y_1 . T_2(x) + \alpha . \beta_1 . T_1 \alpha . \beta =$

$()(y_1 .(\alpha T_2)(x) + \beta_1 .(\alpha T_1 \beta =$

.is a linear transformation$_1$ Thus, αT □

.1.21Definition

Let L be a linear space. A linear transformation $T : L \rightarrow F$ is said to be **Linear functional**. (Note that F can be regarded as a linear space over .(F

.1.22Example

= Let L $F^n, ..., x_n \in F\}$ be a linear space over$_1, ..., x_n) : x_1 x)\} =$ .the field F $+ ... + {}_1 x_1, ..., x_n) = \alpha_1$Let $T : F^n \rightarrow F$ defined by $T(x$ Prove that T is a linear$, ..., \alpha_n \in F._1, ..., x_n) \in F^n$ and $\alpha_1 x)\forall \alpha_n x_n$ .transformation

$, ..., y_n) \in F^n$ and $\alpha, \beta \in F$. Then$_1, ..., x_n), y = (y_1$**Solution:** Let $x = (x$

$[(, ..., y_{n1}, ..., x_n) + \beta(y_1 T(\alpha x + \beta y) = T[\alpha(x$

$(, ..., \alpha x_n + \beta y_{n1} + \beta y_1 \; T(\alpha x =$

$() + ... + \alpha_n(\alpha x_n + \beta y_{n1} + \beta y_1 \; (\alpha x_1 \alpha =$

$(+ ... + \alpha_n y_{n1} \; y_1 + ... + \alpha_n x_n) + \beta(\alpha_1 \; x_1 \alpha(\alpha =$

$.(, ..., y_{n1}, ..., x_n) + \beta T(y_1 \alpha T(x =$

.(Thus, T is a linear transformation (i.e., linear functional

Chapter 2

Normed Linear Space

.2.1Definition

Let $L(F)$ be a linear space over a field F. A mapping $\| \| : L \to \mathrm{R}$ is called

norm if the following conditions hold

((Positivity $x \in L.\forall 0 \|x\| \geq$ (١)

.0if and only if $x = 0$ $\|x\| =$ (٢)

((Triangle Inequality $x, y \in L.\forall \|x + y\| \leq \|x\| + \|y\|$ (٣)

.$x \in L,\ \forall \alpha \in F \forall \|\alpha x\| = |\alpha| \|x\|$ (4)

$L, \| \|$) is called **normed linear space**.)

.2.2 Remark

.From now on, the field F is either R or C

.2.3Theorem

Let $(L, \| \|)$ be a normed linear space. Then, for each $x, y \in L$

.$0_L\| =$ **o**$\|$ (1)

.$\|x\| = \|-x\|$ (2)

. $\|x - y\| = \|y - x\|$ (3)

((Reverse Triangle Inequality$\|x\| - \|y\|$ $| \le \|x - y\|$. | (٤)

)Every 6(Reverse Triangle Inequality) ($\|x\| - \|y\|$ $| \le \|x + y\|$. |(٥)

subspace of a normed space is itself normed space with respect

.to the same norm

((1(1.12see Theorem)$_{\mathbf{L}}\|0\mathbf{o}_{\mathbf{L}}\| = \|\mathbf{o})\ \|1$ *Proof.* (

$.0_{\mathbf{L}}\| = \mathbf{o}\|\ 0 =$

$.x \in L\forall |\ \|x\| = \|x\|\ 1\|-x\| = |-$ (2)

.((2by part ()$\|x - y\| = \|-(y - x)\| = \|y - x\|$ (3)

We must prove $-\|x - y\| \le \|x\| - \|y\| \le \|x - y\|$ (٤)

)). 3(2.1by Definition)$\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$

(I) Hence, $\|x\| - \|y\| \le \|x - y\|$

)). 3(2.1by Definition)Similarly, $\|y\| = \|y - x + x\| \le \|y - x\| + \|x\|$

(II) Hence, $\|y\| - \|x\| \le \|x - y\|$

$.\forall x, y \in L$Hence, by (I) and (II), we get $\|x - y\| \ge |\|x\| - \|y\||$

We must prove $-\|x + y\| \le \|x\| - \|y\| \le \|x + y\|$ (٥)

)). 3(2.1by Definition)$\|x\| = \|x + y - y\| \le \|x + y\| + \|-y\|$

(III) Hence, $\|x\| - \|y\| \le \|x + y\|$

.((3(2.1by Definition)Similarly, $\|y\| = \|y + x - x\| \le \|y + x\| + \|-x\|$

Hence, $\|y\| - \|x\| \le \|x + y\|$

(IV) $\|x\| - \|y\| \ge -\|x + y\|$

Hence, by (III) and (IV), we get $-\|x + y\| \le \|x\| - \|y\| \le \|x + y\|$

$.x, y \in L\forall$

□

Examples of Normed Linear Space2.1

.2.4Example

Let $L=\mathbb{R}$ be a linear space over $\mathbb{R}$ with $\|\ \|: L \to \mathbb{R}$ such that $\|x\| =$ $.|x|$. Show that $(\mathbb{R}, \|\ \|)$ is a normed space

Solution: We show that

$.0 x \in \mathbb{R}$; hence $\|x\| \geq \forall 0 \|x\| = |x| \geq$ (1)

$.0 \Leftarrow\Rightarrow\ x = 0 \iff |x| = 0$ Let $x \in \mathbb{R}$, $\|x\| =$ (2)

$, x \in \mathbb{R}, \forall \alpha \in \mathbb{R} \forall$ (3)

$.\|\alpha x\| = |\alpha x| = |\alpha|\,|x| = |\alpha|\,\|x\|$

$.x, y \in \mathbb{R} \forall \|x+y\| = |x+y| \leq |x| + |y| = \|x\| + \|y\|$ (4)

.2.5Example

Let $L = C$ be a complex linear space over C with $\|\ \|: C \to \mathbb{R}$ such that $.z = a + ib$. Show that $(C, \|\ \|)$ is a normed space$\forall^2 + b^2$ $\|z\| = |z| = \sqrt{a}$

Solution: We show that

$.0 z = \overline{a + ib} \in C$; hence $\|z\| \geq \forall 0 \geq {}^2 + b^2$ $\|z\| = |z| = \sqrt{a}$ (1)

Let $z = a + ib \in C$(٢)

$.0 i = \overline{0 + 0} \Leftarrow\Rightarrow\ z = 0 \iff a = b = 0 = {}^2 + b^2$ $\|z\| = |z| = \sqrt{a}$

Let $z, w \in C$(٣)

$\|z + w\|^2 = \overline{(z + w)(z + w)}$ where $z + w$ =conjugate of $z + w$

$- \quad - \ z + w)(z + w) = zz + ww + wz + wz) =$

$z\overline{z} + w\overline{w} + w\overline{z} + \overline{\overline{w}z}$

$- \quad - \quad - Re\ wz 2zz + ww + =$

$\|w\|\,\|z\| = (\|z\| + 2\|z\|^2 + \|w\|^2 + \geq$ $.^2\|w\|)$

, hence, $\|z + w\| \leq \|z\| + {}^2 \leq (\|z\| + \|w\|)^2$ Thus, $\|z + w\|$
$.\|w\|$

,Let $z \in C, \alpha \in C$ (4)

$|(\|\alpha z\| = |\alpha z| = |\alpha(a + ib$
$= |= |\alpha||z^2 + b^2 \sqrt{a^2) =} \sqrt{\alpha^2 + b^2} (a^2 = \sqrt{\alpha^2 + (\alpha b)^2} \alpha a)) =$

$.\alpha|\|z\||$

$-i$, then1 i, $w = 3+ 2$: Let $z =$ **2.5As an application to Example**
$.\sqrt{13} = \sqrt{2^2 + 3^2} i\| = \sqrt{2+ 3} i - i)\| = \|3) + (1+ 2\|z + w\| = \|($
$.\sqrt{13}\sqrt{5} = \sqrt{325} = \sqrt{15^2 + 10^2} i\| = \sqrt{15+ 10} z\| = \|5\|$
$.\sqrt{13}\sqrt{5} = \sqrt{3^2 + 2^2} \sqrt{5}\|z\| = 5$

.2.6Example

Show that the linear space $C^b(\mathbb{R})$ is a normed space under the norm

$.(\forall f \in C^b(\mathbb{R}\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\},$

.0. Hence, $\|f\| \geq 0 \forall x \in \mathbb{R}$. Then, $\|f\| = \sup|f(x)| \geq 0$ Since $|f(x)| \geq$ (1)

$0 \Longleftrightarrow \sup\{|f(x)| : x \in \mathbb{R}\} = 0 \; \|f\| =$ (2)

$\forall x \in \mathbb{R} 0 \; f(x)| = | \Rightarrow \Leftarrow$

((zero mapping$0 \; \forall x \in \mathbb{R} \iff f = \hat{0} \; f(x) = \Rightarrow \Leftarrow$

Let $f, g \in C^b(\mathbb{R})$. Then (٣)

$\{\|f + g\| = \sup\{|f(x) + g(x)| : x \in \mathbb{R}$

$\{\sup\{|f(x)| + |g(x)| : x \in \mathbb{R} \geq$

$.\sup\{|f(x)| : x \in \mathbb{R}\} + \sup\{|g(x)| : x \in \mathbb{R}\} = \|f\| + \|g\| \geq$

.Hence, $\|f + g\| \leq \|f\| + \|g\|$

Let $f \in C^b(\mathbb{R}), \alpha \in \mathbb{R}$. Then (٤)

$\{\|\alpha f\| = \sup\{|(\alpha f)(x)| : x \in$ R

$\{\sup\{|\alpha||f(x)| : x \in$ R =

below where $A = |f2.7\ \alpha|\sup\{|f(x)| : x \in$ R$\}$ (By Theorem | =

$(|(x)|$ and $\beta = |\alpha$

$.\alpha|\,\|f\||$ =

.2.7Theorem

, then βA is bounded above and 0If A is a bounded above set and $\beta >$

$.(\sup(\beta A) = \beta\sup(A$

= (**:** Let $f, g \in C^b$(R) such that $f(x$**<u>2.6As an application to Example</u>**

,. Hence1 $cos(x) + 2sin(x)$ and $g(x)$ =

.(∀$x \in$ R,1since $|sin(x)| \leq$)1$\|f\| = \sup\{|sin(x)| : x \in$ R$\}$ =

.{| : $x \in$ R1$cos(x)$ + 2$\|f\| = \sup\{|$

.3. So $\|g\|$ = 3= 1 $|cos(x)| + 2| \leq 1cos(x) +$ 2But |

.2.8Example

] is 1, 0] of all real valued continuous functions on [1, 0The linear space C^b[

(.. (**H.W**2.6a normed space under the norm defined in Example

.2.9Example

] is 1, 0] of all real valued continuous functions on [1, 0The linear space C[

a normed space with the norm defined as

.[1, 0]$\int_0^1 |f(x)|\ dx\ \forall f \in C$[$\|f\|$ =

,. Thus0$f(x)|\ dx \geq$ || , then1, 0∀$x \in$ [,0) Since$\int_0 |f(x)| \geq$ 1**solution:** (

$.0\|f\| \geq$

$\Rightarrow\Leftarrow$ 0$\|f\|$ = (2) $\int_0^1 f(x)|\ dx = |$

$[1,0 \forall x \in [0 \ f(x)| = | \ \Rightarrow\Leftarrow$

$[1,0 \forall x \in [0 \ f(x) = \ \Rightarrow\Leftarrow$

.((zero mapping0 $f = \ \Rightarrow\Leftarrow$

]. Then1,0Let $f, g \in C[$ (3)

$|f(x) + \int_0^1 g(x)| dx \|f + g\| =$

$\int_0^1 f(x)| + |g(x)|) dx|) \geq$

$\int_0^1 |g(x)| \ dx = \int_0^1 \|f\| + \|f(x)| \ dx + | = g\|$

], $\alpha \in$ R . Then1,0Let $f \in C[$ (4)

$|(|f(\int_0^1 x \quad |\alpha| |f(x)| \int_0^1 dx = |\alpha| |\alpha f(x)| \ dx = \int_0^1 \|\alpha f\| = \ .dx = |\alpha| \|f\|$

= () such that $f(x1,0$: Let $f \in C[$**2.9As an application to Example**

.. Find $\|f\|$, $\|g\|$ and $\|f + g\|^2$and $g(x) = -x^3 \ x$

$= |f\int_0^1 x)| \ dx \|f\| = \int_0^1 \ .3. \quad = \int_0^1 x \ dx dx = \frac{3}{0}x$

$= |\int_0^1 g(x)| \ dx \|g\| = \int_0^1 \ . \ 2. \quad = \int_0^1 x_0 \ dx dx = x\frac{2}{3}$

$= \|f + g\| \int_0^1 = f + g)(x)| \ dx)| \quad \int_0^1 . x^2 x \ x^3 . \ dx$

$\int_0^1 \ .12 = \frac{3}{4} \ \frac{2}{3} \ x \ - x \frac{1}{\ } dx = \frac{1}{\ } \frac{1}{\ } =$

.2.10Example

Consider the linear space F^n over F ($F =$ R or C). Define $\| \ \| : F^n \rightarrow$ R

$, ..., x_n) \in F^n$. Then $(F^n, \| \ \|)$ is a $_1| , ..., |x_n|\} \forall X = (x_1$by $\|X\| = \max\{|x$

.normed space

$., ..., n1, \forall i = 0, ..., x_n) \in F^n, \ |x_i| \geq \ _1)$ For any $X = (x1$**solution:** (

.0, then $\|X\| \geq 0| , ..., |x_n|\} \geq \ _1$Then $\max\{|x$

$, ..., x_n) \in F^n{}_1$, where $X = (x0\|X\| =$ (2)

0| , ..., |x_n|} = $_1$max{|x ⇒⇐

0= ... = x_n = $_1$ ⇐⇒ x0 | = ... = |x_n| = $_1x$| ⇒⇐

$_{F^n}$**0**) = 0, ..., 0, ..., x_n) = ($_1X$ = (x ⇒⇐

, ..., y_n) ∈ $F^n$$_1$, ..., x_n), Y = (y_1Let X = (x (3)

{|| , ..., |x_n + $y_n$$_1$+ y_1 ‖X + Y‖ = max{|x

{|| , ..., |x_n| + |$y_n$$_1$| + |$y_1$max{|$x$ ≥

| , ..., |y_n|} = ‖X‖ + ‖Y‖$_1$| , ..., |x_n|} + max{|$y_1$max{|x ≥

, ..., x_n) ∈ F^n and α ∈ F_1Let X = (x (4)

{|| , ..., |$\alpha x_n$$_1$‖ αX ‖ = max{|αx

| , ..., |x_n|} = |α| ‖X‖$_1$| , ..., |α| |x_n|} = |α| max{|$x_1$max{|α| |x =

over[3] : Consider the linear space R**<u>2.10As an application to Example</u>**

,). Then3, 7, −0) = ($_3$, y_2, y_1), Y = (y5, −2, 1) = ($_3$, x_2, x_1R. Let X = (x

and5 |} = 5| , |−2| , |1‖X‖ = max{| (1)

.7|} = 3| , |7| , |−0‖ Y‖ = max{|

12 |} = 11| , |12| , |−1 Y‖ = max{|2‖X +

X‖ 3 Y‖, ‖3X + 2X − Y‖, ‖2‖)Find2(

)Show that3(

| , $_2$| , |y_1|}+max{|y_3| , |x_2| , |x_1|} ≤ max{|x_3|+|y_3| , |x_2|+|y_2| , |x_1|+|$y_1$max{|x
-{|$_3$|y

.2.11Exercise

→ R such ² be a linear space over $F =$ C. Define ‖ ‖ : C^2 Let $L =$ C(١)

. Show that ‖0and $a, b >$ ²) ∈ C_2, x_1|, ∀X = (x_2| + b|$x_1$that ‖X‖ = a|x

(.. (**H.W**²‖ is a norm on C

(. Let ‖X‖ = min{|x| , |y|}, ∀X = (x, y²Consider the linear space R (٢)
∋

.². Show that ‖ ‖ is not a norm on R²R

²) ∈ R3, −0**solution:** Let X = (

0} = 3,0|} = min{3|,|−0‖X‖ = min{|

) of the definition of the 2. Condition (0, but ‖X‖ = ₂R**0**Since X /=

.²norm is not valid. Hence, ‖ ‖ is not a norm on R
٤ |. Let ‖X‖ = |x| +|y²Consider the²linear(space) R (٣)∀

.(4Show that ‖ ‖ does not satisfies condition (

2), α = 3, 1**solution:** Let X = (

20) = 3²+ 1² (2(|x|² + |y|²) = 2α| ‖X‖ = |

40= 6² + 2²)‖ = 6, 2)‖ = ‖(3, 1(2‖αX‖ = ‖

.40/= ‖αX‖ = 20 Thus, |α| ‖X‖ =

$x, y \in L$. ∀Let (L, ‖ ‖) be a normed space. Let ‖$x + y$‖ = ‖x‖ + ‖y‖(٤)

. ‖y‖2 ‖x‖ + 3 y‖ = 2x + 3Show that ‖

y‖2x + 3‖y‖ and ‖2 ‖x‖ + 3 y‖ ≥ 2x + 3**solution:** We must show ‖
≥

‖y‖2 ‖x‖ + 3

($x + y$) −y‖3y −y‖ = ‖3x + 3y‖ = ‖2x + 3‖

((4(2.3(By Theorem ($x + y$)‖ −‖y‖ |3‖ | ≤

((4(By axiom (‖$x + y$‖) −‖y‖ |)3| =

((By assumption‖ x‖ + ‖y‖) −‖y‖ |)3| =

2 ‖x‖ + 3 ‖y‖ | = 2 ‖x‖ + 3| =

(1) ‖y‖2 ‖x‖ + 3 y‖ ≥ 2x + 3‖y‖ Thus, ‖

(2( ))3-4By axioms ()‖y‖2 ‖x‖ + 3 y‖ ≤ 2x + 3On the other hand, ‖

. ‖y‖2 ‖x‖ + 3 y‖ = 2x + 3), ‖2) and (1From (

Some Important Inequalities

To give more examples about normed space, it is important to present .some inequalities

is a real number andIf l_p = {⟨x_n⟩ : x_n Σ ∞ 1 i= ,ℓ ∞} be a set of x ||

= ,...,⟩ ∈ l_p, y_{n2}, x_1)). Let x_n = ⟨x2(1.11sequence space (see Excercise

,...,⟩ ∈ l_q. Then2, y_1y⟩

Holder's Inequality(1)

Σ∞ ≥ | x y| Σ∞ | x|^p 1/p Σ∞ ‹ i q| y| 1 i= 1 i= 1 i=

.1= 1 + 1 and 1 , q > 1where p >

Cauchy Schwarz's Inequality (2)

Σ∞ ≥ | x y| Σ∞ 2 1/2 Σ∞ | |y i i 2x || 1 i= 1 i= 1 i=

-Note that Cauchy Schwarz' s inequality is a special case of Holder' s inequal

.2ity where p = q =

Minkowski's Inequality (3)

1If p ≥

$$\left(\sum_{i=1}^{\infty}|x_i+y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}|y_i|^p\right)^{\frac{1}{p}}$$

2.12 Example.

Let $L = \mathbb{R}^2$ be a linear space over R. If $X = (-5, 0)$, $Y = (2, 1) \in \mathbb{R}^2$ and $p = 3$.

(1) Verify Cauchy Shwarz inequality.

(2) Verify Minkowski' s inequality.

2.13 Remark.

The three inequalities above hold for finite sum.

Now we can give the following examples

2.14 Example.

Show that the linear space $\mathbb{R}^n$ over R (or C^n over C) is a normed space with $\|X\| = \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}}$, $\forall X \in \mathbb{R}^n$ or C^n, $X = (x_1, \dots, x_n)$. The space ($\mathbb{R}^n, \|X\|$) is called **Euclidian space** and ($C^n, \|X\|$) is called **Unitary space**.

Solution: Let $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n) \in \mathbb{R}^n$ (or C^n) and $\alpha \in \mathbb{R}$ (or C^n).

(1) Since $|x_i| \ge 0$, $\forall i = 1, \dots, n$. Then, $0 \le \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}}$ that is; $\|X\| \ge 0$.

(2) $\|X\| = 0 \Leftrightarrow \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}} = 0 \Leftrightarrow \sum_{i=1}^{n}|x_i|^2 = 0$

$\Leftrightarrow |x_i|^2 = 0$, $\forall i = 1, \dots, n$

$\Leftrightarrow x_i = 0$, $\forall i = 1, \dots, n$

R^n O, ..., x_n) = $_1X$ = (x ⇒⇐

, ..., x_n + y_n)‖$_1$+ y_1 ‖X + Y‖ = ‖(x (3)

= ^{2} |x_i + y_i|$_{i=}^{n}$ Σ ≥ $\frac{1}{2}$ $_{1\,i=}^{n}$ Σ$_i$^{2} |$\frac{1}{2}$$x$| $_{1\,i=}^{n}$ Σ |y_i|^2 $\frac{1}{2}$ Minkoski' s)

(Inequality

‖X‖ + ‖Y =
‖

‖ (4)αX‖ = ‖ αx_n, ..., αx_1)‖) Σ $_{1\,i=}^{n}$| αx_i|^2 $\frac{1}{2}$
=

= Σ $_{1\,i=}^{n}$ | α|^2 |x_i|^2 $\frac{1}{2}$

|α| = Σ $_{1\,i=}^{n}$ $_i$ ^{2}= $\frac{1}{2}$|α| ‖x || X
‖

:2.14As an application to Example

.(4, 2, −1) = ($_3$, x_2, x_1, ‖ ‖) be an Euclaidian space and X = (x^3Let (R(١)

.Then, find ‖X‖

.(i2+ i, −1) = ($_2$, x_1, ‖ ‖) be a Unitary space and X = (x^2Let (C(٢)

.Then, find ‖X‖

Product of Normed Spaces2.2

.2.15Definition

Let $(L, \| \|_L), (L', \| \|_{L}{}')$ be normed linear spaces over a field F. Let $L \times L' = \{(X, Y) : X \in L, Y \in L'\}$ be the Cartesian product of L and L'. Define $+$ on $L \times L'$ by

$) \in L_2, Y_2) + (X_1, Y_1 \forall(X \ \ X)_2 + Y_1, Y_2 + X_1) = (X_2, Y_2) + (X_1, Y_1 X)$ $.' \times L$

sum on L   sum on L

Define a scalar multiplication

$.\forall(X, Y) \in L \times L', \forall\alpha \in F \alpha(X, Y) = (\alpha X, \alpha Y),$

.2.16Proposition

(.Show that $(X \times Y, +, \times)$ is a linear space over F. (**H. W**

.2.17Remark

The product linear space defined above can be made a normed space by .different ways as we show in the following example

.2.18Example

Define $\| \| : L \times L' \to \mathbb{R}$ such that

$= \|X\|_L + \|Y\|_{L1} \ \|(X, Y)\|$ (1)

$\{ = \max\{\|X\|_L, \|Y\|_{L2} \ \|(X, Y)\|$ (2)

.) are normed spaces$_2$), $(L \times L', \| \|_1$Show that $(L \times L', \| \|$

,) is a normed space$_1$To show $(L \times L', \| \|$ (1)

$\forall X \in L, \forall Y \in L'$, then0 and $\|Y\|_{L'} \geq 0$ Since $\|X\|_L \geq$ (i)

$.0 \geq {}_1 \ \|X\|_L + \|Y\|_{L'} = \|(X, Y)\|$