$\|g\|^2 2\|f\|^2 + 2$ By (**I**) and (**II**), we get $\|f+g\|^2 + \|f-g\|^2 \neq$

$4 \neq 5$ i.e.,

.4.17Example

Then the .^{2}) $\in R_2, x_1 \forall X = (x|_2| + |x_1$and let $\|X\| = |x^2$ Let $L = R$

,.. i.e4.13converse of Theorem

,$\|\ \|$) is a normed linear space (**H.W.**) ^{2}Show that (R(1)

(is not I.P.S^2 is not generated by I.P.S (i.e, R^2 **R**)Show that2(

) To show that L is not I.P.S, we shall show that parallelogram 2**<u>Solution</u>:** (

for $^2 \|Y\|2 + {}^2 \|X\|2 \neq {}^2 + \|X - Y\|^2$ law does not hold. i.e., $\|X + Y\|$

^{2}some $X, Y \in R$

(1, 6) and $Y = (-3, 2$Let $X = ($

$50\|X\|^2 = 2 \Longrightarrow 5\ | = 3| + |2\|X\| = |$

$98\|Y\|^2 = 2 \Longrightarrow 7\ | = 1| + |6\|Y\| = |-$

$8| = 4| + |4)\| = |-4, 4\|X + Y\| = \|(-$

$64\|X + Y\|^2 =$

$10| = 2| + |8)\| = |2, 8\|X - Y\| = \|($

$100\|X - Y\|^2 =$

$= 100 + 64 = {}^2 + \|X - Y\|^2$ Thus, $\|X + Y\|$

$148 = 98 + 50 = {}^2 \|Y\|2 + {}^2 \|X\|2$ and 164

$\|Y2 + {}^2 \|X\|2 \neq {}^2 + \|X - Y\|^2$ Hence, $\|X + Y\|$

.i.e., $\|\ \|$ does not satisfy paralleogram law$^2\|$

.4.18Example

. Then^2) $\in$ R$_2$, $x_1\forall(x|\}_2|$, $|x_1$and let $\|X\| = \max\{|x^2$ Let $L =$ R

, $\| \|$) is a normed linear space (**H.W.**) ^{2}Show that (R(1)

(.generated by I.P.S? (**H.W**^{2} R)Is2 (

4.19 Theorem

Let $(L, \langle\, ,\, \rangle)$ is an I.P.S. Then

(1) If $(x_n) \to x$ and $(y_n) \to y$ then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

(2) If (x_n) and (y_n) are Cauchy sequences in L then $\langle x_n, y_n \rangle$ is a Cauchy sequence in F.

Proof. (1)

$$\langle x_n, y_n \rangle = \langle x + (x_n - x), y + (y_n - y) \rangle$$

$$= \langle x, y \rangle + \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle$$

$$\langle x_n, y_n \rangle - \langle x, y \rangle = \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle$$

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle|$$

$$\le |\langle x, y_n - y \rangle| + |\langle x_n - x, y \rangle| + |\langle x_n - x, y_n - y \rangle|$$

$$\le \|x\|\,\|y_n - y\| + \|x_n - x\|\,\|y\| + \|x_n - x\|\,\|y_n - y\| \quad \text{(By Cauchy Schwarz)}$$

But $(x_n) \to x$ and $(y_n) \to y$ then $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$

Hence, $|\langle x_n, y_n \rangle - \langle x, y \rangle| \to 0$, and hence, $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

(2) for any $n, m \in Z_+$

$$\langle x_n, y_n \rangle = \langle (x_n - x_m) + x_m, (y_n - y_m) + y_m \rangle$$

$$= \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_n - x_m, y_m \rangle$$

$$\langle x_n, y_n \rangle - \langle x_m, y_m \rangle = \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle$$

$$|\langle x_n, y_n \rangle - \langle x_m, y_m \rangle| = |\langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle|$$

$$\le |\langle x_n - x_m, y_n - y_m \rangle| + |\langle x_m, y_n - y_m \rangle| + |\langle x_m, y_n - y_m \rangle|$$

By) $\|x_n - x_m\| \, \|y_n - y_m\| + \|x_m\| \, \|y_n - y_m\| + \|x_n - x_m\| \, \|y_m\| \geq$

(Cauchy Schwarz

$\to$ and $\|y_n - y_m\|$ 0 But (x_n) and (y_n) are Cauchy sequences, then $\|x_n - x_m\| \to$

$\infty \to$ as $n \to \infty$. Also, (x_n) and (y_n) are bounded sequences, then as n 0

0 x_n, y_n − x_m, y_m □ $\to$.

4.20 Corollary

Let $(L, \langle\,,\,\rangle)$ is an I.P.S. Then

If $(x_n) \to x$ then $\|x_n\| \to \|x\|$ (١)

If (x_n) is a Cauchy sequences in L then $\|x_n\|$ is a convergent se- (٢)

.quence in R

(4.19By Theorem)) Since $(x_n) \to x$ then $x_n, x_n \to x, x$ 1 *Proof.* (

Hence, $\|x_n\|^2 \to \|x\|^2$, i.e., $\|x_n\| \to \|x\|$

,(2(4.19Since (x_n) is a Cauchy sequences in L, then by Theorem (2)

$_n$ is a Cauchy sequence in F. Since $F =$ R or C then F is com x , x

is a convergent sequence in F. Thus, $\|x_n\|^2$ is a $\|x\|$ plete. Thus,

convergent sequence in F □

4.3 Hilbert Space

4.21 Definition.
Hilbert space is an I.P.S. $(L, \langle\, ,\, \rangle)$ which is a Banach space with respect to $\|x\| = \sqrt{\langle x, x \rangle}$.

4.22 Example.
Consider the I.P.S. $(\mathbb{R}^n, \langle\, ,\, \rangle)$ (or $(C^n, \langle\, ,\, \rangle)$) such that $\langle X, Y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ where $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n) \in \mathbb{R}^n$ (or C^n). (see Example 4.6)

Show that $(\mathbb{R}^n, \langle\, ,\, \rangle)$ (or $(C^n, \langle\, ,\, \rangle)$) is a Hilbert space.

Solution: Since $\langle X, X \rangle = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2 = \|X\|^2$,

From Example 3.2, $\mathbb{R}^n$ (or C^n) is a Banach space w.r.t. $\|X\| = \sqrt{\langle X, X \rangle}$, and thus, $(\mathbb{R}^n, \langle\, ,\, \rangle)$ (or $(C^n, \langle\, ,\, \rangle)$) is a Hilbert space.

4.23 Example.
The space $C[-1, 1]$ with the inner product defined by $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$ is not a Hilbert space.

Solution: Let

$$f_n(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ nx & \text{if } 0 < x < \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

$$\|f_n - f_m\|^2 = \langle f_n - f_m, f_n - f_m \rangle$$

(We must find $f_n(x) - f_m(x)$.) Suppose $n > m$, then $\frac{1}{m} > \frac{1}{n}$.

$$f_n(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ nx & \text{if } 0 < x < \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

and

$$f_m(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ mx & \text{if } 0 < x < \frac{1}{m} \\ 1 & \text{if } \frac{1}{m} \le x \le 1 \end{cases}$$

Then

$$(f_n(x) - f_m(x)) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ (n-m)x & \text{if } 0 < x < \frac{1}{n} \\ 1 - mx & \text{if } \frac{1}{n} \ge x \ge \frac{1}{m} \\ 0 & \text{if } \frac{1}{m} \le x \le 1 \end{cases}$$

$$\|f_n - f_m\|^2 = \int_{-1}^{1} (f_n(x) - f_m(x))^2 dx = \int_0^{\frac{1}{n}} ((n-m)x)^2 dx + \int_{\frac{1}{n}}^{\frac{1}{m}} (mx - 1)^2 dx$$

$$= \left[\frac{(n-m)^2 x^3}{3}\right]_0^{\frac{1}{n}} + \left[\frac{(mx-1)^3}{3m}\right]_{\frac{1}{n}}^{\frac{1}{m}}$$

$$= \frac{1}{3}(n-m)^2 \frac{1}{n^3} + \frac{1}{3m}\left[0 - \left(\frac{m}{n} - 1\right)^3\right]$$

$$= \frac{n-m}{3n3} \frac{1}{n} \frac{2n-m))}{m3}(3$$

$$= \frac{2n-m))}{m^2 n3}$$

= Thus, $\|f_n - f_m\|^2 \frac{2n-m))}{m^2 n3}$

Since $n > m$, then $n = m + t$

$\|f_n - f_m\|^2 \quad \frac{2t2 \quad 0 \to}{m^2 m + t))3} = \quad \infty \to$ as m

.. Thus, $\langle f_n \rangle$ is a Cauchy sequence0Hence, $\|f_n - f_m\| \to$

But $f_n \to f$ where

$$(f(x = \begin{cases} 0 & \text{if } 0x \leq \geq 1- \\ 1 & \text{if } 1x \leq > 0 \end{cases}$$

], i.e., The1,1]. Then, $\langle f_n \rangle$ is not convergent in $C[-1,1$Thus, $f \notin C[-$

.space is not Hilbert space

.4.24Remark

Every Hilbert space is a Banach space but the converse is not true. For example, the space $C[a, b]$ with $\|f\| = \max\{|f(x)| : x \in [a, b]\}$ is a Banach). However, $C[a, b]$ is not a Hilbert space since it 3.5space (see Example does not satisfy parallelogram law; that is $\| \ \|$ can not be obtained from

.(4.16inner product (see Example

-Orthogonality and Orthonormality in Inner Prod4.4

uct Space

. *orthogonal Elements*4.25Definition

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S and $x, y \in L$. Then x is said to be **orthogonal**

.0on y (denoted by $x \perp y$) if and only if $\langle x, y\rangle =$

.4.26Example

is usual inner product$_2$ y_2+ x_1 $y_1$is I.P.S such that $\langle X, Y\rangle =$ x^2 Let $L =$ R

.(2, 1), $Z =$ (1, −2), $Y =$ (3, 6. Let $X =$ $(-^2) \in$ R$_2$, y_1), $Y =$ (y_2, x_1 $X =$ (x∀

.Show that $X \perp Z$, $Y \perp Z$ and $Y \not\perp X$

.. Hence, $X \perp Z$0= 6 + 6)) = −2, 1), (3, 6**<u>Solution:</u>** $\langle X, Z\rangle =$ ⟨(−

$= \langle Y, Z\rangle$

$= \langle Y, X\rangle$

.4.27Proposition

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S and $x, y \in L$. Then

.$x \perp y$ then $y \perp x$If (i)

$_L\perp$ x $\forall x \in L$. **(H.W.)** **o(ii)**

(.$_L$ **(H.Wo**$x \perp x$ then $x =$ (iii)if

), we have3(4.1From Definition .0) Let $x \perp y$ then $\langle x, y\rangle =$ 1*Proof.* (

.. i.e., $y \perp x$0$y, x\rangle = \langle x, y\rangle =$ □

4.28 Proposition.

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S and $x, x_1, \ldots, x_n \in L$ such that x is orthogonal on $x_1, \ldots, x_n$. Prove that x is orthogonal on any linear combination of $x_1, \ldots, x_n$.

Proof. Let w be a linear combination of $x_1, \ldots, x_n$, i.e., there exists $\alpha_i \in F$ such that $w = \sum_{i=1}^{n} \alpha_i x_i$. We must show $\langle x, w\rangle = 0$.

$$\langle x, w\rangle = \langle x, \sum_{i=1}^{n} \alpha_i x_i\rangle = \sum_{i=1}^{n} \bar{\alpha}_i \langle x, x_i\rangle \quad \text{(by Corollary 4.9(ii))}$$

$$= \sum_{i=1}^{n} \bar{\alpha}_i 0 \quad \text{(From the assumption)}$$

$$= 0. \qquad \square$$

4.29 Example.

(1) Find the value of a that makes the vectors $X = (a, 1, -2)$, $Y = (2, 6, -3)$ orthogonal vectors in $\mathbb{R}^3$ with usual inner product. (**H.W.**)

(2) Let $(L, \langle\ ,\ \rangle)$ be an I.P.S over $\mathbb{R}$ and let $x, y \in L$ such that $\|x\| = \|y\| = 1$ (i.e., x and y are normal elements). Prove that $x + y \perp x - y$.

Answer: $\langle x + y, x - y\rangle = \langle x, x\rangle - \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle = \|x\|^2 - \langle x, y\rangle + \langle x, y\rangle - \|y\|^2 = 0$. Hence, $x + y \perp x - y$.

(3) Let $(L, \langle\ ,\ \rangle)$ be an I.P.S and let $x, y \in L$ such that $x \perp y$. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = \|x - y\|^2$.

Answer: $\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + 0 + 0 + \|y\|^2 = \|x\|^2 + \|y\|^2$.

Similarly, $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

(4) Let $(L, \langle\ ,\ \rangle)$ be an I.P.S and let $x, y \in L$ such that $x \perp y$. Prove that

$\|x + \lambda y\| = \|x - \lambda y\|$.

Answer: (H.W.)

(5) Let $(L, \langle\ ,\ \rangle)$ be an I.P.S and let $x_1, x_2, \ldots, x_n \in X$ such that $x_i \perp x_j \ \forall i \neq j$. Prove that $\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$.

Answer: We prove using induction. If $n = 1$, the statement is true.

If $n = 2$. Since $x_1 \perp x_2$ then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ (by part (3)).

Suppose the statement is true for $n = k$, i.e., $\left\| \sum_{i=1}^{k} x_i \right\|^2 = \sum_{i=1}^{k} \|x_i\|^2$.

To prove the statement is true when $n = k + 1$, i.e.,

T.P. $\left\| \sum_{i=1}^{k+1} x_i \right\|^2 = \sum_{i=1}^{k+1} \|x_i\|^2$.

$$\left\| \sum_{i=1}^{k+1} x_i \right\|^2 = \left\| \sum_{i=1}^{k} x_i + x_{k+1} \right\|^2 = \left\| \sum_{i=1}^{k} x_i \right\|^2 + \|x_{k+1}\|^2$$

$$= \sum_{i=1}^{k} \|x_i\|^2 + \|x_{k+1}\|^2 \quad \text{(by induction } n = k\text{)}$$

$$= \sum_{i=1}^{k+1} \|x_i\|^2.$$

4.30 Definition. ***Orthogonal to Set***

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S, $x \in L$, and $A \subseteq X$. Then, x is said to be orthogonal on A $(x \perp A)$ if $x \perp a \ \forall a \in A$.

4.31 Example.

Consider the space $\mathbb{R}^2$ with usual product space and $A = \{(a, 0) : a \in \mathbb{R}\}$.

Then $(0, 2) \perp A$ because $\langle (0, 2), (a, 0) \rangle = 0.a + 2.0 = 0$.

. *Orthogonal Sets*4.32Definition

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S, and $A, B \subseteq L$. Then, A is said to be orthogonal

$, \forall a \in A, \forall b \in B$ to B $(A \perp B)$ if $a \perp b$,

.4.33Example

$\{, a) : a \in$ R0with usual inner product and $A = \{($^{2} Consider the space R

$.) : b \in$ R$\}$. Show that $A \perp B$0and $B = \{(b,$

$) \in B$, then0$, a) \in A$ and for each $(b,$0**Answer:** for each (

.. Thus, $A \perp B$0$.b =$ 0+ 0 $, b)\rangle =$ a.0$), (0a,)\rangle$

.4.34Proposition

$.\{$**o**Let $(L, \langle\ ,\ \rangle)$ be an I.P.S, and $A, B \subseteq L$ such that $A \perp B$ then $A \cap B = \{$

Proof. Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$ (**I**)

$\forall a \in A, \forall b \in B.$,0Since $A \perp B \Rightarrow \langle a, b\rangle =$

.0From (**I**), $\langle a, b\rangle = \langle x, x\rangle =$

$.\{$**o**, then $A \cap B = \{$**o**$\}$, $x =$ 2(4.1Using Definition ☐

.4.35Definition

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $\varphi \neq A \subseteq L$. Then, the set

$\{\forall a \in A A^{\perp} = \{x \in L : x \perp a,$

.is called the orthogonal complement on A

.4.36Proposition

,Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $\varphi \neq A, B \subseteq L$. Then

.{**o**$L^{\perp}$ = { (1)

(.L. (**H.W** = $^{\perp}${**o**} (2)

.{**o**$A \cap A^{\perp}$ = { (3)

.$^{\perp\perp}A \subseteq A$ (4)

$A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$. (**H.W.**) If(5)

.$^{\perp}A \subseteq B^{\perp}$ then $B \subseteq A$)If6(

.{**o**, $\forall l \in L\}$ = {0⟩ $L^{\perp} = \{x \in L : x \perp L\} = \{x \in L : \langle x, l\rangle =$ 1*Proof.* (

((I Let $x \in A \cap A^{\perp} \Rightarrow x \in A$ and $x \in A^{\perp}$ (٢)

(II) Since $x \in A^{\perp}$ then $x \perp A$

, thus $x =$ 0From **(I)** and **(II)**, $x \perp x$. i.e., $\langle x, x\rangle =$

.{**o**. Then, $A \cap A^{\perp}$ = {**o**

.To prove $A \subseteq A^{\perp\perp}$. Let $x \in A$ (٤)

For any $y \in A^{\perp} \Rightarrow y \perp A$. In particular, $y \perp x$ $(x \in A)$

.$^{\perp\perp}\forall y \in A^{\perp}$. Thus, $x \in A$), $x \perp y$,1(4.27From Proposition

), $B^{\perp\perp} \subseteq A^{\perp}$. 5Let $A \subseteq B^{\perp}$, then from part ((6)

), $B \subseteq B^{\perp\perp} \subseteq A^{\perp}$. Then, $B \subseteq$ 4Now, from part (□

.$^{\perp}A$

.4.37Theorem

$A \subseteq L$. Then, $A^{\perp}$ is a closed subspaceLet $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $\varphi \neq$

.of L

.) To prove $A^{\perp}$ is a subspace of L1*Proof.* (

$^{\perp}$Let $x, y \in A^{\perp}$ and $\alpha, \beta \in F$. T.P. $\alpha x + \beta y \in A$

.$\forall a \in A$,0T.P. $\langle \alpha x + \beta y, a \rangle =$

(**I**) 0Since $x, y \in A^{\perp} \Rightarrow \langle x, a \rangle = \langle y, b \rangle =$

$0 = 0 + \beta.0\ \alpha x + \overline{\beta} y, a \rangle = \alpha \langle x, \overline{a} \rangle + \beta \langle y, a \rangle =$[(from (*I*)

.Thus, $A^{\perp}$ is a subspace of L

T.P. $A^{\perp}$ is a closed set (i.e., $A^{\perp} \subseteq \overline{A^{\perp}}$ and $\overline{A^{\perp}} \subseteq$ (2)
($^{\perp}A$

((**I** It is clear that $A^{\perp} \subseteq A^{\perp}$

T.P. $\overline{A^{\perp}} \subseteq A^{\perp}$. Let $x \in \overline{A^{\perp}}$ then $\exists (x_n) \in A^{\perp}$ such that $(x_n) \rightarrow$
$.x$

$,^{\perp}$Since $(x_n) \in A$ $\quad \forall a \in A\, n \in N \Rightarrow x_n \perp A \Rightarrow x_n \perp a, \forall$

.$\forall a \in A$,0$x_n, a \rangle = \rangle \Rightarrow$

.(), $\langle x_n, a \rangle \rightarrow \langle x, a$1(4.19But $(x_n) \rightarrow x$ and $a \rightarrow a$. Thus, from Theorem
0=

$\forall a \in A$. Then, $x \in A^{\perp}$. Thus, $A^{\perp} \subseteq$ 0 $\overline{x}, a \rangle = \rangle$.(**II**)
$^{\perp}A$

.From (**I**) and (**II**), $A^{\perp}$ is a closed set

□

. *Orthonormal Set*4.38Definition

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $A \subseteq L$. Then, A is called **orthonormal set**

if

.$x \neq y$ A is said to be **orthogonal** if $x \perp y$ $\forall x, y \in A$, (١)

$\forall x$1 $= \|x\| =$ ½ Each element $x \in A$ is a normal element. i.e., $\langle x, x \rangle$ (٢)
∋

4.39 Remark

1/ Orthonarmal set has no zero element (is not normal element) because $\|0\| \neq 1$, ($0 \notin A$).

4.40 Example

Let $L = \mathbb{R}^3$ with usual inner product and $A = \{(1,2,-2),(2,-1,2),(2,2,1)\} \subset L$. Show that A is orthogonal but not orthonormal.

Solution: T.P. A is orthogonal set **(H.W.)**.

To show not every vector in A is normal, i.e.,

$$\|(2,2,1)\|^2 = \langle (2,2,1),(2,2,1)\rangle = 4+4+1 = 9 \neq 1 \Rightarrow \|(2,2,1)\| \neq 1.$$

Thus, A is not orthonormal.

4.41 Theorem

Let L be an I.P.S. and $x_1, \dots, x_n$ be orthonormal vectors in L. Then

$$\sum_{i=1}^{n} |\langle x, x_i\rangle|^2 \le \|x\|^2 \quad \forall x \in L.$$

4.42 Example

Let $L = \mathbb{R}^3$, $X_1 = \frac{1}{3}(1,2,-2)$, $X_2 = \frac{1}{3}(2,-1,2)$ and $X_3 = \frac{1}{3}(2,2,1)$. Let $X = (3,1,2)$. Then

$$|\langle X, X_3\rangle|^2 = [\tfrac{1}{3}(6+2+2)]^2 = \frac{100}{9}$$

$$|\langle X, X_2\rangle|^2 = [\tfrac{1}{3}(6-1+4)]^2 = \frac{1}{9}$$

$$|\langle X, X_1\rangle|^2 = [\tfrac{1}{3}(3+2-4)]^2 = \frac{25}{9}$$

$$\sum_{i=1}^{3} |\langle X, X_i\rangle|^2 = \frac{25}{9} + \frac{1}{9} + \frac{100}{9} = 14.$$

on the other hand, $\|X\|^2 = \langle X, X\rangle = 4 + 1 + 9 = 14$.

As in Theorem 4.41, $\|X\|^2 = |\langle X, X\rangle| = \sum_{i=1}^{3} \ldots$

Take $X = (1, 1, 1)$ and apply Theorem 4.41 (H.W.).

Theorem 4.43.

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. Let (x_n) be an orthonormal sequence in L and (λ_n) be a sequence in F such that $\sum_{i=1}^{+\infty} |\lambda_i|^2 < +\infty$. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$. Then, (y_n) is a Cauchy sequence.

Proof. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$, $y_m = \sum_{i=1}^{m} \lambda_i x_i$. Assume that $n < m$ then $m = n + k$ for some $k \in N$. We must prove $\|y_m - y_n\| \to 0$.

$$y_m - y_n = \sum_{i=1}^{m} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n+k} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=n+1}^{n+k} \lambda_i x_i$$

$$\begin{aligned}
\|y_m - y_n\|^2 &= \left\langle \sum_{i=n+1}^{n+k} \lambda_i x_i, \sum_{i=n+1}^{n+k} \lambda_i x_i \right\rangle \\
&= \sum_{i=n+1}^{n+k} \lambda_i \sum_{i=n+1}^{n+k} \overline{\lambda_i} \langle x_i, x_i \rangle \\
&= \sum_{i=n+1}^{n+k} \lambda_i \overline{\lambda_i} \langle x_i, x_i \rangle \\
&= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \|x_i\|^2 \\
&= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \qquad (\forall i\ \|x_i\|^2 = 1)
\end{aligned}$$

As $n \to +\infty$, $\sum_{i=n+1}^{n+k} |\lambda_i|^2 \to 0$ ($\sum_{i=1}^{+\infty} |\lambda_i|^2$ convergent).

Thus, $\|y_m - y_n\|^2 \to 0$ which means $\|y_m - y_n\| \to 0$. Hence, (y_n) is a Cauchy sequence. □

جامعة ديالى
كلية التربية المقداد
قسم رياضيات – المرحلة الرابعة

٢٠٢٠ - ٢٠٢١

ملخص
محاضرات التحليل الدالي

أعداد أستاذة المادة
م. م إيناس حسن عبد كاظم

التحليل الدالي

Functional Analysis

EX.: let $L = C[0,1]$, $< , > : L \times L \to R$

$$<f,g> = \int_0^1 f(t)g(t)\,dt$$

show that $< , >$ is I.P.S. on L.

Solution:

① $<f,f> = \int_0^1 f(t).f(t)\,dt = \int_0^1 (f(t))^2 dt \geq 0$

② $<f,f> = 0 \Leftrightarrow \int_0^1 (f(t))^2 dt \Leftrightarrow (f(t))^2 = 0$

$\Leftrightarrow f(t) = 0 \quad \forall\, t \in [0,1]$. ($f = \hat{O}$ دالة صفرية)

③ let $\alpha, \beta \in R$, $f, g, h \in L$

$$<\alpha f+\beta g, h> = \int_0^1 (\alpha f+\beta g)(t)\, h(t)\,dt$$

$$= \int_0^1 (\alpha f(t)+\beta g(t))\, h(t)\,dt$$

$$= \int_0^1 [\alpha f(t).h(t)]\,dt + [\beta g(t).h(t)]\,dt$$

$$= \alpha \int_0^1 f(t).h(t)\,dt + \beta \int_0^1 g(t).h(t)\,dt$$

$$= \alpha <f,h> + \beta <g,h>$$

$<f,g> = <g,f>$ H.w.

H.w: let $f(x) = x+1$, $g(x) = x^2$, $h(x) = 3x+2$ $\forall x \in [0,1]$.

Find: $\langle f,f \rangle$, $\langle f+g, h \rangle$, $\langle f,h \rangle$, $\langle 2f+3g, h \rangle$.

Theorem: Every inner product space is a normed space and hence a metric space.

Proof: let $(L, \langle , \rangle)$ is an I.P.S. and let $\| \ \| : L \longrightarrow \mathbb{R}$ is defined by

$$\|x\| = \sqrt{\langle x,x \rangle} \quad \forall x \in L$$

① since $\langle x,x \rangle \geq 0 \ \forall x \in L \Rightarrow$

$$\|x\| = \sqrt{\langle x,x \rangle} \geq 0 \quad \forall x \in L$$

② $\|x\| = 0 \Leftrightarrow \sqrt{\langle x,x \rangle} = 0 \Leftrightarrow \langle x,x \rangle = 0$

$\Leftrightarrow x = 0_x$ $\boxed{x = 0_X}$

③ let $\alpha \in F$ and $x \in L$

$$\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle = \alpha \bar{\alpha} \langle x,x \rangle = |\alpha|^2 \|x\|^2$$

Thus: $\|\alpha x\| = |\alpha| \, \|x\|$

EX. : $X = \overset{x_1\ x_2}{(2, 1)}$, $Y = \overset{y_1\ \ y_2}{(0, -3)}$, $Z = \overset{z_1\ z_2}{(3, 4)}$

Find $\langle X, Z \rangle$, $\langle X, X \rangle$, $\langle X+Y, Z \rangle$

تعريف I.P.S حسب المثال السابق.

Solution :

$$\langle X, Z \rangle = x_1 z_1 + x_2 z_2 = 2.3 + 1.4$$

$$= 6 + 4 = 10$$

$$\langle X, X \rangle = x_1 . x_1 + x_2 . x_2 = x_1^2 + x_2^2 = (2)^2 + (1)^2$$

$$= 4 + 1 = 5$$

$$\langle X+Y, Z \rangle = ?$$

$$X = (2, 1) \ , \ Y = (0, -3) \Rightarrow X+Y = x_1 y_1 + x_2 y_2$$

$$= 2.0 + 1.{}^{-}3$$

$$= -3$$

H.w : $\langle 2X + 3Y, Z \rangle$

EX.: $L = C^2$ and

$$\langle X, Y \rangle = \sum_{i=1}^{2} x_i \bar{y}_i \quad \forall X, Y \in C^2$$

where: $X = (x_1, x_2)$, $Y = (y_1, y_2)$

If $X = (2+3i, 1+i)$, $Y = (1+i, 1-i)$

$Z = (2, 1+i)$

Find: $\langle X, X \rangle$, $\langle X+Y, Z \rangle$, $\langle X, Y+Z \rangle$

Solution:

$$\langle X, X \rangle = \sum_{i=1}^{2} x_i \bar{x}_i = x_1 \bar{x}_1 + x_2 \bar{x}_2$$

$$= (2+3i)(2-3i) + (1+i)(1-i)$$

$$= (2)^2 + (3)^2 + (1)^2 + (1)^2$$

$$= 4 + 9 + 1 + 1 = 15$$

$$\langle X+Y, Z \rangle = ?$$

$$X = (x_1, x_2) = (2+3i, 1+i)$$

$$Y = (y_1, y_2) = (1+i, 1-i)$$

$$X+Y = \big((2+3i) + (1+i), (1+i) + (1-i)\big)$$

$$= (3+4i, 2) = (r_1, r_2) \qquad r_1 = 3+4i, \; r_2 = 2$$

$$\therefore \langle X+Y, Z \rangle = \sum_{i=1}^{2} r_i \bar{z}_i = r_1 \bar{z}_1 + r_2 \bar{z}_2$$

$$= (3+4i)(2) + (2)(1-i)$$

H.W $\langle X, Y+Z \rangle$

Inner Product Space :

Let L is a linear space over F.

$< , > : L \times L \longrightarrow F$ is called an inner product on L if

(1) $<x,x> \geq 0 \ \forall x \in L$.

(2) $<x,x> = 0 \Leftrightarrow x = 0$

(3) $\overline{<x,y>} = <y,x> \ \forall x,y \in L$, where,

$\overline{<x,y>}$ = conjugate of $<x,y>$

(4) $<\alpha x + \beta y, z> = \alpha <x,z> + \beta <y,z> \ \forall x,y,z \in L$

$\therefore (L, < , >)$ is called inner product space or Pre-Hilbert space. (I.P.S.)

Examples: let $L = \mathbb{R}^2$, and let $X = (x_1, x_2)$, $Y = (y_1, y_2)$

$< , > : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow F$ is defined as $Z = (z_1, z_2)$

$$<X,Y> = x_1 y_1 + x_2 y_2 \ \forall X, Y \in \mathbb{R}^2.$$

$X = (x_1, x_2)$, $Y = (y_1, y_2)$

show that $< , >$ is I.P.S. ?

Solution : $X = \langle x_1, x_2 \rangle$ $\quad X = (x_1, x_2)$

الشرط الأول : $\langle X, X \rangle = x_1 \cdot x_1 + x_2 \cdot x_2 = x_1^2 + x_2^2 \geq 0$

الشرط الثاني : $\langle X, X \rangle = 0 \Leftrightarrow x_1^2 + x_2^2 = 0 \Leftrightarrow$

$x_1 = x_2 = 0 \Leftrightarrow X = (0, 0).$

الشرط الثالث : $\langle X, Y \rangle = x_1 y_1 + x_2 y_2 = y_1 x_1 + y_2 x_2 = \overline{\langle X, Y \rangle}$

الشرط الرابع : $\langle \alpha X + \beta Y, Z \rangle \overset{?}{=} \alpha \langle X, Z \rangle + \beta \langle Y, Z \rangle$

$X = (x_1, x_2) \Rightarrow \alpha X = \alpha (x_1, x_2) = (\alpha x_1, \alpha x_2)$

$Y = (y_1, y_2) \Rightarrow \beta Y = \beta (y_1, y_2) = (\beta y_1, \beta y_2)$

$Z = (z_1, z_2)$

$\therefore \alpha X + \beta Y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$

$$\begin{aligned}
\therefore \langle \alpha X + \beta Y, Z \rangle &= (\alpha x_1 + \beta y_1) z_1 + (\alpha x_2 + \beta y_2) z_2 \\
&= \alpha x_1 z_1 + \beta y_1 z_1 + \alpha x_2 z_2 + \beta y_2 z_2 \\
&= (\alpha x_1 z_1 + \alpha x_2 z_2) + (\beta y_1 z_1 + \beta y_2 z_2) \\
&= \alpha (x_1 z_1 + x_2 z_2) + \beta (y_1 z_1 + y_2 z_2) \\
&= \alpha \langle X, Z \rangle + \beta \langle Y, Z \rangle
\end{aligned}$$

(4) $\|x+y\| \leq \|x\| + \|y\|$?

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x,x \rangle + \langle y,x \rangle + \langle x,y \rangle + \langle y,y \rangle$$

$$= \|x\|^2 + \overline{\langle x,y \rangle} + \langle x,y \rangle + \|y\|^2$$

$$= \|x\|^2 + 2\operatorname{Re}\langle x,y \rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle x,y \rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{(By Cauchy Schwarz)}$$

$$= (\|x\| + \|y\|)^2$$

$\therefore \|x+y\| \leq \|x\| + \|y\|$.

محاضرة تحليل دالي

Product of Normed Space

Def/ Let $(L, \| \|_L)$, $(L', \| \|)$ be a normed space over field F and $L \times L' = \{(x, y), x \in L, y \in L'\}$ is said to be cartesian product of L and L'

Define + on $L \times L'$ by تعريف عملية الجمع

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$\forall (x, y_1) + (x_2, y_2) \in L \times L'$$

Define scalar multiplication

$$\alpha(x, y) = (\alpha x, \alpha y) \quad \forall (\alpha x, \alpha y) \in L \times L'$$
$$\forall \alpha \in F$$

Ex // 2.18

Define $\| \| : L \times L' \longrightarrow \mathbb{R}$ s.t

1) $\|(x, y)\|_1 = \|x\|_L + \|y\|_{L'}$

2) $\|(x,y)\|_2 = \max\{\|x\|_L , \|y\|_{L'}\}$

Show that $(L\times L', \|\ \|_1)$, $(L\times L', \|\ \|_2)$ are normed space ?

Sol/

1) To show $(L\times L', \|\ \|_1)$ is normd space

i) since $\|x\|_L \geq 0$ and $\|y\| \geq 0$ $\forall x\in L, \forall y\in L'$

هنا تحقق شروط (normed) حسب التعريف المعطى في المثال

then

$$\|x\|_L + \|y\|_{L'} = \|(x,y)\| \geq 0$$

ii) $\|(x,y)\|_1 = 0 \Leftrightarrow \|x\|_L + \|y\|_{L'} = 0$

$\Leftrightarrow x = y = 0$, $(L, \|\ \|_L)$, $(L', \|\ \|_{L'})$ are anormed space

iii) For each $(x_1,y_1) . (x_2,y_2) \in L\times L'$

$$\begin{aligned}
\|(x_1,y_1)+(x_2,y_2)\|_1 &= \|(x_1+x_2, y_1+y_2)\| \\
&= \|x_1+x_2\|_L + \|y_1+y_2\|_{L'} \\
&\leq \|x_1\|_L + \|x_2\|_L + \|y_1\|_{L'} + \|y_2\|_{L'} \\
&= (\|x_1\|_L + \|y_1\|_{L'}) + (\|x_2\|_L + \|y_2\|_{L'}
\end{aligned}$$

$\left((L, \| \|_{L'})\right), \left((L, \| \|_{L'})\right)$ are normed space

$$\Leftrightarrow (x, y) = (0, 0)$$

iii) For each $(x_1, y_1), (x_2, y_2) \in L \times L'$

$$\|(x_1, y_1) + (x_2, y_2)\|_2 = \|(x_1 + x_2, y_1 + y_2)\|_2$$

$$= \max\{\|x_1 + x_2\|_L, \|y_1 + y_2\|_{L'}\}$$

$$\leq \max\{\|x_1\|, \|y_1\|_{L'}\} +$$

$$\max\{\|x_2\|_L, \|y_2\|_{L'}\}$$

$$= \|(x_1, y_1)\|_2 + \|(x_2, y_2)\|_2$$

IV) For each $(x, y) \in L \times L'$ and $\forall \alpha \in F$

$$\|\alpha(x, y)\|_2 = \|(\alpha x, \alpha y)\| = \max\{\|\alpha x\|_L, \|\alpha y\|_{L'}\}$$

$$= \max\{|\alpha| \|x\|_L, |\alpha| \|y\|_{L'}\}$$

$$= |\alpha| \max\{\|x\|_L, \|y\|_{L'}\}$$

$$= |\alpha| \|(x, y)\|$$

$$= \|(x_1, y_1)\|_L + \|(x_2, y_2)\|_{L'}$$

iv) For each $(x,y) \in X \times Y$ and for each $\alpha \in F$

$$\|\alpha(x,y)\| = \|\alpha x\|_L + \|\alpha y\|_{L'}$$

$$= |\alpha|\|x\|_L + |\alpha|\,\|y\|_{L'}$$

$$= |\alpha|\,\|(x,y)\|$$

تحققت جميع شروط Normed في هذا المثال, لنقطة رقم (1)

الآن نحقق الشرط الثاني من المثال (2) ونحقق عليه شروط (Normed).

2- Now, We show that $\|\,\|_2$ is norm on $L \times L'$

((Norm شروط))

i) Since $\|x\|_L \geq 0$ and $\|y\|_{L'} \geq 0$ $\forall x \in L$ $\forall x \in L$ and $y \in L'$ then

$$\max\{\|x\|_L, \|y\|_{L'}\} = \|(x,y)\|_2 \geq 0$$

ii) $\|(x,y)\|_2 = 0 \Leftrightarrow \max\{\|x\|_L, \|y\|_{L'}\}$

$$\Leftrightarrow \|x\|_L = \|y\|_{L'} = 0$$

$$\Leftrightarrow x = y = 0$$

مثال تطبيقي عن مثال 2.18

Ex //

Let $L = (\mathbb{R}, \| \|_{\mathbb{R}})$ and $L' = (\mathbb{R}^2, \| \|_{\mathbb{R}^2})$

where $\| \|_{\mathbb{R}^2}$ is Euclaidian norm if $x = 3 \in L \in \mathbb{R}$

$y = (1,-2) \in \mathbb{R}^2 \in L'$. Find $\|(x, y)\|_1$ and

$\|(x, y)\|_2$?

Solution

$$\|(x, y)\|_1 = \|x\| + \|y\|_2$$

$$\|(3, (1,-2))\| = \|3\|_{\mathbb{R}} + \|(1,-2)\|_{\mathbb{R}^2}$$

$$\|3\| = \sqrt{3^2 + 0} = \sqrt{3^2} = |3| = 3$$

$$\|(1,-2)\| = \sqrt{1^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}$$

$$\|(3, (1,-2))\|_1 = \|3\|_{\mathbb{R}} + \|(1,-2)\|_{\mathbb{R}^2}$$

$$= |3| + \sqrt{5}$$

$$= 3 + \sqrt{5}$$

Find $\|(X_1, Y)\|_2$ H.W

Sol/ $\|(X,Y)\| = \max\{\|X\|_L, \|Y\|_{L'}\}$

$\|(3,(1,-2))\| = \max\{\|3\|_{\mathbb{R}}, \|(1,-2)\|_{\mathbb{R}^2}\}$

$= \max\{3, \sqrt{5}\}$

محاضرة تحليل دالي رقم ( )

Normed space and Metric space :

Def // Let X be a non empty set and $d: X \times X \longrightarrow \mathbb{R}$ be a mapping then d is called metric if

1) $d(x,y) \geq 0 \quad \forall\, x,y \in X$
2) $d(x,y) = 0 \iff x = y \quad \forall\, x,y \in X$
3) $d(x,y) = d(y,x) \quad \forall\, x,y \in X$
4) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\, x,y,z \in X$

then (X,d) is called metric space

ملاحظة / طلاب هنا تعريف metric space مهم جدا

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مدرس المادة :
التوقيع :

رئيس القسم :
التوقيع :

محاضرة تحليل دالي

Theorem // 2.20 // Let $(L, \|\ \|)$ be a normed space.

Let $d: L \times L \longrightarrow \mathbb{R}$ define by

$d(x,y) = \|x-y\| \quad \forall x,y \in X$. Prove that

(L, d) is a metric space (i.e بمعنى every normed space is metric space).

Then metric d is called metric induced ناتج عن by the norm.

solution//

By using the definition of norm

1) $\|x-y\| \geq 0 \quad \forall x,y \in L$, then $d(x,y) = \|x-y\| \geq 0$

2) $d(x,y) = 0 \Leftrightarrow \|x-y\| = 0 \Leftrightarrow x-y = 0 \Leftrightarrow x = y$
$\forall x,y \in L$

3) $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

4) $d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$ (نضيف ونطرح z)
$= d(x,z) + d(z,y)$

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التوقيع :

Remark ٧٣ // Not every metric space is normed space

سوف نوضح هذه الملاحظة بالمثال الآتي :

Ex / 2.23

Let d be the discrete metric space X. Then d can't be obtained from a norm on X (i.e $(X, \|\ \|)$)
where

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

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مدرس المادة :

التوقيع :

رئيس القسم :

التوقيع :

Lemma 2.21

Let d be a metric induced by a normd space $(L, \|\ \|)$ (i.e $d(x,y) = \|x-y\|$). Then satisfies the following

1) $d(x+a, y+a) = d(x,y) \quad \forall x, y, a \in L$
2) $d(\alpha x, \alpha y) = |\alpha|\, d(x,y) \quad \forall x, y \in L, \forall \alpha \in F$

Solution//

1) $d(x+a, y+a) = \|(x+a) - (y+a)\|$

$= \|x + a - y - a\|$

$= \|x - y\| = d(x,y)$

2) $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = \|\alpha(x-y)\|$

$= |\alpha|\, \|x - y\| = |\alpha|\, d(x,y)$

Remark: Not every metric space is normd space

مثال تطبيقي عن مجاميع الجداء لديكارتي

Ex/2.18/ let $L = (\mathbb{R}, \| \|_{\mathbb{R}})$ and $L' = (\mathbb{R}^2, \| \|_{\mathbb{R}^2})$

where $\| \|_{\mathbb{R}^2}$ is the Enclaidian norm.

If $x = 3 \in L = \mathbb{R}$ and $y = (1,-2) \in L' = \mathbb{R}^2$

Find $\|(x,y)\|_1$ and $\|(x,y)\|_2$

Ex 2.18/ ملاحظة/ حسب المثال السابق

تعريف $\|(x,y)\|_1$ و $\|(x,y)\|_2$ هو كالاتي

1) $\|(x,y)\|_1 = \|x\|_L + \|y\|_{L'}$

2) $\|(x,y)\|_2 = \max \{\|x\|_L, \|y\|_{L'}\}$

Solution//

1) $\|(x,y)\|_1 = \|3\| + \|(1,-2)\|$

$\|x\|_1 = \sqrt{3^2} = 3$ ← (Norm تعريف)

$\|(1,-2\|_1 = \sqrt{1^2+(-2)^2} = \sqrt{1+4} = \sqrt{5}$

$\therefore \|(x,y)\|_1 = \|3\| + \|(1,-2)\| = 3 + \sqrt{5}$

(2) $\|(x,y)\|_2 = \max\{\|3\|, \|(1,-2)\|\}$
$= \max\{3, \sqrt{5}\} = 3$

$$\|\alpha(x,y)\| = |\alpha|\ \|(x,y)\|$$

$$\|\alpha(x,y)\| = \|(\alpha x, \alpha y)\|$$

$$= \|\alpha x\| + \|\alpha y\|$$

$$= |\alpha|\ \|x\| + |\alpha|\ \|y\|$$

$$= |\alpha| (\|x\| + \|y\|)$$

$$= |\alpha| (\|(x,y)\|)$$

محاضرة تحليل دالي رقم (1)

Convergence in Normed Space

Definition 2.27: Let $\langle X_n \rangle$ be a sequance in normed space $(L, \| \|)$. Then $\langle X_n \rangle$ is said to be convergent in L if $\exists x \in L$ such that $\forall \epsilon > 0$, $\exists k \in Z_+$ such that $\| X_n - X \| < \epsilon$, $\forall n > k$

We write $X_n \longrightarrow x$ as $n \longrightarrow \infty$ or $\lim_{n \to \infty} (X_n) = X$,

that is $\| x_n - x \| \to 0 \Leftrightarrow x_n \longrightarrow X$

$\langle X_n \rangle$ is divergent (متباعدة) if it is not Convergent (متقاربة).

ملاحظة // التعريف مهم جداً حفظ التقارب في normed space

(1)

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قائمة الدرجات الفرعية للأمتحانات النهائية للعام الدراسي ٢٠١٨ - ٢٠١٩م

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محاضرة تحليل دالي

Theorem 2.30 "إثبات"

Let $\langle x_n \rangle$, $\langle y_n \rangle$ be two sequences in normed space $(L, \|\ \|)$ such that $x_n \to x$ and $y_n \to y$.

Then

(1) $\langle x_n \rangle + \langle y_n \rangle \to x + y$ as $n \to \infty$

$\langle x_n \rangle - \langle y_n \rangle \to x - y$

(2) $\lambda \langle x_n \rangle \to \lambda x$ for any scalar λ

$\|y_n - y\| < \frac{\epsilon}{2}$

(3) $\|\langle x_n \rangle\| \to \|x\|$

$\|x_n - x\| < \frac{\epsilon}{2}$

proof// Since $x_n \to x$, then $\forall \epsilon > 0$, $\exists k_1 \in Z_+$ s.t $\|x_n - x\| < \frac{\epsilon}{2}$, $\forall n > k_1$. Also since $y_n \to y$, then for $\epsilon > 0$, $\exists k_2 \in Z_+$ s.t $\|y_n - y\| < \frac{\epsilon}{2}$ $\forall n > k_2$

Let $k = \max\{k_1, k_2\}$. Then $\forall n > k$

$\|x_n - x\| < \frac{\epsilon}{2}$ and $\|y_n - y\| < \frac{\epsilon}{2}$ ---- (1)

Now, for each $n > k$,

$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \le \|x_n - x\| + \|y_n - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ (from (1))

Thus $x_n + y_n \to x + y$.

وزارة التعليم العالي والبحث العلمي
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اللجنة الأمتحانية
قسم :

المادة :
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عدد الوحدات :

قائمة الدرجات الفرعية للأمتحانات النهائية للعام الدراسي ٢٠١٨ - ٢٠١٩م

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(2) Let $\epsilon > 0$. Since $x_n \to x$, $\exists k \in Z_+$ s.t $\|x_n - x\| < \frac{\epsilon}{\lambda}$, $\forall n > k$ ---(I)

But $\|\lambda x_n - \lambda x\| = |\lambda| \, \|x_n - x\| < \frac{\epsilon}{|\lambda|} |\lambda| = \epsilon$

Thus $\lambda \langle x_n \rangle \to \lambda x$

(3) Let $\epsilon > 0$. Since $x_n \to x$, $\exists k \in Z_+$ s.t $\|x_n - x\| < \epsilon$, $\forall n > k$ ---(I)

$x_n \to x$
$\|x_n - x\|$
$\|x_n\| \to \|x\|$

But $\big| \|x_n\| - \|x\| \big| \leq \|x_n - x\| < \epsilon \quad \forall n > k$.

Hence, $\|x_n\| \longrightarrow \|x\|$

$\|x_n\| - \|x\| =$
$\|x_n - x\| < \epsilon$

Definition 2-31 // Let $\langle x_n \rangle$ be a sequance in normed space $(L, \| \|)$. Then $\langle x_n \rangle$ is said to be Cauchy sequance if $\forall \epsilon > 0, \exists k \in Z_+$ s.t $\|x_n - x\| < \epsilon$ $\forall n, m > k$.

ملاحظة تعريف Cauchy sequance
مهم جداً حفظ