# Chapter 4

## Inner Product Space

**.4.1Definition**

Let $L$ is a linear space over $F$. A mapping $\langle\ ,\rangle : L \times L \to F$ is called an

**inner product on** $L$ if the following axioms hold

$. x \in L \forall 0 x, x\rangle \geq \rangle$ (1)

$.0 \Longleftrightarrow\ x = 0\ \ x, x\rangle = \rangle$ (2)

——.$\langle \forall x, y \in L$, where $\langle x, y\rangle$ =conjugate of $\langle x, y x, y\rangle = \langle y, x\rangle\rangle$ (3)

$. \forall x, y, z \in L \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle\rangle$ (4)

$L, \langle\ ,\rangle)$ is called **inner product space** (briefly, I.P.S) or **Pre-Hilbert**)

**.space**

**.4.2Remark**

$\forall x, y \in L.$ ) becomes $\langle x, y\rangle = \langle y, x\rangle 3 F =$ R then axiom (If(1)

.)Every subspace of inner product space is an inner product space2(

## Examples of Inner Product Space4.1

**.4.3Example**

$_1 y_1 \rightarrow$ $F$ is defined as $\langle X, Y\rangle = x^2 \times \mathbb{R}^2$ and let $\langle\ ,\ \rangle : \mathbb{R}^2$ Let $L = \mathbb{R}$

). Show that $\langle\ ,\ \rangle$ is $_2, y_1), Y = (y_2, x_1$ where $X = (x^2\ \forall X, Y \in \mathbb{R}_2\ y_2 +\ \ x$

.$^2$an inner product on R

**Solution:** (i) We check the I.P.S axioms

$^2) \in \mathbb{R}_2, x_1 X = {}^2_2(x \forall {}^2_1 0 X, X\rangle = x\ + x\ \geq \rangle$ (1)

$, 0 \Longleftarrow\Longrightarrow\ X = (0\ \Longleftarrow\Longrightarrow\ x^2_2 = {}^2_1 x\ = 0 \Longleftarrow\Longrightarrow\ x_2 + {}_1 x\ = 0\ X, X\rangle = \rangle$ (2)

$(0$

$((\text{since } \overline{F} = \mathbb{R} = \langle X, Y\rangle_2\ y_2 +\ x_1\ y_1 X, Y\rangle = x\rangle$ (3)

$(_2, z_1), Z = (z_2, y_1), Y = (y_2, x_1$ Let $\alpha, \beta \in \mathbb{R}$ and let $X = (x$ (4)

$\langle(_2, z_1), (z_2 +\ \beta y_2\ , \alpha x_1 +\ \beta y_1\ \alpha X + \beta Y, Z\rangle = \langle(\alpha x\rangle$

$_2) z_2 +\ \beta y_2 + (\alpha x_1\ ) z_1 +\ \beta y_1\ \alpha x) =$

$(_2 z_2 +\ \beta y_1\ z_1) + (\beta y_2 z_2 +\ \alpha x_1\ z_1 \alpha x) =$

$(_2 z_2 +\ y_1\ z_1) + \beta(y_2 z_2 +\ x_1\ z_1 \alpha(x =$

$\langle \alpha\langle X, Z\rangle + \beta\langle Y, Z =$

.$^2$Thus, $\langle\ ,\ \rangle$ is an inner product on R

**:4.3As an application to Example**

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). Find $\langle X, Z\rangle, \langle X, X\rangle, \langle X +\ Y, 4, 3), Z = (3, -0), Y = (1, 2$Let $X = ($

$.\langle Z$

**.4.4Remark**

, let $L = \mathrm{R}^n$ and $\langle\ ,\rangle : \mathrm{R}^n \times \mathrm{R}^n \to F$4.3As a generalization of Example

where $\ni$ $X, Y\forall + \dots + x_n y_n$2 $y_2 +$ $x_1$ $y_1$is defined as $\langle X, Y\rangle$ R$=$ $x$

$, \dots, y_n)$. Then, $(\mathrm{R}^n, \langle\ ,\rangle)$ is an inner product $_1, \dots, x_n), Y = (y_1 X = (x$
space

.**check!**). The space $(\mathrm{R}^n, \langle\ ,\rangle)$ is called **usual inner space**)

**.4.5Example**

., which of the following is an inner product on $L^2$Let $L = \mathrm{R}$

(.**H.W**)$_2 y_2 +$ $x_1$ $y_1 x$3i) $\langle X, Y\rangle =$ )

ii) $\langle X, Y\rangle = x_1^2 y_1^2 + x_2^2)_2^2$
$y$

$(_2, y_1), Y = (y_2, x_1$where $X = (x$

__**Solution:**__ (i) We check the I.P.S axioms

ii) The first three axioms of the definition of inner product hold but the )

.forth condition does not satisfy

). Then2, 2), $Z = (-0, 1), Y = (-1, -1$and let $X = (1$ If $\alpha = \beta =$

$4 = {}^2 2^2\ )1 + (-^2\ )2(-0^2)\rangle = 2, 2), (-1, -0\alpha X + \beta Y, Z\rangle = \langle(\rangle$

$\langle(2, 2), (-0, 1)\rangle + \beta\langle(-2, 2), (-1, -1$and $\alpha\langle X, Z\rangle + \beta\langle Y, Z\rangle = \langle($

$12 = 0^2.2 + {}^2.2^2(1-) + {}^2.2^2(1-) + {}^2(2-).1^1 =$

$.\langle\alpha\langle X, Z\rangle + \beta\langle Y, Z$Thus, $\langle\alpha X + \beta Y, Z\rangle \neq$

**.4.6Example**

Let $L = F^n$ be a linear space and let $\langle\ ,\rangle : F^n \times F^n \to F$ defined as

$\langle X, Y\rangle \sum_{1 i=}^{n}$ $.\bar{\ }_i \dots, y_n 1, \dots, x_n), Y = (y_1 X, Y \in F^n$ where $X = (x \forall x_i y$

.Show that $\langle\, ,\rangle$ is an inner product on $F^n$

**:<u>Solution</u>**

$\langle X, X\rangle$ (1) $\sum_{1 i=}^{n} = \bar{x}_i\, x \;\sum_{1 i=}^{n} 0 x_i|^2 \geq |$

$=$

$0 X, X\rangle = \rangle$ (2) $\sum_{1 i=}^{n}, \ldots, n 1 \forall i = 0 \iff x_i = 0 \;\; x_i|^2 = |$

$\Rightarrow\Leftarrow$

$_{F^n}0) = 0, \ldots, 0, \ldots, x_n) = (_1 X = (x \;\; \Rightarrow\Leftarrow$

$\langle \overline{X, Y}\rangle$ (3) $\overline{\sum_{i}} = \overline{\sum_{1 i=}^{n}} x_i y \;\sum_{1 i=}^{n} \equiv_i x\, y \;\sum_{1 i=}^{n} \langle \bar{y}_i x_i = \langle Y, X$

$=$

Let $\alpha, \beta \in F$ and let $X, Y, Z \in F^n$ (4)

$, \ldots, \alpha x_n + {}_1 + \beta y_1 \;\alpha X + \beta Y = (\alpha x$

$(\beta y_n$

$\langle \alpha X + \beta Y, Z\rangle \;\sum_{1 i=}^{n} \alpha x_i + \beta y_i) \bar{z}_i = ) \;\sum_{1 i=}^{n} x_i \bar{z}_i + \beta \;\sum_{1 i=}^{n} - \; y_i z_i = \alpha \langle X,$

$= \qquad \alpha \qquad + \langle Z$

$.\langle \beta \langle Y, Z$

.Thus, $\langle\, ,\rangle$ is an inner product on $F^n$

**<u>:4.6As an application to Example</u>**

$\langle$ Let $L =^2 C$ and $\langle X, Y \sum = \;), \bar{Y}_2, x_1$ where $X = (x^2\; X, Y \in C \forall x_i y_{i1} \;{}^{2}_{i=}$

$=$

$(+ i1, 2 - i), Z = (1 + i, 1 + i), Y = (1\, i, 3 + 2\,)$. If $X = (_2, y_1 y)$

$\langle$Find $\langle X, X\rangle, \langle X + Y, Z\rangle, \langle X, Y + Z$

$(+ i1 + i)(1\, i) + (3 + \overline{2\, i)}(3 + 2$ **<u>Solution:</u>** $\langle X, X\rangle = ($

$(- i1 + i)(1\, i) + (3 - 2\, i)(3 + 2) =$

$15 = (1 + 1) + (9 + 4) =$

$(2i, 4 + 3\; X + Y = ($

$i6 + 8 - i) = 1\,(2i) + 8 + 6 + \overline{i) =} (1\,(2 + 2\, i)4 + 3\; X + Y, Z\rangle = ()$

$= \langle X, Y + Z\rangle$

**.4.7Example**

-] be a linear space over R, and let $\langle\ ,\ \rangle : L \times L \to$ R is de1,0Let $L = C[$

$.f(x)g(x)\ dx.$ Prove that $\langle\ ,\ \rangle$ is an inner product on $L$fined by $\langle f, g\rangle = \int_0^1$

$= \langle\rangle\ \langle f, f$1**Solution:** ( $\int_0^1$ 0$[f(x)]$ $dx \geq \int_0^1 f(x)f(x)\ dx =$ 2

$\Rightarrow\Leftarrow$ 0$f, f\rangle = \rangle$ (2) $\int_0^1$ [1,0$\forall x \in$ [0 $=$ 2 $\Longleftrightarrow$ $[f(x)]$0 $dx =$ 2 $f(x)]]$

0] $\Longleftrightarrow$ $f = \hat{1}$,0$\forall x \in$ [0 $f(x) =$ $\Rightarrow\Leftarrow$

Let $\alpha, \beta \in$ R and $f, g, h \in$ (3) $L$

$(\alpha f + \beta g)(x)h(x)\ dx\alpha f + \beta g, h\rangle = \rangle \int_0^1$

$\int_0^1 \alpha f(x) + \beta g(x)\ h(x)\ dx) =$

$\alpha = \int_0^1 f(x)\ h(x)\ dx + \beta \int_0^1 g(x)\ h(x)\ dx$

$\langle\alpha\langle f, h\rangle + \beta\langle g, h =$

$= \langle f, g\rangle$ (4) $\int_0^1 f(x)g(x)\ dx = \int_0^1 g(x)f(x)\ dx = \langle g, f \langle$

**:4.7As an application to Example**

] $1, 0x \in [\forall 2x + 3h(x) = ,^2 g(x) = x, 1$Let $f(x) = x +$

$\langle g, h\rangle, \langle f - g, h - g 3f + 2$Find $\langle f, f\rangle, \langle f + g, h\rangle, \langle f, h\rangle, \langle$

**.4.8Example**

Let $X = R$ and $\langle \, , \, \rangle : R \times R \to R$ such that $\langle x, y\rangle =$ $\quad x, y \in R.$ $\forall$
$||xy$ Is

$(.X, \langle \, , \, \rangle)$ I.P.S? (**H.W**)

**.4.9Theorem**

Let $(L, \langle\ ,\ \rangle)$ be an inner product space (I.P.S). Then, $\forall x, y, z \in L$

$0\mathbf{x}, x\rangle = \mathbf{0x}\rangle = \langle \mathbf{0}x, \rangle$ (1)

$.\langle \overline{x}, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ (2)

$\langle \mathbf{x}, x\mathbf{0x} + \mathbf{0x}, x\rangle = \langle \mathbf{0}\rangle$ $\langle 1$ *Proof.* (

$\langle \mathbf{x}, x\mathbf{0x}, x\rangle + \langle \mathbf{0}\rangle =$

$\langle \mathbf{x}, x\mathbf{0x}, x\rangle + \langle \mathbf{0} = \langle 0\ \mathbf{x}, x\rangle + \mathbf{0}$Hence, $\langle$

(I) Now, $\mathbf{x}, x\rangle\mathbf{0} = \langle 0$ Thus,

$\langle \mathbf{x0x}, x\rangle = \langle x, \mathbf{0}\langle$

$\langle \mathbf{x0}x, \rangle = 0$

$\langle \mathbf{x0}x, \rangle = 0$

$\langle x, \alpha y + \beta z\rangle = \langle \alpha y + \beta z, x\rangle$ (2)

$\langle \alpha\langle y, x\rangle + \beta\langle z, x =$

$\langle \alpha\ \langle y, x\rangle + \beta\ \langle z, x =$

$\langle \alpha\ \langle x, y\rangle + \beta\ \langle x, z =$ □

**.4.10Corollary**

If $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

(i) $\sum_{1 i=}^{n} = \alpha\ x\ , y\ \sum_{1 i=}^{n} \alpha_i\ x_i, y\ , \ldots, x_n, y \in L_1$where $x$

ii) $x, \sum )_{1 i=}^{n} = _i \beta\ y\ \sum_{1 i=}^{n} x, y\beta_i\ , \ldots, y_n \in L_1$where $x, y$

(iii) $\sum_{i=1}^{n} \langle \alpha_i x_i , \sum_{j=1}^{m} \beta_j y_j \rangle = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \bar{\beta}_j \langle x_i , y_j \rangle$

$, ..., y_m \in L_1, ..., x_n, y_1$ where $x$

.*Proof.* (i) We proof using induction

((by definition of norm, $y\rangle_1 \langle x_1, y\rangle = \alpha_1 x_1$ then $\langle \alpha 1$ If $n =$

(by definition of $, y\rangle_2 \langle x_2, y\rangle + \alpha_1 \langle x_1, y\rangle = \alpha_2 x_2 + \alpha_1 x_1$ then $\langle \alpha 2$ If $n =$

(norm

Suppose (i) hold when $n = k$

$$\sum_{i=1}^{k} \langle \alpha_i x_i , y \rangle = \sum_{i=1}^{k} \alpha_i \langle x_i , y \rangle \qquad \textbf{(I)}$$

1To prove (i) hold when $n = k +$

$$.T.p \quad \sum_{i=1}^{k+1} \langle \alpha_i x_i , y \rangle = \sum_{i=1}^{k+1} \alpha_i \langle x_i , y \rangle$$

$$\sum_{i=1}^{k+1} \langle \alpha_i x_i , y \rangle = \langle \sum_{i=1}^{k} \alpha_i x_i + \alpha_{k+1} x_{k+1} , y \rangle$$

$$= \langle \sum_{i=1}^{k} \alpha_i x_i , y \rangle + \langle \alpha_{k+1} x_{k+1} , y \rangle$$

$$= \sum_{i=1}^{k} \alpha_i \langle x_i , y \rangle + \alpha_{k+1} \langle x_{k+1} , y \rangle$$

$$= \sum_{i=1}^{k+1} \alpha_i \langle x_i , y \rangle$$

.(ii) The proof is similar to the proof of (i)

$=$ iii) Let $z) \sum_{j=1}^{m} \beta_j y_j$

$$\sum_{i=1}^{n} \langle \alpha_i x_i , \sum_{j=1}^{m} \beta_j y_j \rangle = \sum_{i=1}^{n} \langle \alpha_i x_i , z \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \langle x_i , z \rangle \qquad \text{((by part (i)}$$

$$= \sum_{i=1}^{n} \alpha_i \langle x_i , \sum_{j=1}^{m} \beta_j y_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \overline{\beta_j} \langle x_i, y_j \rangle \quad \text{(by part (ii))}$$ □

**4.11. Theorem**

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. such that $\langle v_1, w\rangle = \langle v_2, w\rangle$, $\forall w \in L$. Then $v_1 = v_2$. Also, if $\langle v, w\rangle = 0$, $\forall w \in L$ then $v = 0$.

*Proof.* By assumption, $\langle v_2 - v_1, w\rangle = \langle v_2, w\rangle - \langle v_1, w\rangle = 0$, $\forall w \in L$.

Put $w = v_2 - v_1$, then $\langle v_2 - v_1, v_2 - v_1\rangle = 0 \Longrightarrow v_2 - v_1 = 0 \Longrightarrow v_1 = v_2$.

Now, $\langle v, w\rangle = 0$, $\forall w \in L \Longrightarrow \langle v, v\rangle = 0 \Longrightarrow v = 0$. □

**4.12. Theorem. *General Cauchy Schwarz's Inequality***

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. and let $\|\ \| : L \to \mathbb{R}$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$, $\forall x \in L$. Then

$$|\langle x, y\rangle| \le \|x\|\,\|y\| \quad \forall x, y \in L.$$

*Proof.* If $x = 0$ or $y = 0$ then $\langle x, y\rangle = 0$, and hence $|\langle x, y\rangle| = 0 \le \|x\|\,\|y\|$

If $y \neq 0$, put $z = \dfrac{y}{\|y\|}$ (**I**)

$$\|z\|^2 = \langle z, z\rangle = \left\langle \frac{y}{\|y\|}, \frac{y}{\|y\|} \right\rangle$$

$$= \frac{1}{\|y\|^2}\langle y, y\rangle = \frac{1}{\|y\|^2}\|y\|^2 = 1 \quad \text{(II)}$$

Next, it is enough to show that $|\langle x, z\rangle| \le \|x\|$

because if $|\langle x, z\rangle| \le \|x\|$ then from (**I**)

$$\left|\left\langle x, \frac{y}{\|y\|}\right\rangle\right| = \frac{1}{\|y\|}|\langle x, y\rangle| \le \|x\|$$

$$\therefore |\langle x, y\rangle| \le \|x\|\,\|y\|$$

Let $\alpha \in F$ then $\langle x - \alpha z, x - \alpha z\rangle \ge 0$

$0 \le \langle x, x\rangle - \bar{\alpha}\langle z, x\rangle - \alpha\langle x, z\rangle + \alpha\bar{\alpha}\langle z, z\rangle$

$0 \le \|x\|^2 - \bar{\alpha}\langle z, x\rangle - \alpha\langle x, z\rangle + \alpha\bar{\alpha}$ (from (I))

$0 \le \|x\|^2 - \langle x, z\rangle\overline{\langle x, z\rangle} + \langle x, z\rangle\overline{\langle x, z\rangle} - \alpha\overline{\langle x, z\rangle} - \bar{\alpha}\langle z, x\rangle + \alpha\bar{\alpha}$

$0 \le \|x\|^2 - |\langle x, z\rangle|^2 + \langle x, z\rangle(\overline{\langle x, z\rangle} - \bar{\alpha}) - \alpha(\overline{\langle z, x\rangle} - \bar{\alpha}))$

$0 \le \|x\|^2 - |\langle x, z\rangle|^2 + \langle x, z\rangle(\overline{\langle x, z\rangle} - \bar{\alpha}) - \alpha(\overline{\langle x, z\rangle} - \bar{\alpha}))$

$0 \le \|x\|^2 - |\langle x, z\rangle|^2 + (\langle x, z\rangle - \alpha)\overline{(\langle x, z\rangle - \alpha)}$

$0 \le \|x\|^2 - |\langle x, z\rangle|^2 + |\langle x, z\rangle - \alpha|^2$ **(III)** $(\forall \alpha \in F)$

Put $\alpha = \langle x, z\rangle$, then (**III**) becomes

$0 \le \|x\|^2 - |\langle x, z\rangle|^2 \Longrightarrow |\langle x, z\rangle|^2 \le \|x\|^2$

$\|x\| \ge |\langle x, z\rangle|$

$\|x\| \ge \left|\left\langle x, \frac{y}{\|y\|}\right\rangle\right|$ ((using (**I**))

$\|x\| \ge \frac{1}{\|y\|}|\langle x, y\rangle|$

$|\langle x, y\rangle| \le \|x\|\,\|y\|.$ □

**As an application to Theorem 4.12:**

If $L = \mathbb{R}^n$, and $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$ for any $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$.

Apply Cauchy Schwarz inequality.

**Sloution:** We have $\|X\| = \langle X, X\rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ and $\|Y\| = \langle Y, Y\rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}$

From Theorem 4.12, $|\langle X, Y\rangle| \le \|X\|\,\|Y\|$; that is

$$\left|\sum_{i=1}^{n} x_i y_i\right| \le \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}$$

## .4.13Theorem

.Every inner product space is a normed space and hence a metric space

*Proof.* Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. and let the function $\|\ \| : L \to \mathbb{R}$ is defined

by

$x \in L$. To prove $\|\ \|$ is a norm on $L \forall \|x\| = \sqrt{\langle x, x \rangle}$

$x \in L \forall 0 \; x \in L \implies \|x\| = \sqrt{\langle x, x \rangle} \geq \forall 0$ Since $\langle x, x \rangle \geq$ (1)

$\iff x = 0 \iff \sqrt{\langle x, x \rangle} = 0 \iff \langle x, x \rangle = 0 \; \|x\| =$ (2)
x0

Let $\alpha \in F$ and $x \in L$ (3)

$^2\|x\|^2 = \langle \alpha x, \alpha x \rangle = \alpha\bar{\alpha}\langle x, x \rangle = |\alpha|^2 \; \|\alpha x\|$

Thus, $\|\alpha x\| = |\alpha| \, \|x\|$

$x, y \in L \forall$T.P. $\|x + y\| \leq \|x\| + \|y\|$ (4)

$= \langle x + y, x + y \rangle^2 \; \|x + y\|$

$\langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle =$

$^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2 \; \|x\| =$

$^2 Re \langle x, y \rangle + \|y\| 2 + {}^2 \; \|x\| =$

$\|y\|^2 + \langle x, y \rangle 2\|x\| + \geq^2$

$\|x\| \, \|y\| + \|y\| 2 \, \|x\|^2 + \geq$ (by Cauchy Schwarz)

$^2 \|x\| + \|y\|)) =$

.Thus, $\|x + y\| \leq \|x\| + \|y\|$ □

### 4.14 Theorem

Let $(I, \langle\ ,\ \rangle)$ is an I.P.S. and $x, y \in I$. Then

$$\|x+y\|^2 = \|x\|^2 + 2\,Re\langle x,y\rangle + \|y\|^2 \quad (1) \qquad \text{(Polarization Identity)}$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad (2) \qquad \text{(Law of Parallelogram)}$$

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) \quad (3)$$

*Proof.* (1) $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + \overline{\langle x,y\rangle} + \langle x,y\rangle + \|y\|^2$$

$$= \|x\|^2 + 2\,Re\langle x,y\rangle + \|y\|^2$$

(2) T.P. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

By part (1), $\|x+y\|^2 = \|x\|^2 + 2\,Re\langle x,y\rangle + \|y\|^2$ **(I)**

$$\|x-y\|^2 = \langle x-y, x-y\rangle$$

$$= \langle x,x\rangle - \langle y,x\rangle - \langle x,y\rangle + \langle y,y\rangle$$

$$= \|x\|^2 - \overline{\langle x,y\rangle} - \langle x,y\rangle + \|y\|^2$$

$$= \|x\|^2 - 2\,Re\langle x,y\rangle + \|y\|^2 \quad \textbf{(II)}$$

By summing up **(I)** and **(II)** we get $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(3) By parts (1) and (2), we have

$$\|x+y\|^2 - \|x-y\|^2 = \|x\|^2 + 2\,Re\langle x,y\rangle + \|y\|^2 - \|x\|^2 + 2\,Re\langle x,y\rangle - \|y\|^2$$

$Re\ x, y 2 Re\ x, y\ + 2 =$

$x, y\ +\ \ x, y\ +\ \ \overline{x, y}\ +\ \ x, y\ =$ ——

$(\mathbf{I})\qquad x, y 2\ \ y, x\ +\ \ 2 =$

$= \ \ x + iy, x + iy^2\ \|x + iy\|$

$x, x\ + i\ \ y, x\ + i\ \ x, y\ +\ \ y, y\ =$

$^2 + \ i\ \ y, x\ \ -i\ \ x, y\ \ + \|y\|^2\ \|x\| =$

$= \ \ x - iy, x - iy^2\ \|x - iy\|$

$x, x\ \ -i\ \ y, x\ \ -i\ \ x, y\ +\ \ y, y\ =$

$\|y\| \overset{2}{+} \|x\|\ \ -i\ \ y, x\ \ + i\ \ x, y = {}^2$

,Hence we get

$-^2 - i\ \ \|x\|^2\ \ + i\ \ y, x\ \ -i\ \ x, y\ \ + \|y\|^2 = \ i\ \ \|x\|^2\ - i\|x - iy\|^2\ i\|x + iy\|$

$i\ \ y, x\ \ + i\ \ x, y$

$^2\|y\| +$

$+\ \ -\ \ y, x^2 - i\|x\|^2 -\ \ y, x\ \ +\ \ x, y\ \ + i\|y\|^2\ i\|x\| =$

$^2 x, y\ \ -i\|y\|$

$y, x 2\ \ x, y\ \ -\ \ 2 = \qquad (\mathbf{II})$

By (**I**) and (**II**), we have

$x, y\ \ 2\ \ y, x\ \ + \ 2\ = \ ^2\ \ - i\|x - iy\|^2 + \ i\|x + iy\|^2 - \|x - y\|^2\ \ \|x + y\|$

$y, x 2\ \ x, y\ -2\ +$

$x, y4 = {}^2 - i\|x - iy\|^2 + i\|x + iy\|^2 - \|x - y\|^2 \|x + y\|$

$\frac{1}{4}\ \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|\ \ x, yx\ {}^2 - iy\| =$ □

**.4.15Remark**

Any normed linear space generated from inner product space must satisfies

.4.14the three laws of Theorem

**.4.16Example**

Let $L = C[a, b]$ and let $\|f\| = \max\{|f(x)| : x \in [a, b]\}$. Then the converse

,.. i.e4.13of Theorem

Show that ( $L, \|\ \|$) is a normed linear space (**H.W.**) (1)

($L$ is not generated by I.P.S (i.e, $L$ is not I.P.S)Show that2(

-) To show that $L$ is not I.P.S, we shall show that parallelo2**<u>Solution</u>:** (

for $^2$ $\|g\|2\|f\|^2 + 2/=$gram law does not hold. i.e., $\|f + g\|^2 + \|f - g\|^2$

.[some $f, g \in C[a, b$

$b - a$ $[x \in [a, b\forall$ $\frac{x-a}{}$ = (and $g(x1$ Let $f(x) =$

.[Note that $f, g$ are continuous on $[a, b]$. Thus, $f, g \in C[a, b$

1and $\|g\| = 1\ \|f\| =$

$+ 1\|f + g\| = \left\|\frac{x-a}{b-a} + 1\right\| \max = \frac{x-a}{b-a}\ \ 2 = x \in [a, b] :$

$-1\|f - g\| = \left\|\frac{x-a}{b-a} - 1\right\| \max = \frac{x-a}{b-a}\ \ 1 = x \in [a, b] :$

$5 = 1 + 4\ \|f + g\|^2 + \|f - g\|^2 =$ **(I)**

$4 = 2.1^2 + 2.1^2\ \|g\|^2 = 2\|f\|^2 + 2$ **(II)**

$\|g\|^2 2\|f\|^2 + 2$ By (**I**) and (**II**), we get $\|f+g\|^2 + \|f-g\|^2 \neq$

$4 \neq 5$ i.e.,

**.4.17Example**

Then the .$^2$) $\in R_2, x_1 \forall X = (x|_2| + |x_1$and let $\|X\| = |x^2$ Let $L = R$

,.. i.e4.13converse of Theorem

,$\|\ \|$) is a normed linear space (**H.W.**) $^2$Show that ( R(1)

(is not I.P.S$^2$ is not generated by I.P.S (i.e, R$^2$ R)Show that2(

) To show that $L$ is not I.P.S, we shall show that parallelogram 2**Solution:** (

for $^2$ $\|Y\|2 +$ $^2$ $\|X\|2 \neq$ $^2$ + $\|X-Y\|^2$ law does not hold. i.e., $\|X + Y\|$

$^2$some $X, Y \in R$

(1,6) and $Y = (-3,$ 2Let $X = ($

$50\|X\|^2 = 2 \Rightarrow 5\ | = 3| + |2\|X\| = |$

$98\|Y\|^2 = 2 \Rightarrow 7\ | = 1| + |6\|Y\| = |-$

$8| = 4| + |4)\| = |-4, 4\|X + Y\| = \|(-$

$64\|X + Y\|^2 =$

$10| = 2| + |8)\| = |2, 8\|X - Y\| = \|($

$100\|X - Y\|^2 =$

$= 100 + 64 =$ $^2$ $+ \|X - Y\|^2$ Thus, $\|X + Y\|$

$148 = 98 + 50 =$ $^2$ $\|Y\|2 +$ $^2$ $\|X\|2$ and 164

$\|Y2 +$ $^2$ $\|X\|2 \neq$ $^2$ $+ \|X - Y\|^2$ Hence, $\|X + Y\|$

.i.e., $\|\ \|$ does not satisfy paralleogram law$^2\|$

**.4.18Example**

. Then$^2$) ∈ R$_2$, $x_1$∀($x$|}$_2$| , |$x_1$and let ‖$X$‖ = max{|$x^2$ Let $L$ = R

, ‖ ‖) is a normed linear space (**H.W.**) $^2$Show that ( R(1)

(.generated by I.P.S? (**H.W**$^2$ R)Is2 (

**.4.19Theorem**

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

If $(x_n) \to x$ and $(y_n) \to y$ then $\langle x_n, y_n\rangle \to \langle x, y\rangle$ (1)

is a Cauchy $\langle x_n, y_n\rangle$ If $(x_n)$ and $(y_n)$ are Cauchy sequences in $L$ then (2)

. sequence in $F$

$\langle x_n, y_n\rangle = \langle x + (x_n - x), y + (y_n - y)\rangle$ *Proof.* (1)

$= \langle x, y\rangle + \langle x, y_n - y\rangle + \langle x_n - x, y\rangle + \langle x_n - x, y_n - y\rangle$

$\langle x_n, y_n\rangle - \langle x, y\rangle = \langle x, y_n - y\rangle + \langle x_n - x, y\rangle + \langle x_n - x, y_n - y\rangle$

$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x, y_n - y\rangle + \langle x_n - x, y\rangle + \langle x_n - x, y_n - y\rangle|$

$\leq |\langle x, y_n - y\rangle| + |\langle x_n - x, y\rangle| + |\langle x_n - x, y_n - y\rangle|$

By $\leq \|x\|\,\|y_n - y\| + \|x_n - x\|\,\|y\| + \|x_n - x\|\,\|y_n - y\|$

(Cauchy Schwarz)

But $(x_n) \to x$ and $(y_n) \to y$ then $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$

Hence, $|\langle x_n, y_n\rangle - \langle x, y\rangle| \to 0$, and hence, $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

(2) for any $n, m \in Z_+$

$\langle x_n, y_n\rangle = \langle (x_n - x_m) + x_m, (y_n - y_m) + y_m\rangle$

$= \langle x_n - x_m, y_n - y_m\rangle + \langle x_m, y_n - y_m\rangle + \langle x_n - x_m, y_m\rangle + \langle x_m, y_m\rangle$

$\langle x_n, y_n\rangle - \langle x_m, y_m\rangle = \langle x_n - x_m, y_n - y_m\rangle + \langle x_m, y_n - y_m\rangle + \langle x_m, y_n - y_m\rangle$

$|\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| = |\langle x_n - x_m, y_n - y_m\rangle + \langle x_m, y_n - y_m\rangle + \langle x_m, y_n - y_m\rangle|$

$\leq |\langle x_n - x_m, y_n - y_m\rangle| + |\langle x_m, y_n - y_m\rangle| + |\langle x_m, y_n - y_m\rangle|$

By) $\|x_n - x_m\| \, \|y_n - y_m\| + \|x_m\| \, \|y_n - y_m\| + \|x_n - x_m\| \, \|y_m\| \geq$

(Cauchy Schwarz

$\rightarrow$ and $\|y_n - y_m\|$ 0 But $(x_n)$ and $(y_n)$ are Cauchy sequences, then $\|x_n - x_m\| \rightarrow$

$\infty \rightarrow$ as $n \rightarrow \infty$. Also, $(x_n)$ and $(y_n)$ are bounded sequences, then as $n$ 0

0 $x_n, y_n - x_m, y_m$ □ · $\rightarrow$ ·

**.4.20 Corollary**

Let $(L, \langle\ ,\ \rangle)$ is an I.P.S. Then

If $(x_n) \rightarrow x$ then $\|x_n\| \rightarrow \|x\|$ (١)

If $(x_n)$ is a Cauchy sequences in $L$ then $\|x_n\|$ is a convergent se- (٢)
.quence in R

(4.19 By Theorem )) Since $(x_n) \rightarrow x$ then $x_n, x_n \rightarrow x, x$ 1 *Proof.* (

Hence, $\|x_n\|^2 \rightarrow \|x\|^2$ . i.e., $\|x_n\| \rightarrow \|x\|$

,(2(4.19 Since $(x_n)$ is a Cauchy sequences in $L$, then by Theorem (2)

$x_n, x_n$ is a Cauchy sequence in $F$. Since $F =$ R or $C$ then $F$ is complete. Thus, $\|x_n\|^2$ is a convergent sequence in $F$. Thus, $\|x_n\|$ is a convergent sequence in $F$ □

## 4.3 Hilbert Space

**4.21 Definition.**

Hilbert space is an I.P.S. $(L, \langle \,,\, \rangle)$ which is a Banach space with respect to $\|x\| = \sqrt{\langle x, x \rangle}$.

**4.22 Example.**

Consider the I.P.S. $(\mathbb{R}^n, \langle \,,\, \rangle)$ (or $(C^n, \langle \,,\, \rangle)$) such that $\langle X, Y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i$ where $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n) \in \mathbb{R}^n$ (or $C^n$). (see Example 4.6)

Show that $(\mathbb{R}^n, \langle \,,\, \rangle)$ (or $(C^n, \langle \,,\, \rangle)$) is Hilbert space.

**Solution:** Since $\sqrt{\langle X, X \rangle} = \left(\sum_{i=1}^{n} x_i \overline{x}_i\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = \|X\|$,

From Example 3.2, $\mathbb{R}^n$ (or $C^n$) is a Banach space w.r.t. $\|X\| = \sqrt{\langle X, X \rangle}$, and thus, $(\mathbb{R}^n, \langle \,,\, \rangle)$ (or $(C^n, \langle \,,\, \rangle)$) is a Hilbert space.

**4.23 Example.**

The space $C[-1, 1]$ with the inner product defined by $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$ is not a Hilbert space.

**Solution:** Let

$$f_n(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ nx & \text{if } 0 < x < \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

$$\|f_n - f_m\|^2 = \langle f_n - f_m, f_n - f_m \rangle$$

Suppose $n > m$, then $\frac{1}{n} < \frac{1}{m}$. (We must find $f_n(x) - f_m(x)$)

$$f_n(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ nx & \text{if } 0 < x < \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

and

$$f_m(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ mx & \text{if } 0 < x < \frac{1}{m} \\ 1 & \text{if } \frac{1}{m} \le x \le 1 \end{cases}$$

Then

$$(f_n(x) - f_m(x)) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ (n-m)x & \text{if } 0 < x < \frac{1}{n} \\ mx - 1 & \text{if } \frac{1}{n} \ge x \ge \frac{1}{m} \\ 0 & \text{if } \frac{1}{m} \le x \le 1 \end{cases}$$

$$\|f_n - f_m\|^2 = \int_{-1}^{1} (f_n(x) - f_m(x))^2\,dx = \int_0^{\frac{1}{n}} (n-m)^2 x^2\,dx + \int_{\frac{1}{n}}^{\frac{1}{m}} (1 - mx)^2\,dx$$

$$= \left[\frac{(n-m)^2 x^3}{3}\right]_0^{\frac{1}{n}} + \left[\frac{(mx-1)^3}{3m}\right]_{\frac{1}{n}}^{\frac{1}{m}}$$

$$= \frac{1}{3}(n-m)^2\left(\frac{1}{n}\right)^3 + \frac{1}{3m}\left[0 - \left(\frac{m}{n} - 1\right)^3\right]$$

$$= \frac{n-m1}{{}^3n3} \quad \frac{{}^2n-m))}{n\quad m3}(^3$$

$$= \frac{{}^2n-m))}{m^2n3}$$

= Thus, $\|f_n - f_m\|^2 \; \frac{{}^2n-m))}{m^2n3}$

Since $n > m$, then $n = m + t$

$\|f_n - f_m\|^2 \quad \frac{{}^2t_2 \; 0 \to}{m^2m+t))3} = \infty \to$ as $m$

.. Thus, $\langle f_n\rangle$ is a Cauchy sequence0Hence, $\|f_n - f_m\| \to$

But $f_n \to f$ where

$$(f(x = \begin{cases} 0 & \text{if} \quad 0x \le \ \ge 1- \\ 1 & \text{if} \quad 1x \le \ > 0 \end{cases}$$

]. i.e., The1, 1]. Then, $\langle f_n\rangle$ is not convergent in $C[-1,1$Thus, $f \notin C[-$

.space is not Hilbert space

**.4.24Remark**

Every Hilbert space is a Banach space but the converse is not true. For example, the space $C[a, b]$ with $\|f\| = \max\{|f(x)| : x \in [a, b]\}$ is a Banach ). However, $C[a, b]$ is not a Hilbert space since it 3.5space (see Example does not satisfy parallelogram law; that is $\| \ \|$ can not be obtained from .(4.16inner product (see Example

# 4.4 -Orthogonality and Orthonormality in Inner Product Space

**Definition 4.25.** ***orthogonal Elements*** **.**

Let $(L, \langle \, , \rangle)$ be an I.P.S and $x, y \in L$. Then $x$ is said to be **orthogonal** on $y$ (denoted by $x \perp y$) if and only if $\langle x, y \rangle = 0$.

**Example 4.26.**

Let $L = \mathbb{R}^2$ is I.P.S such that $\langle X, Y \rangle = x_1 y_1 + x_2 y_2$ is usual inner product $\forall X = (x_1, x_2), Y = (y_1, y_2) \in \mathbb{R}^2$. Let $X = (-2, 1), Y = (3, 6), Z = (1, -2)$.

Show that $X \perp Z$, $Y \perp Z$ and $Y \not\perp X$.

**Solution:** $\langle X, Z \rangle = \langle (-2, 1), (3, 6) \rangle = 6 + 6 = 0$. Hence, $X \perp Z$.

$\langle Y, Z \rangle =$

$\langle Y, X \rangle =$

**Proposition 4.27.**

Let $(L, \langle \, , \rangle)$ be an I.P.S and $x, y \in L$. Then

(i) If $x \perp y$ then $y \perp x$.

**(ii)** $0_L \perp x \quad \forall x \in L$. **(H.W.)**

(iii) if $x \perp x$ then $x = 0_L$. **(H.W.)**

*Proof.* (i) Let $x \perp y$ then $\langle x, y \rangle = 0$. From Definition (4.13), we have $\langle y, x \rangle = \overline{\langle x, y \rangle} = \bar{0} = 0$, i.e., $y \perp x$. ∎

**.4.28Proposition**

$, \ldots, x_n \in L$ such that $x$ is orthogonal on${}_1$Let $(L, \langle\, ,\rangle)$ be an I.P.S and $x, x$

$., \ldots, x_{n1}, \ldots, x_n$. Prove that $x$ is orthogonal on any linear combination of $x_1 x$

$, \ldots, x_n$. i.e., there exists $\alpha_i \in F_1$*Proof.* Let $w$ be a linear combination of $x$

$=$ such that $w \sum_{i=1}^{n} .0\alpha_i x_i$. We must show $\langle x, w\rangle =$

$x, w\rangle = \rangle \sum_{i=1}^{n} = {}_i\langle {}_i\alpha\, x \sum_{i=1}^{n}$ — (((ii4.9(by Corollary $\alpha_i\langle x, x_i\rangle$
$\langle x$

$= \sum_{i=1}^{n} \overline{0}\alpha_i$. (From the assumption)

$.0 =$ □

**.4.29Example**

$=$ ), $Y$1, −2Find the value of $a$ that makes the vectors $X$ = $(a,$(١)

(.with usual inner product. (**H.W**[3] orthogonal vectors in R (2 ،5− ،3)

1 Let $(L, \langle\, ,\rangle)$ be an I.P.S over R and let $x, y \in L$ such that $\|x\| = \|y\| =$ (٢)

.(i.e., $x$ and $y$ are normal elements). Prove that $x + y \perp x - y$

$+ \langle -\langle x, y^2$ **Answer:** $\langle x + y, x - y\rangle = \langle x, x\rangle - \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle = \|x\|$

.. Hence, $x + y \perp x - y 0 x, \overset{2}{y}\rangle - \|y\| = \rangle$

Let $(L, \langle\, ,\rangle)$ be an I.P.S and let $x, y \in L$ such that $x \perp y$. Prove that (٣)

$. \overset{2}{\|}x + y\| = \overset{2}{\|}x\| + \overset{2}{\|}y\| \overset{2}{=} \|x - y\|$

$\langle = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y^2$ **Answer:** $\|x + y\|$

$+ {}^2 = \|x\|^2 + \|y\| 0 + 0 + {}^2 \|x\| =$

$.^2 + \|y\|^2 = \|x\|^2$ Similarly, $\|x - y\|^2 \|y\|$

**(4)** Let $(L,\langle\ ,\ \rangle)$ be an I.P.S and let $x, y \in L$ such that $x \perp y$. Prove that $\|x+\lambda y\| = \|x-\lambda y\|$.

**Answer: (H.W.)**

**(5)** Let $(L,\langle\ ,\ \rangle)$ be an I.P.S and let $x_1, x_2, \ldots, x_n \in X$ such that $x_i \perp x_j$ $\forall i \neq j$. Prove that $\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$.

**Answer:** We prove using induction. If $n = 1$, the statement is true.

If $n = 2$. Since $x_1 \perp x_2$, then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ (by part (3)).

Suppose the statement is true for $n = k$, i.e., $\|\sum_{i=1}^{k} x_i\|^2 = \sum_{i=1}^{k} \|x_i\|^2$.

To prove the statement is true when $n = k+1$, i.e.,

T.P. $\|\sum_{i=1}^{k+1} x_i\|^2 = \sum_{i=1}^{k+1} \|x_i\|^2$.

$$\Big\|\sum_{i=1}^{k+1} x_i\Big\|^2 = \Big\|\sum_{i=1}^{k} x_i + x_{k+1}\Big\|^2 = \Big\|\sum_{i=1}^{k} x_i\Big\|^2 + \|x_{k+1}\|^2$$

$$= \sum_{i=1}^{k} \|x_i\|^2 + \|x_{k+1}\|^2 \quad \text{(by induction } n = k)$$

$$= \sum_{i=1}^{k+1} \|x_i\|^2.$$

**Definition 4.30. *Orthogonal to Set***

Let $(L,\langle\ ,\ \rangle)$ be an I.P.S, $x \in L$, and $A \subseteq X$. Then, $x$ is said to be orthogonal on $A$ $(x \perp A)$ if $x \perp a$ $\forall a \in A$.

**Example 4.31.**

Consider the space $\mathbb{R}^2$ with usual product space and $A = \{(a, 0) : a \in \mathbb{R}\}$.

Then $(0, 2) \perp A$ because $\langle (0, 2), (a, 0)\rangle = 0.a + 2.0 = 0$.

**. *Orthogonal Sets*4.32Definition**

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S, and $A, B \subseteq L$. Then, $A$ is said to be orthogonal

.$\forall a \in A, \forall b \in B$to $B$ $(A \perp B)$ if $a \perp b$,

**.4.33Example**

$\{, a) : a \in$ R0with usual inner product and $A = \{($2 Consider the space R

.) : $b \in$ R}. Show that $A \perp B$0and $B = \{(b,$

) $\in B$, then0, $a) \in A$ and for each $(b,$0**Answer:** for each (

.. Thus, $A \perp B$0.$b =$ 0+ 0 , $b)\rangle =$ $a$.0), (0$a$, )⟩

**.4.34Proposition**

.{**0**Let $(L, \langle\ ,\ \rangle)$ be an I.P.S, and $A, B \subseteq L$ such that $A \perp B$ then $A \cap B$ = {

*Proof.* Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$ (**I**)

$\forall a \in A, \forall b \in B$. ,0Since $A \perp B \Rightarrow \langle a, b\rangle =$

.0From (**I**), $\langle a, b\rangle = \langle x, x\rangle =$

.{**0**, then $A \cap B$ = {**0**), $x$ = 2(4.1Using Definition □

**.4.35Definition**

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $\varphi \neq A \subseteq L$. Then, the set

$\{\forall a \in A A^{\perp} = \{x \in L : x \perp a,$

.is called the orthogonal complement on $A$

**.4.36Proposition**

,Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $\varphi \neq A, B \subseteq L$. Then

.$\{\mathbf{o}L^{\perp} = \{$ (1)

(.$L$. (**H.W** = $^{\perp}\{\mathbf{o}\}$ (2)

.$\{\mathbf{o}A \cap A^{\perp} = \{$ (3)

.$^{\perp\perp}A \subseteq A$ (4)

$A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$. (**H.W.**) If(5)

.$^{\perp}A \subseteq B^{\perp}$ then $B \subseteq A$)If6(

.$\{\mathbf{o}, \forall l \in L\} = \{0)$ $L^{\perp} = \{x \in L : x \perp L\} = \{x \in L : \langle x, l\rangle =$ 1*Proof.* (

**((I** Let $x \in A \cap A^{\perp} \Rightarrow x \in A$ and $x \in A^{\perp}$ (٣)

**(II)** Since $x \in A^{\perp}$ then $x \perp A$

, thus $x =$ 0From **(I)** and **(II)**, $x \perp x$. i.e., $\langle x, x\rangle =$

.$\{\mathbf{o}$. Then, $A \cap A^{\perp} = \{\mathbf{o}$

.To prove $A \subseteq A^{\perp\perp}$. Let $x \in A$ (٤)

For any $y \in A^{\perp} \Rightarrow y \perp A$. In particular, $y \perp x$ $(x \in A)$

.$^{\perp\perp}\forall y \in A^{\perp}$. Thus, $x \in A$), $x \perp y$,1(4.27From Proposition

), $B^{\perp\perp} \subseteq A^{\perp}$. 5Let $A \subseteq B^{\perp}$, then from part ( (6)

), $B \subseteq B^{\perp\perp} \subseteq A^{\perp}$. Then, $B \subseteq$ 4Now, from part ( □

.$^{\perp}A$

**.4.37Theorem**

$A \subseteq L$. Then, $A^{\perp}$ is a closed subspaceLet $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $\varphi \neq$

.of $L$

.) To prove $A^{\perp}$ is a subspace of $L$1*Proof.* (

$^{\perp}$Let $x, y \in A^{\perp}$ and $\alpha, \beta \in F$. T.P. $\alpha x + \beta y \in A$

.$\forall a \in A$,0T.P. $\langle \alpha x + \beta y, a \rangle =$

(**I**) 0Since $x, y \in A^{\perp} \Rightarrow \langle x, a \rangle = \langle y, b \rangle =$

0= $0 + \beta.0$ $\alpha x + \bar{\beta} y, a \rangle = \alpha \langle x, a \rangle + \bar{\beta} \langle y, a \rangle =$[from (*I*]

.Thus, $A^{\perp}$ is a subspace of $L$

T.P. $A^{\perp}$ is a closed set (i.e., $A^{\perp} \subseteq \overline{A^{\perp}}$ and $\overline{A^{\perp}} \subseteq$ (2)

($^{\perp}A$

((**I** It is clear that $A^{\perp} \subseteq A^{\perp}$

T.P. $\overline{A}^{\perp} \subseteq A^{\perp}$. Let $x \in \overline{A}^{\perp}$ then $\exists (x_n) \in A^{\perp}$ such that $(x_n) \rightarrow$

.$x$

،$^{\perp}$Since $(x_n) \in A$ $\forall a \in A$ $n \in N \Rightarrow x_n \perp A \Rightarrow x_n \perp a, \forall$

.$\forall a \in A$,0$x_n, a \rangle = \rangle \Rightarrow$

.(), $\langle x_n, a \rangle \rightarrow \langle x, a$1(4.19But $(x_n) \rightarrow x$ and $a \rightarrow a$. Thus, from Theorem

0=

$\forall a \in A$. Then, $x \in A^{\perp}$. Thus, $A^{\perp} \subseteq$ 0 $\overline{x}, a \rangle = \rangle$ .(**II**)

$^{\perp}A$

.From (**I**) and (**II**), $A^{\perp}$ is a closed set

□

**. *Orthonormal Set*4.38Definition**

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. and $A \subseteq L$. Then, $A$ is called **orthonormal set**

if

.$x \neq y$$A$ is said to be **orthogonal** if $x \perp y$ $\forall x, y \in A$, (١)

$\forall x$1 = $\|x\|$ = $^{2}$ Each element $x \in A$ is a normal element. i.e., $\langle x, x \rangle$ $^{1}$ (٢)

$\ni$

### 4.39 Remark

$0 \notin A$ because $\|0\| \neq 1$ (Orthonarmal set has no zero element (is not normal element).

### 4.40 Example

Let $L = \mathbb{R}^3$ with usual inner product and $A = \{(1,2,-2),(2,-1,2),(2,2,1)\} \subset L$. Show that $A$ is orthogonal but not orthonormal.

**Solution:** T.P. $A$ is orthogonal set **(H.W.)**.

To show not every vector in $A$ is normal. i.e.,

$\|(2,2,1)\|^2 = \langle (2,2,1),(2,2,1)\rangle = 4+4+1 = 9 \neq 1 \Rightarrow \|(2,2,1)\| \neq 1$.

Thus, $A$ is not orthonormal.

### 4.41 Theorem

Let $L$ be an I.P.S. and $x_1, \ldots, x_n$ be orthonormal vectors in $L$. Then

$$\sum_{i=1}^{n} |\langle x, x_i\rangle|^2 \leq \|x\|^2 \quad \forall x \in L.$$

### 4.42 Example

Let $L = \mathbb{R}^3$, $X_1 = \frac{1}{3}(1,2,-2)$, $X_2 = \frac{1}{3}(2,-1,2)$ and $X_3 = \frac{1}{3}(2,2,1)$. Let $X = (3,1,2)$. Then

$$|\langle X, X_1\rangle|^2 = \left[\tfrac{1}{3}(6+2+2)\right]^2 = \frac{100}{9}$$

$$|\langle X, X_2\rangle|^2 = \left[\tfrac{1}{3}(6-1+4)\right]^2 = \frac{1}{9}$$

$$|\langle X, X_3\rangle|^2 = \left[\tfrac{1}{3}(3+2-4)\right]^2 = \frac{25}{9}$$

$$\sum_{i=1}^{3} |\langle X, X_i\rangle|^2 = \frac{25}{9} + \frac{1}{9} + \frac{100}{9} = 14.$$

on the other hand, $\|X\|^2 = \langle X, X\rangle = 4 + 1 + 9 = 14$.

As in Theorem 4.41, $|\langle X, X\rangle| = \|X\|^2 = \sum_{i=1}^{3} X^2$

Take $X = (1, 1, 1)$ and apply Theorem 4.41 (**H.W.**).

**Theorem 4.43.**

Let $(L, \langle\ ,\ \rangle)$ be an I.P.S. Let $(x_n)$ be an orthonormal sequence in $L$ and $(\lambda_n)$ be a sequence in $F$ such that $\sum_{i=1}^{+\infty} |\lambda_i|^2 < +\infty$. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$. Then, $(y_n)$ is a Cauchy sequence.

*Proof.* Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$, $y_m = \sum_{i=1}^{m} \lambda_i x_i$. Assume that $n < m$ then $m = n + k$ for some $k \in N$. We must prove $\|y_m - y_n\| \to 0$.

$$y_m - y_n = \sum_{i=1}^{m} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n+k} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=n+1}^{n+k} \lambda_i x_i$$

$$\|y_m - y_n\|^2 = \left\langle \sum_{i=n+1}^{n+k} \lambda_i x_i, \sum_{i=n+1}^{n+k} \lambda_i x_i \right\rangle$$

$$= \sum_{i=n+1}^{n+k} \lambda_i \sum_{i=n+1}^{n+k} \overline{\lambda_i} \langle x_i, x_i\rangle$$

$$= \sum_{i=n+1}^{n+k} \lambda_i \overline{\lambda_i} \langle x_i, x_i\rangle$$

$$= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \|x_i\|^2$$

$$= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \quad (\forall i \ \|x_i\| = 1)$$

As $n \to +\infty$, $\sum_{i=n+1}^{n+k} |\lambda_i|^2 \to 0$ ($\sum_{i=1}^{+\infty} |\lambda_i|^2$ convergent).

Thus, $\|y_m - y_n\|^2 \to 0$ which means $\|y_m - y_n\| \to 0$. Hence, $(y_n)$ is a Cauchy sequence. □

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# ملخص محاضرات

# التحليل الدالي

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