2.29. Exercise

Let $(L, \|\ \|)$ be a normed space. Prove that

i) The set $A_1 = \{x \in L : \|x\| \le 1\}$ is closed.

(ii) The set $A_2 = \{x \in L : \|x\| < 1\}$ is open.

(iii) The set $C = \{x \in L : \|x\| = 1\}$ is closed.

Solution:

i) $A_1 = \{x \in L : \|x\| \le 1\} = \bar{B}_1(0)$.

So, A_1 is a closed set (by Definition 2.25).

ii) $A_2 = \{x \in L : \|x\| < 1\} = B_1(0)$.

So, A_2 is an open set (by Definition 2.25).

iii) $C = \{x \in L : \|x\| = 1\}$

$L \setminus C = \{x \in L : \|x\| < 1\} \cup \{x \in L : \|x\| > 1\}$

Let $C_1 = \{x \in L : \|x\| < 1\}$ is open set.

Let $C_2 = \{x \in L : \|x\| > 1\}$

So, $L \setminus C_2 = \{x \in L : \|x\| \le 1\}$ which is closed set. Hence, C_2 is an open set.

Thus, $L \setminus C = C_1 \cup C_2$ is an open set (by Remark 2.27(2)).

2.30. Definition

Let L be a normed space and $A \subseteq L$. A point $x \in L$ is called **limit point** of A if for each open set G containing x, we have $(G \cap A) \setminus \{x\} \neq \varphi$.

The set of all limit points of A is denoted by A' and is called **derived set**. The closure of A is denoted by $\bar{A}$ and is defined as $\bar{A} = A \cup A'$.

.2.31Proposition

Let L be a normed linear space and $A \subseteq L$. Then $x \in \overline{A}$ if and only if

$., \exists y \in A, \|x - y\| < r0 r > \forall$

$-$ $'$*Proof.* $(\Rightarrow)$ Let $x \in A = A \cup A$

$.\varphi$If $x \in A'$ then for each open set G, $x \in G, (G \cap A) \setminus \{x\} \neq$

$, \varphi$. Thus, we have $B_r(x) \cap A \setminus \{x\} \neq 0$Since $B_r(x)$ is an open set then $\forall r >$

(I)$y \in B_r(x) \cap A, y /= x \implies \|y - x\| < r \exists$

(II)$< r0$ If $x \in A$ then $\exists y = x$ such that $\|y - x\| =$

.From **(I)** and **(II)**, we get the required result

$\ni , \exists y0, \exists y \in A$ such that $\|y - x\| < r$, that is $\forall r > 0$If for each $r >$ $(\Leftarrow)$

$(A, y \in B_r(x$

$.\varphi \implies x \in A'$. Thus, $x \in A, (B_r(x) \cap A) \setminus \{x\} \neq 0 r > \forall \Rightarrow$ □

Convergence in Normed Space2.5

.2.32Definition

Let $\langle x_n \rangle$ be a sequence in a normed space $(L, \| \|)$. Then $\langle x_n \rangle$ is said to

$, \exists k \in Z_+$ such that0be **convergent** in L if $\exists x \in L$ such that $\forall \epsilon >$

$\forall n > k \|x_n - x\| < \epsilon,$

We write $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty}(x_n) = x$; that is

$. \Longleftrightarrow x_n \to x 0 \; \|x_n - x\| \to$

$.x_n\rangle$ is **divergent** if it is not convergent$\rangle$

.2.33Theorem

If $\langle x_n \rangle$ is a convergent sequence in $(L, \| \, \|)$, then its limit is unique. i.e.,

.If $\langle x_n \rangle \to x$ and $\langle x_n \rangle \to y$ then $x = y$

$\in Z_+$ such $_2$, k_1. Since $\langle x_n \rangle \to x$ and $\langle x_n \rangle \to y$, then $\exists k$0*Proof.* Let $\epsilon >$

that

$$\frac{\epsilon}{2} \forall n > k_2, \text{ and } \|x_n - y\| < \frac{\epsilon}{2} \forall n > k_1, \|x_n - x\| <$$

$\}$, so $\forall n > k_2, k_1$Let $k = \max\{k$

$$\|x - y\| = \|x_n - y - x_n + x\| = \|(x_n - y) - (x_n - x)\|$$

$$= \epsilon \frac{\epsilon}{2} + \frac{\epsilon}{2} \|x_n - y\| + \|x_n - x\| < \geq$$

., so $x = y$0. Thus, $\|x - y\| = 0 \forall \epsilon > \|x - y\| < \epsilon, \Rightarrow =$ □

.2.34Theorem

Let $A \subseteq L$ where L is a normed space, let $x \in L$. Then

$.x \in \bar{A} \iff \exists \langle x_n \rangle$ a sequence in A such that $\langle x_n \rangle \to x$

Proof. $(\Rightarrow)$ Let $x \in \bar{A} = A \cup A'$

(I) If $x \in A$ then the sequence $\langle x, x, x, \ldots \rangle \to x$

φ. If $x \notin A$, *i.e.*, $x \in A'$ then for each open set G, $x \in G, (G \cap A) \setminus \{x\} \neq$

φ. Set, we have $B_r(x) \cap A \setminus \{x\} \neq$0Since $B_r(x)$ is an open set then $\forall r >$

$\neq (x)\frac{1}{n} \cap A) \setminus \{x\}_1 \in Z$. Then $\forall n \in Z_+, (B\, r = \frac{}{n} > 0 \varphi /$

$B \ni$ Let x_n ,x, hence $= x)_n \cap A$, s.t $x)_{\frac{1}{n}}$ $_n \forall n \in Z, \frac{1}{n} \| x_n - x \| <$ **(*)**

$\epsilon >$ Thus, $\exists \langle x_n \rangle \in A$ such that $\|x_n - x\| \frac{1}{n}$ $._+ n \in Z \forall$

$0 \forall \epsilon >$ To show $\langle x_n \rangle \to x$; that is $\|x_n - x\| < \epsilon$,

so by Archmedian theorem $\exists k \in Z_+$ such that $\frac{1}{k} < \epsilon$0 Let $\epsilon >$

Hence, $\forall n > k, \frac{1}{n} < \frac{1}{k} < \epsilon$

From (*), $\forall n > k, \|x_n - x\| < \frac{1}{n} < \frac{1}{k} < \epsilon$. Thus, $x_n \to x$ **(II)**

.From **(I)** and **(II)**, we get the required result

$'$If $\exists\langle x_n\rangle$ a sequence in A such that $\langle x_n\rangle \to x$. To prove $x \in A = \overline{A} \cup A$ $(\Leftarrow)$

—

If $x \in A$ then $x \in A$

0 If $x \notin A$. Let G be an open set in L such that $x \in G$. Then $\exists r >$

and $x_n \to x$, $\exists k \in Z_+$ such that 0 such that $B_r(x) \subseteq G$. Since $r >$

$.\|x_n - x\| < r, \forall n > k$

$\forall n > k$ and since $x_n \in A$ $\forall n \in Z_+$. Then This implies, $x_n \in B_r(x)$

$(B_r(x) \cap A) \setminus \{x\} \neq \varphi$. Since $B_r(x) \subseteq G$, then $(G \cap A) \setminus \{x\} \neq$

—

$.\varphi$. So, $x \in A'$, and therefore $x \in A$ □

.2.35Theorem

Let $\langle x_n\rangle, \langle y_n\rangle$ be two sequences in normed space $(L, \| \|)$ such that $x_n \to x$

and $y_n \to y$. Then

$x_n\rangle \pm \langle y_n\rangle \to x \pm y\rangle$ **(1)**

for any scalar $\lambda\lambda\langle x_n\rangle \to \lambda x$ **(2)**

$\|\langle x_n\rangle\| \to \|x\|$ **(3)**

) Since $x_n \to x$, then**1***Proof.* **(**

$_1\forall n > k, \frac{\epsilon}{2} \in Z_+$ such that $\|x_n - x\| < _1, \exists k 0$ for each $\epsilon >$

Also since $y_n \to y$, then

$_2\forall n > k, \frac{\epsilon}{2} \in Z_+$ such that $\|y_n - y\| < _2, \exists k 0$ for each $\epsilon >$

}. Then, for each $n > k_2, k_1$Let $k = \max\{k$

(I) $\frac{\epsilon}{2}$ and $\|y_n - y\| < \frac{\epsilon}{2}$ $\|x_n - x\| <$

Now, for each $n > k$,

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \le \|x_n - x\| + \|y_n - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (\text{from } \mathbf{(I)})$$

Thus, $x_n + y_n \to x + y$ as required.

(2) Let $\epsilon > 0$. Since $x_n \to x$, $\exists k \in Z_+$ s.t $\|x_n - x\| < \frac{\epsilon}{|\lambda|}$ $\forall n > k$ **(II)**

But $\|\lambda x_n - \lambda x\| = |\lambda| \, \|x_n - x\| < |\lambda| \frac{\epsilon}{|\lambda|} = \epsilon$ (using (II))

Thus, $\lambda \langle x_n \rangle \to \lambda x$

(3) Let $\epsilon > 0$. Since $x_n \to x$, $\exists k \in Z_+$ s.t $\|x_n - x\| < \epsilon$, $\forall n > k$. **(III)**

But $|\,\|x_n\| - \|x\|\,| \le \|x_n - x\| < \epsilon$ (using (III)) $\forall n > k$. Hence, $\|x_n\| \to \|x\|$. □

2.36 Definition.

Let $\langle x_n \rangle$ be a sequence in a normed space $(L, \| \, \|)$. Then $\langle x_n \rangle$ is said to be **Cauchy sequence** if $\forall \epsilon > 0$, $\exists k \in Z_+$ s.t $\|x_n - x_m\| < \epsilon$, $\forall n, m > k$.

2.37 Theorem.

Every convergent sequence in a normed space $(L, \| \, \|)$ is a Cauchy sequence.

Proof. Let $\langle x_n \rangle$ be a convergent sequence in L. Then $\exists x \in L$ such that $x_n \to x$ and so $\forall \epsilon > 0$, $\exists k \in Z_+$ such that $\|x_n - x\| < \frac{\epsilon}{2}$, $\forall n > k$ **(I)**

Now, for $n, m > k$,

$$\|x_n - x_m\| = \|(x_n - x) + (x - x_m)\| \le \|x_n - x\| + \|x_m - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (\text{using } \mathbf{(I)})$$

Thus, $\langle x_n \rangle$ is a Cauchy sequence. □

2.38 Definition.

Let $\langle x_n \rangle$ be a sequence in a normed space $(L, \| \, \|)$. Then $\langle x_n \rangle$ is said to be **bounded sequence** if $\exists k \in \mathbb{R}$, $k > 0$ such that $\|x_n\| \le k$, $\forall n \in Z_+$.

.2.39Theorem

.Every Cauchy sequence $\langle x_n \rangle$ in a normed space $(L, \| \ \|)$ is bounded

. Since $\langle x_n \rangle$ is a Cauchy sequence, $\exists k \in Z_+$ such that 1*Proof.* Let $\epsilon =$

, $\forall n > k$ (by 1$\| < {}_1$, $\forall n, m > k$. Hence, $\| x_n - x_{k+}1 \| x_n - x_m \| <$

() (I1considering $m = k +$

$n > k \forall 1 \| < {}_1 \| \ | \leq \| x_n - x_{k+1}$), we have $| \ \| x_n \| - \| x_{k+}$4(2.3By Theorem

X (using ()

1$\| < {}_1$Thus, $\| x_n \| - \| x_{k+}$ $n > k \forall$

$\|_1 + \| x_{k+}1$ Then, $\| x_n \| <$ $n > k \forall$

$\{ \|_1 + \| x_{k+}1 \|, \ldots, \| x_k \|, {}_2 \|, \| x_1$Let $M = \max\{ \| x$

.Hence, $\| x_n \| \leq M$ $\forall n \in Z_+$. So, $\langle x_n \rangle$ is bounded□

.2.40Corollary

.Every convergent sequence in a normed space $(L, \| \ \|)$ is bounded

, Every convergent sequence in a normed space 2.37*Proof.* From Theorem

, every Cauchy sequence in 2.39 $(L, \| \ \|)$ is Cauchy, and from Theorem

.a normed space $(L, \| \ \|)$ is bounded□

Convexity in Normed Linear Space2.6

.2.41Definition

,0A subset A of a linear space L is said to be **convex** if $\forall x, y \in A, \lambda \in [$
[1

.$-\lambda)y \in A$1 then $\lambda x + ($

.2.42Example

?) ⊂ R. Is A convex set3, 1Let A = (

[1, 0**Solution:** Let $x, y \in A, \lambda \in$ [

(I) $\lambda 3\lambda < \lambda x <$ 1=⇒ 3 < x < 1 Since

($-\lambda$1 (3$-\lambda$)y < 1 $-\lambda$) < (1 (1=⇒ 3 < y < 1 Since (II)

(By summing up (I) and (II

($-\lambda$1 (3λ + 3$-\lambda$)y < 1 $-\lambda$) < λx + (1 λ + (

3$-\lambda$)y << 1 λx + (> 1

.$-\lambda$)$y \in A$. Hence, A is convex set1 Thus, λx + (

.2.43Proposition

Let L linear space. Then

Every subspace of L is convex (١)

(.If $A, B \subset L$ are convex sets then $A \cap B$ is convex (**H.W** (٢)

If $A, B \subset L$ are convex sets then $A + B$ is convex (٣)

) Let L be a linear space over a field F = R or C, let A be a **1***Proof.* **(**

, $\forall x, y \in A$ and $\forall \alpha, \beta \in F$ we 1.13subspace of L. Hence, by Theorem

.have $\alpha x + \beta y \in A$

$-\lambda$)$y \in A$.1 $-\lambda$. Hence, $\alpha x + \beta y = \lambda x$ + (1] and β = 1, 0Take $\alpha = \lambda \in$ [

. Thus, A is a convex set

$\in B$. 2 , $b_1 \in A$ and b_2 , $a_1 \in A + B$, then $a_2 + b_2$, $a_1 + b_1$ Let a **(3)**

.[1, 0$\forall\lambda \in$ [) $\in A + B$,2+ b_2 $-\lambda$)(a1) + (1+ b_1 To prove λ(a

((I]1, 0$\lambda \in$ [$\forall \in A_2 - \lambda$)$a$1+(1 $\in A$ =⇒ λa_2 , a_1Since A convex and a

$\in B_2 - \lambda)b1 + (_1 \in B \implies \lambda b_2$, b_1Since B convex and b $[1,0\lambda \in [\forall$ (II)

By summing up (I) and (II) we get

$\in A + B_2 - \lambda)b1 + (_1 + \lambda b_2 - \lambda)a1 + (_1 \lambda a$

.) $\in A + B$. Thus, $A + B$ is a convex set$_2$+ $b_2 - \lambda)(a1$) + ($_1$+ b_1 i.e., λ(a

.2.44Remark

The union of two convex sets is not necessary convex. For example, let

, y 6). Then A is not convex. To show this, take x = 12, 7) $\cup$ (7, 3A = (

_ . $A \cup B \not\ni 7\frac{1}{2} = (8) + \frac{1}{2}(6(-\lambda)y = 1$ then $\lambda x + \frac{1=}{2}(\lambda =$ 8

.2.45Proposition

) and $_0 \in L$. Then $B_r(x_0$ Let $(L, \| \|)$ be a normed linear space, let x

.) are convex sets$_0 B_r(x$

,0), and let $\lambda \in [_0)$ is a convex set. Let $x, y \in B_r(x_0$ *Proof.* To prove $B_r(x$

,]. Then1

(I) $\| < r_0 \| < r$ and $\|y - x_0\| x - x$

); that is we must prove$_0 - \lambda)y \in B_r(x1$ We must prove λx + (

$\| < r_0 - \lambda)y - x1 \| \lambda x$ + (

$\|$ (adding and $_0 - \lambda)y - x1 + (_0 - \lambda \mathbf{x_0} \| = \| \lambda x + \lambda \mathbf{x}_0 - \lambda)y - x1 \| \lambda x$ + (

($_0$subtracting λx

$)\|_0 - \lambda)(y - x1) + (_0 \| \lambda(x - x =$

$-\lambda)r = r1 \| < \lambda r + (_0 - \lambda| \|y - x1)\| + |_0 \lambda| \|(x - x| \geq$

$(-\lambda 1 - \lambda| = 1$ then $|\lambda| = \lambda$, $|0$ by (I) and since $\lambda >$)

) is convex. Similarly, $_0$) and hence $B_r(x_0 - \lambda)y \in B_r(x1$ Thus, λx+(

.) is a convex set$_0 B_r(x$

□

.2.46Proposition

Let $(L, \| \|)$ be a normed linear space and $A \subseteq L$ and convex then $\overline{A}$ is

.a convex set

$-\lambda)y \in \overline{A}$1]. To prove $\lambda x +$ (1,0*Proof.* Let $x, y \in \overline{A}$ and $\lambda \in$ [

, $\exists a, b \in A$ such that2.31. Since $x, y \in A$ then by Proposition 0Let $r >$

(I) $\|x - a\| < r$ and $\|x - b\| < r$

$-\lambda)b \in A$1 Since A is convex then $\lambda a +$ (

$-\lambda)(y - b)\|1 -\lambda)b$ ¨ $= \|\lambda(x - a) + (1 - \lambda)y -$ $\lambda a +$ (1 Now, ¨$\lambda x +$ (

$-\lambda) \|y - b\|1$ $\lambda \|x - a\| +$ ($\geq$

((from (I)$-\lambda)r$1 $\lambda r +$ ($>$

$r =$

$=$1$\lambda a +$ ($-$ $-\lambda)y$1 $\lambda x +$ (Thus, $r > \lambda)b$

$A \ni$ —

.$-\lambda)y \in A$1 , $\lambda x +$ (2.31Thus, from Proposition □

.2.47Remark

The converse of the above proposition is not true. For example, let A =

,] is a convex set. But A is not convex5, 1] $\subset (\mathbb{R}, | |)$ then $\overline{A}$ = [5, 2) ∪ (2, 1[

$\exists 2 =$ (3(+ $-\lambda)y = \frac{1}{2}$1 then $\lambda x +$ (, $\lambda = 3, y = \frac{1}{2}$1since if $x =$.A /

Continuity in Normed Linear Space2.7

.2.48Definition

A mapping $f : L \to L'$ is calledLet L, L' be normed linear spaces.

) such that0(depend on $x0$, $\exists \delta > 0 \in L$ if for each $\epsilon >$ 0 **continuous** at x

.)$\| < \epsilon_0 \| f(x) - f(x \| < \delta$ then0$\| x - x$if$x \in L, \forall$

(($_0$) then $f(x) \in B_\epsilon(f(x_0 x \in B_\delta(x$ifi.e., $\forall x \in L$,

.2.49Theorem

Let L, L' be normed linear spaces. A mapping $f : L \to L'$ is continuous at

.(0implies that $f(x_n) \to f(x_0 \in L$ iff $\forall \langle x_n \rangle \in L$ with $x_n \to x_0$ x

and let $\langle x_n \rangle$ be a 0 *Proof.* ($\Rightarrow$) Let f be a contiunuous mapping at x

.(0. To prove $f(x_n) \to f(x_0$sequence in L such that $x_n \to x$

such that $\forall x \in X0$, then $\exists \delta > 0$Let $\epsilon >$

). 0From continuity of f at x))$\| < \epsilon_0 \| < \delta$ then $\| f(x) - f(x_0$if $\| x - x$

.$\forall n > k \| < \delta$,0, $\exists k \in Z_+$ such that $\| x_n - x0$and $\delta >$ 0 Since $x_n \to x$

.(0$\forall n > k$; that is $f(x_n) \to f(x)\| < \epsilon$,0Hence, $\| f(x_n) - f(x$

To prove f is).0implies that $f(x_n) \to f(x_0$ Suppose that $x_n \to x$ ($\Leftarrow$)

.0continuous at x

, $\exists x \in X0$such that $\forall \delta > 0$, so $\exists \epsilon >$ 0Assume that f is not continuous at x

and

.)$\| \geq \epsilon_0 \| < \delta$ but $\| f(x) - f(x_0 \| x - x$

, then $\exists x_n \in L$ such that0Now, $\forall n \in Z_+$, $\frac{1}{n} >$

~ (but $f(x_{n0})\| \geq \epsilon$. This means $x_n \to x_0 \| < \frac{1}{n}$ but $\| f(x_n) - f(x_0 \| x_n - x$

.0) in L' which is a contradiction. Thus, f is continuous at $x_0 f(x$

.2.50Theorem

Let $(L, \| \|)$ be a normed space and let $f : (L, \| \|) \to (\mathrm{R}, | |)$ such that

.$_0 x \in L$. Then f is continuous at $x \forall f(x) = \|x\|$

, $\exists k \in Z_+$ such that0in L. Then $\forall \epsilon > 0$ *Proof.* Let $x_n \to x$

(I) $n > k \forall \| < \epsilon_0 \| x_n - x$

$n > k \forall \|_0 \| \mid \leq \| x_n - x_0$But $\mid \| x_n \| - \| x$

((Using **(I)**)$n > k \forall \| \mid < \epsilon_0 \| x_n \| - \| x \mid \Rightarrow =$

((Using (since $f(x) = \|x\|$)$n > k \forall) \mid < \epsilon_0 f(x_n) - f(x \mid \Rightarrow =$

.$_0$); that is f is continuous at $x_0 f(x_n) \to f($x □

.2.51Remark

be a $_3 \to L_2 \times L_1$ be normed spaces and let $f : L_3$ and L_2 , L_1Let L

iff whenever $\langle (x_n, y_n) \rangle_2 \times L_1) \in L_0, y_0$mapping. Then f is continuous at $(x$

.($_0, y_0$) then $f(x_n, y_n) \to f(x_0, y_0 \to (x$

.2.52Theorem

Let L be a normed space over a field F. Then

The mapping $f : L \times L \to L$ such that $f(x, y) = x + y$ **(1)** $x, y \in L \forall$ is

.($_0, y_0$continuous at $(x$

$x \in L, \forall \lambda \in F \forall$The mapping $g : F \times L \to L$ such that $g(\lambda, x) = \lambda x$ **(2)**

.($_0$is continuous at $(\lambda, x$

such that$_0$ and $y_n \to y_0$). Then, $x_n \to x_0, y_0$**)** Let $(x_n, y_n) \to (x$**1***Proof.* **(**

.$\infty \to$ as $n 0 \| \to {}_0$and $\| y_n - y 0 \| \to {}_0 \| x_n - x$

$) \|_0 + y_0) \| = \| (x_n + y_n) - (x_0, y_0$Now, $\| f(x_n, y_n) - f(x$

$) \|_0) + (y_n - y_0 \| (x_n - x =$

$\|_0\| + \|y_n - y_0\| x_n - x \geq$

as $n \to \infty$; that is f is continuous at 0 $)\| \to {}_0, y_0$Thus, $\| f(x_n, y_n) - f(x$

.$(_0, y_0)$ is arbitrary, f is continuous at (x_0, y_0). Since $(x_0, y_0(x$

). Then, $\lambda_n \to \lambda$ and $x_n \to {}_0$Let $(\lambda_n, x_n) \to (\lambda, x$ **(2)**

.$\infty \to$ as n0 , $\|x_n - x\| \to 0$. Hence, $|\lambda_n - \lambda| \to {}_0 x$

$$\|_0)\| = \|\lambda_n x_n - \lambda x_0\| g(\lambda_n, x_n) - g(\lambda, x$$

$$\|_0 - \lambda x_\mathbf{0} + \lambda_\mathbf{n} \mathbf{x_0} \| \lambda_n x_n - \lambda_\mathbf{n} \mathbf{x} =$$

$$\|_0) + (\lambda_n - \lambda) x_0 \| \lambda_n (x_n - x =$$

$$\|_0\| + |\lambda_n - \lambda| \| x_0 \lambda_n| \| x_n - x| \geq$$

so that0 and $|\lambda_n - \lambda| \to 0$ $\| \to {}_0$But $\| x_n - x$

,). Thus$_0$as $n \to \infty$; that is $g(\lambda_n, x_n) \to g(\lambda, x$0 $)\| \to {}_0 \| g(\lambda_n, x_n) - g(\lambda, x$

.$(_0 g$ is continuous at $(\lambda,$□

.2.53Theorem

Let L, L' be normed spaces and let $f : L \to L'$ be a linear

.the f is continuous at any point0 transformation. If f is continuous at

$\in X$ be an arbitrary point and let $x_n \to {}_0$ *Proof.* Let x

.(2.49) (using Theorem $_0$. To prove $f(x_n) \to f(x_0 x$

$0 \to {}_0$, then $x_n - x_0$Since $x_n \to x$

$(0) \to f(_0$, thus $f(x_n - x$0But f is continuous at

) 0) $\to f(_0$Since f is a linear transformation, then $f(x_n) - f(x$

.$(_0$It follows that $f(x_n) \to f(x$ □

.2.54Remark

The condition f is a linear transformation in the above theorem is necessary condition. For example: consider the normed space (R, | |). Let f is defined as

$$f(x) = \begin{cases} 1 \text{ if } x \leq \\ . 1x > \text{if} 1 \quad x + \end{cases}$$

.1and discontinuous at 0 It is clear that f is continuous at

Also f is not linear transformation because

12= 1 + 11) = 11) = f(6+ 5 f(

13) = 1+ 6) + (1+ 5) = (6) + f(5and f(

(6) + f(5) /= f(6+ 5 Hence f(

.2.55Theorem

Let L and L' be normed spaces and let $f : L \to L'$ be a linear transforma- tion. Then either f is continuous at each point or .discontinuous at each point

$\in L$ be $_2$. Let $x_1 \in L$ and assume that f is continuous at x_1 *Proof.* Let x in L. Then,$_2$. Let $x_n \to x_2$any point. To prove that f is continuous at x $_1$. Since f is continuous at $x_1 \to x_1 + x_2$ and hence $x_n - x0 \to {}_2 \; x_n - x$.($_1$) $\to f(x_1 + x_2$ then $f(x_n - x$) $\to f_1) + f(x_2$Since f is a linear transformation, then $f(x_n) - f(x$.($_2$, and thus, $f(x_n) \to f(x0) \to {}_2$). Hence, $f(x_n) - f(x_1(x$. Thus, f can not be continuous at $_2$Therefore, f is continuous at x .some points and discontinuos at some points

□

2.56 Example.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is not continuous at $(0,0)$.

Solution: Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$.

Then, $x_n \to 0$ and $y_n \to 0$. Thus, $(x_n, y_n) \to (0,0)$. But

$$f(x_n, y_n) = \frac{(\frac{1}{n})(\frac{1}{n})}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \frac{1}{2}.$$

Hence, $f(x_n, y_n) \to \frac{1}{2} \neq 0$. Thus, $f(x_n, y_n) \nrightarrow f(0,0) = 0$ but $(x_n, y_n) \to (0,0)$.

Thus, f is not continuous at $(0,0)$.

2.8 Boundedness in Normed Linear Space

2.57 Definition. ***Bounded Set*.**

Let L be a normed space and let $A \subset L$. A is called a **bounded set** if there exists $k > 0$ such that $\|x\| \leq k$ $\forall x \in A$.

2.58 Example.

Consider $(\mathbb{R}, |\,|)$ and let $A = [-1, 1]$. Since $|x| \leq 1$, then A is bounded.

2.59 Example.

Consider $(\mathbb{R}^2, \|\,\|)$ be a normed space such that $\|X\| = \left(\sum_{i=1}^{2} |x_i|^2\right)^{\frac{1}{2}}$ be the Eucledian norm, for each $X = (x_1, x_2) \in \mathbb{R}^2$.

Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x \leq 1, y \geq 0\}$. Then, A is unbounded.

.2.60Theorem

Let L be a normed space and let $A \subseteq L$. Then the following statements .are equivalent

.A is bounded *(١)*

If $\langle x_n \rangle$ is a sequence in A and $\langle \alpha_n \rangle$ is a sequence in F such that $\alpha_n \to$ (٢)

.0then $\alpha_n x_n \to 0$

$x_n \in A$. $\forall$such that $\|x_n\| \leq k$0) Since A is bounded, $\exists k >$ 2)⇒(1*Proof.* (

,. Hence0as $n \to \infty$, then $|\alpha_n| \to 0$ Since $\alpha_n \to$

(since $\|x_n\| \leq k$)$\|$ = $\|\alpha_n x_n\| = |\alpha_n| \, \|x_n\| \leq |\alpha_n| \, k0\|\alpha_n x_n -$

and hence0 $\|$ → 0Therefore, $\|\alpha_n x_n -$.0, thus $|\alpha_n| k \to$ 0But $|\alpha_n| \to$

.x**0**$\alpha_n x_n \to$

Suppose A is not bounded. Then, $\forall k \in Z_+, \exists x_k \in A$ such that (1)⇒(2)

.$\|x_k\| > k$

. But0Put $\alpha_k = \frac{1}{k}$. Hence, $\alpha_k \to$

$\| \alpha_k x_k \| = \frac{1}{k} \| x_k \| > \frac{1}{k} . k = 1$

.(2which contradicts (0 , thus $\alpha_k x_k \sim$ 1Then, $\|\alpha_k x_k\|$ ☒

. *Bounded Mapping*2.61Definition

.Let L, L' be two normed space and $f : L \to L'$ be a linear transformation f is called **bounded mapping** if for each $A \subseteq L$ bounded then $f(A)$ is .$A \subseteq L$∀bounded set in Y

.2.62Example

. Show that f is a^2$x, y) \in$ R)∀→ R such that $f(x, y) = x + y^2$ Let f : R

and A is bounded. Show that^2 linear transformation (**H.W.**). Let $A \subseteq$ R

.$f(A)$ is bounded

such that $\|(x, y)\|$0 and A is bounded, then $\exists k \geq$ 2 **Solution:** Let $A \subseteq \mathbb{R}$ $\geq$

$\leq k \implies x + y \leq k^2 x, y)^2 \in A^2 \implies (x + y)^2 \forall k^2$

$\implies |x| \leq k^2 \leq k^2$, then $x^2 \leq k^2 + y^2 \leq x^2$ Since x **(I)**

$\implies |y| \leq 2 \leq k^2$, then $y^2 \leq k^2 + y^2 \leq x^2$ Similarly, y k **(II)**

(Note that $\forall (x, y) \in A \implies f(x, y) = x + y \in f(A$
$k2 f(x, y)| = |x + y| \leq |x| + |y| \leq k + k = |$
by **(I)** and **(II)**

.k. Thus, $f(A)$ is bounded, and hence, f is bounded2i.e., $|f(x, y)| \leq$

.2.63Theorem

.Let L, L' be normed spaces and $f : L \to L'$ be a linear transformation

such that $\|f(x)\| \leq k\|x\|$0 Then f is bounded iff $\exists k >$ $x \in \forall$ .X

Proof. ($\Rightarrow$) If f is bounded and let $A = \{x \in L : \|x\| \leq$

}. It is clear A is bounded, and hence, $f(A)$ is bounded 1 by definition)

.(of bnd function in Y

(I) $x \in A \forall$such that $\|f(x)\| \leq k$0 Thus, $\exists k >$

.0$\mathbf{x}\| = \mathbf{0} \leq k\|0\ \mathbf{x})\| = \mathbf{0_Y}$, and thus, $\|f(\mathbf{0_X}) = \mathbf{0_X}$ then $f(\mathbf{0}$If $x =$ (1)

$\frac{1}{}$ $\frac{x}{\mathbf{x}}$ = such that $\|\frac{x}{y}\|$, put $y = \mathbf{0}$ If $x \neq$ (2) $\|x\| = \frac{}{\|x\|}$ $\frac{}{\|x\|}$ $\frac{}{x\|}$.=

Hence, $y \in A$. Thus, $\|f(y)\| \leq$ k **(II)**

$= (\ ) \|f(y)\| \frac{}{\|x\|} = f \frac{1}{\|x\|} \|f(x)\| \frac{}{\|x\|} f(x) =$

By **(II)**, $\|f(y)\| \leq k$, $\frac{1}{\|x\|}$ thus $\|f(x)\| \leq$ .k i.e., $\|f(x)\| \leq k.\|x\|$ as

.required

$x \in A \forall_1$such that $\|x\| \leq k$0 $>_1$ Let A be a bounded set. Then, $\exists k$ ($\Leftarrow$)

Since $\|f(x)\| \leq k\|x\|$ $\forall x \in X$, hence $\|f(x)\| \leq k\|x\|$ $\forall x \in A$. Then we

; $_1$= kk_2 $\forall x \in A$ where k_2 $\forall x \in A$. Thus, $\|f(x)\| \leq k_1$ get $\|f(x)\| \leq kk$

.that is, $f(A)$ is a bounded set □

.2.64Theorem

Let L, L' be normed spaces and $f : L \to L'$ be a linear transformation.

.Then f is bounded if and only if f is continuous

Proof. ($\Leftarrow$) Suppose that f is continuous and not bounded,

.hence $\forall n \in Z_+, \exists x_n \in L$ such that $\|f(x_n)\| > n\|x_n\|$

= Let y_n $\frac{\|f(x_n)\|}{n\|x_n\|}$ $\frac{x}{n\|x_n\|}$ ()Then, $\|f(y_n)\| = \frac{x}{n\|x_n\|} f$. $\frac{n\|x_n\|}{n\|x_n\|}$ 1 =

(0, i.e., $f(y_n) \sim f(1)\| = \|f(y_n)\| > 0$Thus, $\|f(y_n) - f($ **(I)**

= n = but $\|y_n\|$ $\frac{1}{n\|x_n\|}$ $= \frac{\|x_n\|}{n\|x_n\|}$ x

. x**0**, and hence, $y_n \to 0$as $n \to \infty$, we get $\|y_n\| \to$

(Since f is a linear transformation) y**0**x) = **0** f(It follows that $f(y_n) \to$

((i1.19By Theorem

.This contradicts **(I)**, thus, f is bounded

Assume that f is bounded to prove f is continuous for all $x \in L$. Let ($\Rightarrow$)

such that0 , to find $\delta > 0 \in L$ and $\epsilon >_0 x$

.)$\| < \epsilon_0\| < \delta \implies \|f(x) - f(x_0 x \in L, \|x - x\forall$

(f is linear transformation))$\|_0)\| = \|f(x - x_0\| f(x) - f(x$

(I) $x \in L \forall$s.t. $\|f(x)\| \leq k\|x\|0$ Since f is bounded, then $\exists k >$

$\|_0)\| \leq k\|x - x_0)\| = \|f(x - x_0$Hence, $\|f(x) - f(x$

By **(I)**

($\| < \delta_0$Since $\|x - x)k\delta >$

(By choosing $\delta = \frac{\epsilon}{k} = \epsilon) k.\frac{\epsilon}{k} =$

.)$\| < \epsilon_0\| < \delta \implies \|f(x) - f(x_0$Thus, $\|x - x$

is an arbitrary, then f is$_0$ Since $x \in L._0$ Hence, f is continuous at x

.cont. $\forall x \in L$ □

.2.65Theorem

Let L, L' be normed spaces and $f : L \to L'$ be a linear transformation. If .(L' is a finite dimensional space then f is bounded (hence, continuous

.2.66Example

. $^2x, y) \in$ R)∀→ R defined as $f(x, y) = x + y^2$ Let f : R

(!f is a linear transformation function (check

.(. Hence, f is bounded (hence, continuous2) = ^{2}and dim(R

Bounded Linear Transformation2.9

.2.67Definition

Let L, L' be normed spaces over a field F. The set of all bounded linear transformation mappings from L to L' is defined as

{.$B(L, L') = \{T : T : L \to L'$ is a linear bounded (hence, cont.) trans

.2.68Theorem

Prove that $B(L, L')$ is a linear subspace (over a field F) of the space of linear transformation mappings with respect to usual addition and usual .scalar multiplication

$\in B(L,_2 + \beta T_1 \in B(L, L')$. To prove αT_2 , T_1*Proof.* Let $\alpha, \beta \in F$ and T (ii), 1.20are linear transformations, then by Theorem $_2$, $T_1 L'$) Since T .are linear trans$_2$, $\beta T_1 \alpha T$

is linear $_2 +$ βT_1 (i),αT1.20are linear trans., by Theorem $_2$, βT_1Now, αT .transformation

.is bounded$_2$ + βT_1 Next, we show αT

such that $\forall x \in X$ we have0 > 2 , $k_1$are bounded, then $\exists k_2$, T_1Since T

(I) $\|x\|_2 (x)\| \leq k_2\|x\|$ and $\|T_1(x)\| \leq k_1\|T$

$)(x)\|_2)(x) + (\beta T_1)(x)\| = \|(\alpha T_2 + \beta T_1$ Then, $\|(\alpha T$

(Definition of scalar multiplication)$(x)\|_2(x) + \beta. T_1\|\alpha. T =$

$(x)\|_2(x)\| + \|\beta. T_1\|\alpha. T \geq$

$(x)\|_2(x)\| + |\beta| \|T_1 \alpha| \|T| =$

$\|x\|_2 \|x\| + |\beta| k_1 \alpha| k| \geq$

$) \|x\| = k_2 + |\beta| k_1 \alpha| k|) =$ $\|x\|$ $(_2 + |\beta| k_1 k = |\alpha| k)$

.is bounded$_2$ + βT_1 Hence, αT

$\exists$ $_2 + \beta T_1$ is bounded and linear transformation, then $\alpha T_2 + \beta T_1$ Since αT

$('B(L,$

.2.69Theorem

Let L, L' be normed space. Prove that $B(L, L')$ is a normed space such that $\forall T \in B(L, L')$ we have

$\{1\|T\| = \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_{\leq}$

$('$*Proof.* To prove $\|\ \|$ is a norm on $B(L, L$

.0, then $\|T\| \geq 1 \forall x \in L, \|x\|_X \leq 0$ since $\|T(x)\|_Y \geq$ (1)

$0\} = 1 \Longleftrightarrow \sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq 0$ $\|T\| =$ (2)

$1 \forall x \in L, \|x\|_L \leq 0$ $\|T(x)\|_{L'} =$ $\Rightarrow\Leftarrow$

$1 \forall x \in L, \|x\|_L \leq 0$ $T(x) =$ $\Rightarrow\Leftarrow$

$0T = \hat{}$ $\Rightarrow\Leftarrow$

$('\in B(L, L_2$, T_1Let T(3)

$$\{1)(x)\|_{L'} : x \in L', \|x\|_L \leq {}_2 + \ T_1 \| = \ \sup\{\|(T_2 + \ T_1 \| T$$

$$\{1(x)\|_{L'} : x \in L, \|x\|_L \leq {}_2(x)\|_{L'} + \| T_1 \sup\{\|(T \geq$$

$$\{1(x)\|_{L'} : \|x\|_L \leq {}_2\} + \sup_{x \in L}\{\|(T1(x)\|_{L'} : \|x\|_L \leq {}_1 \sup_{x \in L}\{\|(T \geq$$

$$\|_2\| + \|T_1\|T =$$

$$\{1\|\alpha T\| = \sup\{\|(\alpha.T(x)\|_{L'} : x \in L, \|x\|_L \leq \quad (4)$$

$$\{1\alpha|\sup\{\|T(x)\|_{L'} : x \in L, \|x\|_L \leq | =$$

$$\alpha|\|T\|| = \qquad \square$$

Chapter 3

Banach Space

3.1 Definition.
Let L be a normed space. Then, L is **complete** if every Cauchy sequence in L is convergent to a point in L. The complete normed space is called **Banach space**.

3.1 Examples of Banach Space

3.2 Example.

The space F^n ($F = \mathbb{R}$ or C) with the norm $\|X\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$, $\forall X = (x_1, \ldots, x_n)$ is a Banach space.

Solution: Let X_m be a Cauchy sequence in F^n

$$X = X_m = X_1, X_2, \ldots, X_m, \ldots$$

$$= (x_{11}, x_{12}, \ldots, x_{1n}), (x_{21}, x_{22}, \ldots, x_{2n}), \ldots, (x_{m1}, x_{m2}, \ldots, x_{mn}), \ldots$$

Then $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|X_m - X_j\| < \epsilon$, $\forall m, j > k$ **(I)**

Since $X_m, X_j \in F^n$, then

$$X_m = (x_{m1}, x_{m2}, \ldots, x_{mn}), \; x_{mi} \in F, \; i = 1, \ldots, n$$

$$X_j = (x_{j1}, x_{j2}, \ldots, x_{jn}), \; x_{ji} \in F, \; i = 1, \ldots, n$$

$(, \ldots, x_{mn} - x_{jn} 2 - x_{j2}, x_{m1} - x_{j1}\ X_m - X_j = (x_m$

From **(I)**, $\|X_m - X_j\| < \epsilon \quad m, j > k\forall$

$^2\|X_m - X_j\| \ < \epsilon \qquad ^2 \quad m, j > k\forall$

$\sum_{1 i=}^{n} \quad m, j > k\forall^2 x_{mi}{}^2 - x_{ji}| \ < \epsilon|$

$, \ldots, n1\forall i = \ m, j > k, \forall^2 x_{mi}{}^2 - x_{ji}| \ < \epsilon|$

$, \ldots, n1\forall i = \ m, j > k, \forall x_{mi} - x_{ji}| < \epsilon|$

$., \ldots, n_{mi}1$is a Cauchy sequence in F, $\forall i = \ x$ Hence,

— $., \ldots, n1$Then, x_{mi} is convergent to x_i $\forall i =$

$> |, \exists k_i \in Z_+$ such that $|x_{mi} - x_i 0$Thus, for any $\epsilon > \ \sqrt{\epsilon}_n \quad m_i > k_i\forall$

$, \ldots, k_n\}$. Then$_1$Put $l = \max\{k$

$x_{mi} - x_i| < \sqrt{\frac{\epsilon}{n}} \quad \cdot m > l\forall \quad , \ldots, n1 i = \forall$

$x_{mi} - x_i|^2 < \frac{^2\epsilon}{n} \quad \cdot m > l\forall \quad , \ldots, n1 i = \forall$

$=_m X\| - {}^2\|X \sum_{1 i=}^{n} \ {}_i x \mid {}_{mi} <^2 n - x|\frac{\epsilon^2}{n} \quad m > l\forall$

$.\forall m > l\|X_m - X\| < \epsilon,$

and$_m X$ be a Cauchy sequence in $F X$Thus,$_m$ X. Thus, F^n is a $\rightarrow$

.Banach space

.3.3Example

= Show that $(F^n, \|\ \|)$ is a Banach space where $F^n = \mathrm{R}^n$ (or C^n and $\|X\|$

$^p \ |\sum_{i=1}^{n} x_i|^{\frac{1}{p}}$ (•. (**H.W**)1or C), $p \geq$)$X = (x, \ldots, x) \in \mathrm{R}\forall$

.3.4Example

$\forall X = \ |, \ldots, |x_n|\},_1$The space R^n (or C^n) with the norm $\|X\| = \max\{|x$

$., \ldots, x_n) \in \mathrm{R}^n$ (or C^n) is a Banach space$_1(x$

Solution: Let X_m be a Cauchy sequence in F^n

$$\dots ‘, \dots, X_{m2}, X_1 X_m = X$$

$$\dots ‘(, \dots, x_{mn2}, x_{m1n}), \dots, (x_{m2}, \dots, x_{22}, x_{21n}), (x_1, \dots, x_{12}, x_{11}x) =$$

$m, j > k\forall, \exists k \in Z_+$ such that $\|X_m - X_j\| < \epsilon 0$Then $\forall \epsilon >$ **(I)**

Since $X_m, X_j \in F^n$, then

$$, \dots, n1i = x_{mi} \in F,, \dots, x_{mn}),_2, x_{m1}X_m = (x_m$$

$$, \dots, n \; X_m \; 1i = x_{ji} \in F,, \dots, x_{jn}),_2, x_{j1}X_j = (x_j$$

$$(, \dots, x_{mn} - x_{jn2} - x_{j2}, x_{m1} - x_{j1} - X_j = (x_m$$

$‘|, \dots, |x_{mn} - x_{jn}|\} < \epsilon_1 - x_{j1}$ Then, $\|X_m - X_j\| = \max\{|x_m .m, j > k\forall$

$., \dots, n$ and $\forall m, j > k1i = \forall$It follows that $|x_{mi} - x_{ji}| < \epsilon$

is a Cauchy sequence in R (or C). So it is convergent to $x\,x$ Hence,$_i$

$., \dots, n1i = \forall$in F

$, \exists k_i \in Z_+$ such that $|x_{mi} - x_i| < \epsilon 0$Hence, for any $\epsilon >$ $m_i > k_i\forall$

$0, \dots, k_n\}$. Then, for each $\epsilon >$ $_1$Put $l = \max\{k$

$$, \dots, n1\forall i = m > l, \forall x_{mi} - x_i| < \epsilon|$$

$|, \dots, |x_{mn} - x_n|\} < \epsilon_1 - x_1$, $\|X_m - X\| = \max\{|x_m 0$for each $\epsilon >$ $m > l\forall$

.Thus, X_m be a Cauchy sequence in R^n (or C^n) and $X_m \to X$

.Thus, R^n (or C^n) is a Banach space

.3.5Example

$\ni \forall f$The space $C[a, b]$ with the norm $\|f\| = \max\{|f(x)| : x \in [a, b]\}$

$.C[a, b]$ is a Banach space

[**Solution:** Let f_n be a Cauchy sequence in $C[a, b$

(I) $n, m > k\forall, \exists k \in Z_+$ such that $\|f_n - f_m\| < \epsilon 0$Then $\forall \epsilon >$

$> \{[, \exists k \in Z_+$ such that $\max\{|f_n(x) - f_m(x)| : x \in C[a, b 0$Hence, $\forall \epsilon >$

$\epsilon \quad \forall n, m > k$

It follows that $|f_n(x) - f_m(x)| < \epsilon \quad \forall x \in C[a,b] \quad \forall n, m > k$

.Hence, $f_n(x)$ is a Cauchy sequence in R

,Since R is a Banach space, then $f_n(x)$ is convergent to $f(x)$ in R. Thus

$n \geq k \forall, \exists k \in N$ such that $|f_n(x) - f(x)| < \epsilon 0 \epsilon > \forall$

$n \geq k \forall$Thus, $\|f_n - f\| = \max\{|f_n(x) - f(x)| : x \in [a,b]\} < \epsilon$

.Hence, $f_n \to f$ as $n \to \infty$. Thus, $C[a,b]$ ia a Banach space

.3.6Example

.$|f(x)|\,dx$ is not a Banach space] with $\int_0^1$ the norm $\|f\| = 1,0$The space $C[$

.(2.9], $\|\ \|$) is a normed space (see, Example 1,0**Solution:** The space ($C[$

Let

$$f_n(x) = \begin{cases} 1 & \text{if } \ \frac{1}{2} \geq x \geq 0 \\ 1n + \frac{1}{2}nx + - & \text{if } \ \frac{1}{2} + \frac{1}{n} \geq x > \frac{1}{2} \\ 0 & \text{if } \ \frac{1}{2} + \frac{1}{n} < x \leq 1 \end{cases}$$

Now, for all].1,0Then, f_n is continuous function on [.2where $n \geq$

we have2 $n, m \geq$

$$\|f_n - f_m\| = \int_0^1 |f_n(x) - f_m(x)|\, dx$$

$$= \int_0^{\frac{1}{2}} |f_n(x) - f_m(x)|\, dx + \int_{\frac{1}{2}}^1 |f_n(x) - f_m(x)|\, dx$$

$$= \int_0^{\frac{1}{2}} |1 - 1|\, dx + \int_{\frac{1}{2}}^1 |f_n(x) - f_m(x)|\, dx$$

$$= \int_{\frac{1}{2}}^1 |f_n(x) - f_m(x)|\, dx$$

$$\int_{\frac{1}{2}}^{1} + f_n(x)| \ dx| \geq \int_{\frac{1}{2}}^{1} \quad \textbf{(I)} \quad f_m(x)| \ dx|$$

$$= \int_{\frac{1}{2}}^{1} f_n(x)| \ dx| \text{But} \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} + dx \frac{n}{2}(1 + nx + -) \int_{\frac{1}{2}+\frac{1}{n}}^{1} dx \, 0$$

$$= \frac{nx}{2} {}^{2}nx - \frac{x}{2} + + \Big|_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}}$$

$$1 \quad \frac{1}{n} \frac{n}{2} - \frac{=}{2} = (\frac{2}{2} + \quad - \frac{1}{n}) + \frac{1}{2} \frac{n}{2}(+ \frac{}{2}) \quad - \frac{1}{n} + \frac{1}{2}(+) \ \frac{1}{n2} \qquad \textbf{(II)}$$

,Similarly $\int_{\frac{1}{2}}^{1} = f_m(x)| \ dx| \ \frac{1}{n2}$ **(III)**

Substitute (**II**) and (**III**) in (**I**) to get $\| \ _m0 \rightarrow _n \ + \ f \frac{1}{m2} - \frac{1}{n2} f \| \leq$

$\infty \rightarrow$ as m, n

Thus, $\langle f_n \rangle$ is a Cauchy sequence. From the definition of f_n, we note that

$f_n \rightarrow g$ where

$$= (g(x \quad \begin{matrix} \geq \leq 1 x 0 \text{ if } \frac{1}{2} \\ 0 \text{ if } \frac{1}{2} . 1 x \leq \geq \end{matrix}$$

]. Then1,0But g is not continuous. Thus, $\langle f_n \rangle$ does not converge in C[

.], $\| \ \|$) is not a Banach space1,0C[)

Some Properties of Banach Space3.2

.3.7Theorem

Let L be a Banach space and let H be a subspace of L. Then, H is a .Banach space if and only if H is a closed set in L

Proof. ⇒) If H is a Banach space T.P. $\bar{H} = H$. We know that $\bar{H} \subseteq H$ ‾, $\exists\langle x_n\rangle \in H$ such that $x_n \to x$2.34Let $x \in H$, then by Theorem Hence, $\langle x_n\rangle$ is a Cauchy sequence in H. Then, $\exists y \in H$ such that $x_n \to y$ Thus, $x \in H$ (i.e., $H \subseteq H$). Thus, $x_n \to x$ and $x_n \to y$, so $x = y$.

— .(Therefore, $H = H$ (i.e., H is closed

If H is a closed set. Let $\langle x_n\rangle$ be a Cauchy sequence in H, so that $\langle x_n\rangle$ (⇐ is a Cauchy sequence in L. Hence, it converges; that is $\exists x \in L$ such that , $x \in H = H$. 2.34$x_n \to x$. But $\langle x_n\rangle$ is a sequence in H. By Theorem .i.e., $x \in H$. Thus, H is a Banach space

□

.3.8Theorem

.Every finite dimensional normed space is a Banach space

.3.9Corollary

.Every finite dimensional subspace of a Banach space is closed set

Proof. Let L be a Banach space and let H be a finite dimensional subspace , H 3.7, H is a Banach space. From Theorem 3.8of L. Then, by Theorem .is a closed set

□

3.10 Definition. *Quotient Space*

Let L be a linear space over F. Let H be a subspace of L.

Let $L/H = \{x + H : x \in L\}$

Define addition and scalar multiplication by

$$(x_1 + H) + (x_2 + H) = (x_1 + x_2) + H \quad \forall\, x_1 + H,\ x_2 + H \in L/H$$

$$\alpha.(x_1 + H) = \alpha.x_1 + H \quad \forall\, x_1 + H \in L/H \text{ and } \forall \alpha \in F.$$

3.11 Proposition.

Prove that $(L/H, +, .)$ is a linear space over F. (**H.W**.)

.3.12Theorem

.Let $(L, \| \|)$ and $H \subseteq L$ be a closed set Then $(L/H, +, .)$ is a normed

where$_1$ space with $\| \|$

.$\{= \inf\{\|x + y\| : y \in H_1 \ \|x + H\|$

$0 \geq {}_1$) T.P. $\|x + H\|$1*Proof.* (

For any $x + H \in L/H$

$\forall y \in H 0 \ \|x + y\| \geq$

$0\|x + y\| : y \in H\} \geq \}$

$0 = \inf\{\|x + y\| : y \in H\} \geq {}_1 \|x + H\|$

${}_{L/H}0 \Leftarrow\Rightarrow x + H = H = 0 = {}_1$ T.P. $\|x + H\|$ (2)

$0 = \Rightarrow \inf\{\|x + y\| : y \in H\} = 0 = {}_1$ If $\|x + H\|$ $(\Rightarrow)$

as0 . Hence, $x + y_n \to 0$as $n \to 0$ Hence, $\exists \langle y_n \rangle \in H$ such that $\|x + y_n\| \to$

.$\infty \to n$

Thus, $y_n \to -x$. Thus, $\exists \langle y_n \rangle \in H$ such that $y_n \to -x$. Thus, by Theorem

—

.,$-x \in H$2.34

.Since $\overline{H}$ is closed, then $-x \in H = H$, i.e., $-x \in H$

${}_{L/H}0$Since H is a subspace then $x \in H$ and $x + H = H$, that is, $x + H =$

$y \in H \forall_{L/H}$ then $x \in H$. i.e., $x + H \in H0$. $(\Leftarrow)$ If $x + H = H =$

$\{= \inf\{\|x + y\| : y \in H\} = \inf\{\|z\| : z \in H_1$ Hence, $\|x + H\|$

.$0 = {}_1$. Thus, $\|x + H\|0$, so $\inf\{\|z\| : z \in H\} = 0\| = 0 \in H$ and $\|0$ Since

${}_1 = |\alpha| \|x + H\|_1$ T.P. $\|\alpha(x + H)\|$ (3)

) holds 3then (0 If $\alpha =$

then0 If $\alpha \neq$

$\{= \inf\{\|\alpha(x + y)\| : y \in H_1 \ \|\alpha(x + H)\|$

$\{\inf\{|\alpha| \, \|x + y\| : y \in H =$

$\{\alpha| \inf\{\|x + y\| : y \in H| =$

((If A is bounded below, then $\inf(\alpha A) = \alpha \inf(A)$

$_1\alpha| \, \|x + H\|| =$

$+ H \in L/H_2 + H, x_1$ Let x (4)

$_1) + H\|_2 + x_1 = \|(x_1 + H)\|_2 + H) + (x_1 \, \|(x$

$\{+ y\| : y \in H_2 + x_1 \inf\{\|x =$

$\{\in H_2 , z_1\| : z_2 + z_1 + z_2 + x_1 \inf\{\|x =$

$\{\in H_2 , z_1\| : z_2 + z_2 \| + \|x_1 + z_1 \inf\{\|x \geq$

$\{\in H_2 \| : z_2 + z_2 \in H\} + \inf\{\|x_1 \| : z_1 + z_1 \inf\{\|x =$

$_1 + H\|_2 + \|x_1 + +H\|_1 \, \|x =$

.Thus, L/H is a normed space□

.3.13Proposition

.If $(L, \| \, \|)$ is a Banach space and H is a closed subspace of L Then

.) is a Banach space$_1 L/H, \| \, \|)$

.*Proof.* $L/H = \{x + H : x \in L\}$. Let X_n be a Cauchy sequence in L/H

$\forall n \in N$Then, $X_n = x_n + H$, where $x_n \in L$,

$< \epsilon \; \forall n, m > k_1$, $\exists k \in Z_+$ such that $\|X_n - X_m\|0\epsilon > \forall$

$< \epsilon \; \forall n, m > k_1$, $\exists k \in Z_+$ such that $\|x_n - x_m + H\|0$so, $\forall \epsilon >$

, $\exists k \in Z_+$ such that0Then, $\forall \epsilon >$

$\inf\{\|x_n - x_m + y\| : y \in H\} < \epsilon \; \forall n, m > k$

This implies, $\forall y \in H$, $x_n + y$ is a Cauchy in L

Since L is a Banach space,then $\exists z \in L$ such that $x_n + y \to z = (z - y) + y$

$w + y \ \forall y \in H =$

.Thus, $x_n + H \to w + H$. Thus, L/H is a Banach space□