Matrices:

A matrix (plural matrices) is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. For example, the dimension of the matrix below is 2 × 3 (read "two by three"), because there are two rows and three columns:

A matrix with *m* rows and *n* columns is called an *m* × *n* matrix or *m*-by-*n* matrix, while *m* and *n* are called its *dimensions*.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}) \in \mathbb{R}^{m\times n}.$$

Square matrix main types of matrices:

1- Square matrix:

A square matrix is a matrix with the same number of rows and columns. An n-by-n matrix is known as a square matrix of order n. a_{ii} .

A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2- Diagonal matrix:

All entries outside the main diagonal are zero. A= $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

3- Upper triangular matrix:

If all entries of A below the main diagonal are zero. A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

4- Lower triangular matrix:

All entries of A above the main diagonal are zero.

A = $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

5- Identity matrix:

The identity matrix I_n of size n is the n-by-n matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0.

$$\mathbf{I}_1 = [1],\ \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \cdots,\ \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Basic operations of matrices:

There are a number of basic operations that can be applied to modify matrices.

1- Addition

The sum A+B of two m-by-n matrices A and B is calculated entrywise:

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$$

2- Subtracting

To subtract two matrices: subtract the numbers in the matching positions:

$$3 - 4 = -1$$

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$$

Note: subtracting is actually defined as the addition of a negative matrix: A + (−B)

3- Multiplication of a matrix by a scalar

Definition Let A be a $K \times L$ matrix and α be a scalar. The product of A by α is another $K \times L$ matrix, denoted by αA, such that its (k,l)-th entry is equal to the product of α by the (k,l)-th entry of A, that is

$$(\alpha A)_{kl} = \alpha A_{kl}$$

Example Let $\alpha = 2$ and define the 2×3 matrix

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

The product αA is

$$\alpha A = 2 \cdot \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \cdot 0 & 2 \cdot 2 & 2 \cdot 3 \\ 2 \cdot 1 & 2 \cdot 1 & 2 \cdot 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 4 & 6 \\ 2 & 2 & 0 \end{bmatrix}$$

Properties

$$\alpha(\beta A) = (\alpha\beta)A$$

for any matrix A and any scalars α and β.

$$\alpha(A + B) = \alpha A + \alpha B$$

4- Matrix multiplication

Matrix multiplication is a binary operation that produces a matrix from two matrices. For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The result matrix, known as the matrix product, has the number of rows of the first and the number of columns of the second matrix.

If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, the matrix product C = AB defined to be the $m \times p$ matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Matrices and Systems of Simultaneous Linear Equations

We now see how to write a system of linear equations using matrix multiplication.

Example 4

The system of equations

$$-3x + y = 1$$

$$6x - 3y = -4$$

can be written as:

$$\begin{pmatrix} -3 & 1 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Properties

AB ≠ BA. are square matrices of the same size. Even in this case, one has in general.

For example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

but

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC},$$

nd (right distributivity)

$$(\mathbf{B}+\mathbf{C})\mathbf{D} = \mathbf{BD}+\mathbf{CD}.$$

$$c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B} \text{ and } (\mathbf{AB})c = \mathbf{A}(\mathbf{B}c).$$

5- Dividing

And what about division? Well we don't actually divide matrices, we do it this way:

$$A/B = A \times (1/B) = A \times B^{-1}$$

where $\mathbf{B}^{-1}$ means the "inverse" of B.

6- Transpose of a matrix:

In linear algebra, the transpose of a matrix is an operator which flips a matrix over its diagonal, that is it switches the row and column indices of the matrix by producing another matrix denoted as A^T.

$[A^T]_{ij} = [A]_{ji}$ If **A** is an $m \times n$ matrix, then $\mathbf{A}^T$ is an $n \times m$ matrix. For example

- $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Properties

For matrices **A**, **B** and scalar *c* we have the following properties of transpose:

1. $\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}.$

   The operation of taking the transpose is an involution (self-inverse).

2. $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}.$

   The transpose respects addition.

3. $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}.$

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{\mathrm{T}}\right)^{\mathrm{T}} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

4. $(c\mathbf{A})^{\mathrm{T}} = c\mathbf{A}^{\mathrm{T}}.$

7- Trace of a matrix

In linear algebra, the trace (often abbreviated to **{A}**) of a square matrix A is defined to be the sum of elements on the main diagonal (from the upper left to the lower right) of A. The trace is only defined for a square matrix (n × n).

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

Let $\mathbf{A}$ be a matrix, with

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 3 \\ 11 & 5 & 2 \\ 6 & 12 & -5 \end{pmatrix}$$

Then

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} = -1 + 5 + (-5) = -1$$

Properties

$$\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$$
$$\operatorname{tr}(c\mathbf{A}) = c\operatorname{tr}(\mathbf{A})$$

A matrix and its transpose have the same trace:

$$\operatorname{tr}(\mathbf{A}) = \operatorname{tr}(\mathbf{A}^{\mathrm{T}}).$$

Trace of a product

$$\operatorname{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = \operatorname{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}}) = \operatorname{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{A}) = \operatorname{tr}(\mathbf{B}\mathbf{A}^{\mathrm{T}})$$

$$\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$$

$$\operatorname{tr}(\mathbf{ABCD}) = \operatorname{tr}(\mathbf{BCDA}) = \operatorname{tr}(\mathbf{CDAB}) = \operatorname{tr}(\mathbf{DABC}).$$

$$\operatorname{tr}(\mathbf{ABC}) \neq \operatorname{tr}(\mathbf{ACB}).$$

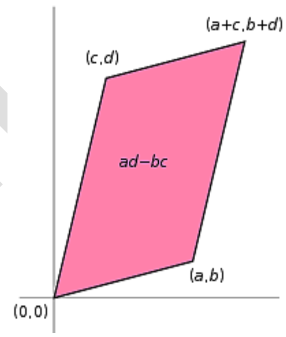
$$\mathrm{tr}(\mathbf{AB}) \neq \mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B})$$

8- Determinant of a matrix

In linear algebra, the determinant is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix. The determinant of a matrix A is denoted |A|.

In the case of a 2 × 2 matrix the determinant may be defined as

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$



The Leibniz formula for the determinant of a 2 × 2 matrix for finding area of plane.

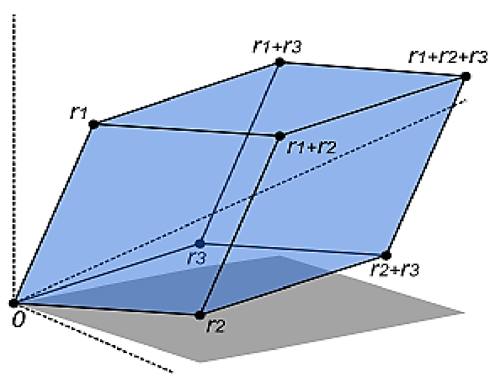
Similarly, for a 3 × 3 matrix A, its determinant is the Laplace formula for the determinant of a 3 × 3 matrix is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

this can be expanded out to give

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$



Properties of the determinant

1. $\det(I_n) = 1$, where I_n is the $n \times n$ identity matrix.
2. $\det(A^{\mathsf{T}}) = \det(A)$, where A^{T} denotes the transpose of A.
3. $\det(A^{-1}) = \dfrac{1}{\det(A)} = [\det(A)]^{-1}$.
4. For square matrices A and B of equal size,

   $$\det(AB) = \det(A) \times \det(B).$$

9- Inverse of a matrix

For a square matrix A, the inverse is written A^{-1}. Non-square matrices do not have inverses.

Note: Not all square matrices have inverses. A square matrix which has an inverse is called invertible or nonsingular, and a square matrix without an inverse is called noninvertible or singular. Why we need the inverse ??? because there is not divided in matrix.

<u>How many methods to find inverse square matrix with ex.??</u>

Properties

Furthermore, the following properties hold for an invertible matrix A:

- $(A^{-1})^{-1} = A$
- $AA^{-1} = A^{-1}A = I$
- $(kA)^{-1} = k^{-1}A^{-1}$ for nonzero scalar k
- $(A^{T})^{-1} = (A^{-1})^{T}$;
- For any invertible n-by-n matrices A and B, $(AB)^{-1} = B^{-1}A^{-1}$.
- $\det A^{-1} = (\det A)^{-1}$.

$$AA^{-1} = A^{-1}A = I$$

Example: For matrix $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$, its inverse is $A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$ since

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $A^{-1}A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In relation to its adjugate

The adjugate of a matrix *A* can be used to find the inverse of *A* as follows:

If *A* is an *n x n* invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

Inversion of 2 × 2 matrices:

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example

Find the inverse of the matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

Solution

Using the formula

$$A^{-1} = \frac{1}{(3)(2)-(1)(4)} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$

This could be written as

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$$

Inversion of 3 × 3 matrices:

Then the inverse matrix is:

$$\underset{3\times3}{B^{-1}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}^{-1} = \frac{1}{det(B)} \begin{bmatrix} (ek-fh) & -(bk-ch) & (bf-ce) \\ -(dk-fg) & (ak-cg) & -(af-cd) \\ (dh-eg) & -(ah-bg) & (ae-bd) \end{bmatrix}$$

Where $det(B)$ is equal to:

$$det(B) = a(ek-fh) - b(dk-fg) + c(dh-eg)$$

<u>Special Matrices</u>

In this section we introduce some important special matrices can be used in necessary application:

1- Diagonally Dominant Matrix:

In mathematics, a square matrix is said to be diagonally dominant if, for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row. More precisely, the matrix A is diagonally dominant if:

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \text{for all } i,$$

If a strict inequality (>) is used, this is called *strict diagonal dominance*.

Examples:

The matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

is diagonally dominant because

$$|a_{11}| \geq |a_{12}| + |a_{13}| \quad \text{since} \quad |+3| \geq |-2| + |+1|$$

$$|a_{22}| \geq |a_{21}| + |a_{23}| \quad \text{since} \quad |-3| \geq |+1| + |+2|$$

$$|a_{33}| \geq |a_{31}| + |a_{32}| \quad \text{since} \quad |+4| \geq |-1| + |+2|.$$

The matrix

$$B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

is *not* diagonally dominant because

$$|b_{11}| < |b_{12}| + |b_{13}| \quad \text{since} \quad |-2| < |+2| + |+1|$$

$$|b_{22}| \geq |b_{21}| + |b_{23}| \quad \text{since} \quad |+3| \geq |+1| + |+2|$$

$$|b_{33}| < |b_{31}| + |b_{32}| \quad \text{since} \quad |+0| < |+1| + |-2|.$$

That is, the first and third rows fail to satisfy the diagonal dominance condition.

The matrix

$$C = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

is *strictly* diagonally dominant because

$$|c_{11}| > |c_{12}| + |c_{13}| \quad \text{since} \quad |-4| > |+2| + |+1|$$

$$|c_{22}| > |c_{21}| + |c_{23}| \quad \text{since} \quad |+6| > |+1| + |+2|$$

$$|c_{33}| > |c_{31}| + |c_{32}| \quad \text{since} \quad |+5| > |+1| + |-2|.$$

Exercises

Classify the following matrices as diagonally dominant, strictly diagonally dominant or unknown:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -4 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} -6 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & -2 & 7 \end{bmatrix}.$$

*2- **Band matrix***

Band matrix is a sparse matrix whose non-zero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on either side.

p is the lower bandwidth if $a_{ij}=0$ for $i > j+p$.

q is upper band width if $j < i+q$.

$$\mathbf{A} = \begin{bmatrix} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix}$$

– The $\times$ denotes an arbitrary nonzero entry

– This 8×6 matrix has lower bandwidth 3 and upper bandwidth 1

Examples:

– *Example 1.* A 5×7 upper triangular matrix $p = 0, q = 6$

$$\begin{bmatrix} \times & \times & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times & \times \end{bmatrix}$$

Some Band Matrices

Matrix	p	q
Diagonal	0	0
Upper Triangular	0	$n-1$
Lower Triangular	$m-1$	0
Tridiagonal	1	1
Upper bidiagonal	0	1
Lower bidiagonal	1	0
Upper Hessenberg	1	$n-1$
Lower Hessenberg	$m-1$	1

- *Example 2.* A 5×6 tridiagonal matrix $p = q = 1$

$$\mathbf{A} = \begin{bmatrix} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & \times \end{bmatrix}$$

- *Example 3.* A 4×6 Lower Hessenberg matrix $p = 3$, $q = 1$

$$\mathbf{A} = \begin{bmatrix} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & \times & 0 \end{bmatrix}$$

3- Monotonic Matrix

A monotonic matrix of order n is an $n \times n$ matrix in which every element is either 0 or contains a number from the set {1,...,n} subject to the conditions.

matrix is monotone if all elements of A^{-1} are nonnegative). For example, the following (2×2) matrices are monotone:

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. The monotone is non-singular matrix.

4-

Pseudo Inverse of a Matrix

The matrix $(A^TA)^{-1}A^T$ is called *pseudo inverse* of a matrix A and denoted by $pinv(A)$. The pseudo inverse can be expressed of a rectangular matrix, or not invertible square matrix.

Example : Find A^{-1} for the following matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}$$

Solution: we see that A is rectangular matrix thatwe cannot be compute A^{-1} director. So, we find pseudo inverse as follows:

Firstly find A^TA,

$$A^TA = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 11 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{adj(A^TA)}{|A^TA|} = \frac{1}{30}\begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix}$$

$$pinv(A) = (A^TA)^{-1}A^T = \frac{1}{30}\begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 5 & -17 & 4 \\ 0 & 12 & 6 \end{bmatrix}$$

That is $pinv(A)$: $\begin{bmatrix} \frac{1}{6} & \frac{-17}{30} & \frac{2}{15} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$

Exercises:

Find pseudo inverse for the following matrices:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \text{ and}$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

Chapter Two

Eigenvalues, Eigenvectors and its Applications

In this chapter we introduce definition of eigenvalues, eigenvectors, how can it's calculating and illustrate the importance of the topic by demonstrating some of its applications.

Eigenvalues and Eigenvectors

Suppose that $\boldsymbol{A}$ is a square $(n \times n)$ matrix. We say that a nonzero vector v is an eigenvector (*ev*) and a *scalar* λ is its eigenvalue (*ew*) if

$$\boldsymbol{A}v = \lambda v \qquad (2.1)$$

Geometrically this means that $\boldsymbol{A}v$ is in the same or appositive direction as v, depending on the sign of λ.

Notice that Equation (2.1) can be rewritten as follows:

$$\boldsymbol{A}v - \lambda v = 0$$

since $Iv = v$, we can do the following:

$$\boldsymbol{A}v - \lambda v = \boldsymbol{A}v - \lambda Iv = (\boldsymbol{A} - \lambda I)v = 0$$

If v is nonzero, then the matrix $(\boldsymbol{A} - \lambda I)$ must be singular and

$|\boldsymbol{A} - \lambda I| = 0$.

This is called the ***characteristic equation*** (or ***characteristic polynomial*** $\boldsymbol{p}(\boldsymbol{\lambda})$).

Calculating Eigenvalues and Eigenvectors

If $\boldsymbol{A}$ is (2×2) or (3×3) matrix then we can find its eigenvalues and eigenvectors by hand.

Note

Let $\boldsymbol{A}$ is a square $(n \times n)$ matrix and λ is an eigenvalue of $\boldsymbol{A}$. The set of all eigenvectors corresponding to λ, together with zero vector is a subspace of R^n and this space is called eigenspace of λ.

Example 1: Find eigenvalues and eigenvectors for the following matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} \end{bmatrix}$.

Solution $\boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 4 \\ 3 & 5-\lambda \end{bmatrix}$,

$|\boldsymbol{A} - \lambda I| = (1-\lambda)(5-\lambda) - 3(4) = \lambda^2 - 6\lambda - 7$ (*This called characteristic polynomial*),

$\lambda^2 - 6\lambda - 7 = 0 \rightarrow (\lambda - 7)(\lambda + 1) = 0 \rightarrow \lambda = 7, \lambda = -1.$

$\lambda = 7$ and $\lambda = -1$ are the eigenvalues of $\boldsymbol{A}$.

To find eigenvectors, if $\lambda = 7$, we solve the equation

$$(\boldsymbol{A} - 7\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -6x + 4y \\ 3x - 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$-6x + 4y = 0, 3x - 2y = 0,$

Hence, $(x, y) = (2, 3)$ is a solution of $3x - 2y = 0$ (or $-6x + 4y = 0$). Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 7$ are non-zero vectors of form $r_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix}, r_1 \in R\{0\}$.

The $S_1 = \{r_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix}, r_1 \in \mathrm{R}\}$ is a subspace of R^2.

To find eigenvectors, if $\lambda = -1$, we solve the equation

$$(\boldsymbol{A}-(-1)\boldsymbol{I})v=0 \rightarrow \left(\begin{bmatrix}1 & 4\\ 3 & 5\end{bmatrix}-\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right)\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

$$\begin{bmatrix}2 & 4\\ 3 & 6\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

$$\begin{bmatrix}2x+4y\\ 3x+6y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

$2x+4y=0, 3x+6y=0,$

Hence, $(x, y) = (-2, 1)$ is a solution of $2x+4y=0$(or $3x+6y=0$). Thus the eigenvectors of A when $\lambda=-1$ are non-zero vectors of form

$r_2\begin{bmatrix}-2\\ 1\end{bmatrix}$, $r_2 \in R\{0\}$. The $S_2=\{r_2\begin{bmatrix}-2\\ 1\end{bmatrix}, r_2 \in \mathrm{R}\}$ is a subspace of R^2.

<u>Example2</u>: Find eigenvalues and eigenvectors for the following matrix $\boldsymbol{A}=\begin{bmatrix}\mathbf{1} & \mathbf{-3} & \mathbf{3}\\ \mathbf{3} & \mathbf{-5} & \mathbf{3}\\ \mathbf{6} & \mathbf{-6} & \mathbf{4}\end{bmatrix}$.

<u>Solution</u>: $\boldsymbol{A}-\boldsymbol{\lambda I}=\begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}$$

$$=\begin{bmatrix}1-\lambda & -3 & 3\\ 3 & -5-\lambda & 3\\ 6 & -6 & 4-\lambda\end{bmatrix},$$

$$|\boldsymbol{A}-\lambda\boldsymbol{I}|=(1-\lambda)\{[(-5-\lambda)(4-\lambda)]-3(-6)\}+3\{3(4-\lambda)-3(6)\}+3\{3(-6)-6(-5-\lambda)\}$$

$$=(1-\lambda)(\lambda^2+\lambda-20+18\}+3\{12-3\lambda-18\}+3\{-18+30+6\lambda\}$$
$$=-\lambda^3+3\lambda-2-9\lambda-18+18\lambda+36=-\lambda^3+12\lambda+16$$

To find the solution to $|A-\lambda I|=0$, i.e. to solve $\lambda^3-12\lambda-16=0$,

$$\lambda^3-12\lambda-16=(\lambda-4)(\lambda^2+4\lambda+4)=0,$$

$\lambda = 4, \lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2} = \frac{-4 \pm 0}{2} = -2$ (repeated root).

To find eigenvectors, if $\lambda = 4$, we solve the equation

$$(\boldsymbol{A} - 4\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -3x - 3y + 3z \\ 3x - 9y + 3z \\ 6x - 6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$-3x - 3y + 3z = 0, 3x - 9y + 3z = 0, 6x - 6y = 0,$

$$x - \frac{1}{2}z = 0, y - \frac{1}{2}z = 0$$

Hence, $(x, y, z) = (\frac{1}{2}, \frac{1}{2}, 1)$ is a solution of $x - \frac{1}{2}z = 0, y - \frac{1}{2}z = 0.$

Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 4$ is non-zero vectors of the form $r_1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}, r_1 \in R\{0\}$. The $S_1 = \{r_1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}, r_1 \in R\}$ is a subspace of R^3.

To find eigenvectors, if $\lambda = -2$, we solve the equation

$$(\boldsymbol{A} - (-2)\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3x - 3y + 3z \\ 3x - 3y + 3z \\ 6x - 6y + 6z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$3x - 3y + 3z = 0, 3x - 3y + 3z = 0, 6x - 6y + 6z = 0,$

$$x - y + z = 0$$

Hence, $(x, y, z) = (0,1,1)$ is a solution of $x - y + z = 0$.

Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = -2$ are non-zero vectors of the form $r_2\begin{bmatrix}0\\1\\1\end{bmatrix}, r_2 \in R\{0\}$. The $S_2 = \{r_2\begin{bmatrix}0\\1\\1\end{bmatrix}, r_2 \in R\}$ is a subspace of R^3.

Complex Eigenvalues

It turns out that the eigenvalues of some matrices are complex numbers, even when the matrix only contains real numbers. When this happens the complex *ew*'s must occur in conjugate pairs, i.e.,

$$\lambda_{1,2} = \alpha \pm \beta i$$

The corresponding *ev*'s must also come in conjugate pairs:

$$w = u \pm vi$$

Example3: Find eigenvalues and eigenvectors for the matrix $\boldsymbol{A} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$.

Solution

$$\boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix} = \begin{bmatrix}-\lambda & -1\\1 & -\lambda\end{bmatrix},$$

$$|A - \lambda \boldsymbol{I}| = (-\lambda)(-\lambda) - 1(-1) = \lambda^2 + 1,$$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \lambda^2 + 1 = 0 \rightarrow \lambda^2 = -1 \rightarrow \lambda = \pm i.$$

$\lambda = i$ and $\lambda = -i$ are the eigenvalues of $\boldsymbol{A}$.

To find eigenvectors, if $\lambda = i$, we solve the equation

$$(\boldsymbol{A} - i\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} - \begin{bmatrix}i & 0\\0 & i\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\begin{bmatrix}-i & -1\\1 & -i\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

$$\begin{bmatrix}-ix - y\\x - iy\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

$-ix - y = 0, x - iy = 0 \rightarrow y = -ix,$

Hence, the eigenvectors of $\boldsymbol{A}$ when $\lambda = \mathrm{i}$ are non-zero vectors of form $r_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}, r_1 \in R\{0\}$. The eigenspace = $\{\begin{bmatrix} z_1 \\ -iz_2 \end{bmatrix}, z_1, z_2 \in \mathrm{R}\}$.

To find eigenvectors, if $\lambda = -\mathrm{i}$, we solve the equation

$$(\boldsymbol{A} - (-i)\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} ix - y \\ x + iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$ix - y = 0, x + iy = 0 \rightarrow y = ix,$

Hence, the eigenvectors of $\boldsymbol{A}$ when $\lambda = -\mathrm{i}$ are non-zero vectors of form $r_2 \begin{bmatrix} 1 \\ i \end{bmatrix}, r_2 \in R\{0\}$. The eigenspace = $\{\begin{bmatrix} z_1 \\ iz_2 \end{bmatrix}, z_1, z_2 \in \mathrm{R}\}$.

<u>Notes</u>

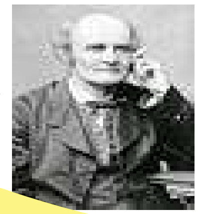
1. An eigenvalue of $A_{n\times n}$ is a root of the characteristic polynomial. Indeed λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$. So there are at most n distinct eigenvalues of A.
2. Similar matrices have the same eigenvalues (**HW**).
3. If $\boldsymbol{A}$ be a diagonal matrix then its eigenvalues are the diagonal elements (**HW**).
4. If $\boldsymbol{A}$ be an upper (lower) triangular matrix then its eigenvalues are the diagonal elements (**HW**).
5. If $\boldsymbol{A}$ be a square matrix then $\boldsymbol{A}$ and $\boldsymbol{A}^{\boldsymbol{T}}$ have the same eigenvalues (**HW**).
6. If $\boldsymbol{A}$ be a square matrix then $|\boldsymbol{A}|$ is equal to the product of all eigenvalues of A (**HW**).
7. $\boldsymbol{A}$ is a singular matrix $\leftrightarrow \lambda = 0$ be an eigenvalue of $\boldsymbol{A}$ (**HW**).
8. If A be an invertible matrix with eigenvalue λ of eigenvector v then λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$ with eigenvector v (**HW**).
9. The set of all the eigenvalues of A is referred to as the spectrum of A and denoted by $\Lambda(A)$.

10. The maximum modulus of the eigenvalues is called spectral radius and denoted by $\rho(A)$, that is:

$$\rho(A) = \max_{\lambda\in\Lambda(A)} |\lambda|.$$

[illegible]

[illegible]hur Cayley (16 August 1821 – 26 January [illegible]s a British mathematician



Let $\boldsymbol{A}$ be a square $(n \times n)$ ma[illegible]ith characteristic polynomial

$p($[illegible]$\lambda^n + c_1\lambda^{n-1} + \cdots. + c_{n-1}\lambda + c_n$ and $\lambda^n + c_1\lambda$ [illegible] $\cdots + c_{n-1}\lambda + c_n = 0$ then $\boldsymbol{A}$[illegible]$^{n-1} + \cdots. + c_{n-1}\boldsymbol{A} + c_n\boldsymbol{I_n} = 0$.

Example4: Apply Cayley[illegible]ilton Theorem on the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$.

Solution: [illegible] $= \lambda^2 - 3\lambda + 2$, by Cayley-Ha[illegible] Theorem

[illegible] $^2 - 3\boldsymbol{A} + 2\boldsymbol{I_2} = 0,$

$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \text{[illegible]} \\ -3 & 7 \end{bmatrix}$ [illegible] $\begin{bmatrix} 0 & -6 \\ 3 & \text{[illegible]} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ \text{[illegible]} & \text{[illegible]} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

Exercises

1. Find eigenvalues and eigenvectors for the following matrices

$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$

$\boldsymbol{E} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{G} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}, \boldsymbol{H} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \boldsymbol{K} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix},$

$\boldsymbol{L} = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$

2. Find eigenvalues and eigenvectors for the following matrices:

$\boldsymbol{A} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{bmatrix}$ and

$\boldsymbol{D} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & -2 & 3 \end{bmatrix}.$

3. [illegible]

$$A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

Eigenvalues and Eigenvectors of symmetric matrix

A matrix is symmetric if it is equal to its own transpose, in symmetric matrix the upper right left and the lower left half of the matrix are mirror images of each other about the diagonal. A$(n \times n)$ symmetric matrix not only has a nice structure, but it also satisfied the following:

- It has exactly n eigenvalues (not necessary distinct).
- There exists a set of n eigenvectors, one for each eigenvalue, that are mutually orthogonal.
- A symmetric matrix has n eigenvalues and there exist n linearly independent eigenvectors (because of orthogonal) even if the eigenvalues are not distinct.

Example5: Find eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

Solution: $A - \lambda I = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix},$$

$$|A - \lambda I| = (5-\lambda)(5-\lambda) - 3(3) = \lambda^2 - 10\lambda + 25 - 9 = \lambda^2 - 10\lambda + 16$$

$$= (\lambda - 8)(\lambda - 2) = 0 \rightarrow \lambda = 8, 2$$

To find eigenvectors, if $\lambda = 8$, we solve the equation

$$(A - 8I)v = 0 \quad \rightarrow \left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -3x + 3y \\ 3x - 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$-3x + 3y = 0, 3x - 3y = 0,$

$x = 1, y = 1$. Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 8$ is nonzero vectors of form $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find eigenvectors, if $\lambda = 2$, we solve the equation:

$$(\boldsymbol{A} - 2I)v = 0 \rightarrow \left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3x + 3y \\ 3x + 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$3x + 3y = 0, 3x + 3y = 0,$

$x = 1, y = -1$. Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 2$ are non-zero vectors of form $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus we have two orthogonal eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (linearly independent).

Example 6: [illegible]

[illegible]

Solution: $\boldsymbol{A}-\boldsymbol{\lambda I}=\begin{bmatrix}3 & 2 & 4\\ 2 & 0 & 2\\ 4 & 2 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}3 & 2 & 4\\ 2 & 0 & 2\\ 4 & 2 & 3\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}$$

$$=\begin{bmatrix}3-\lambda & 2 & 4\\ 2 & -\lambda & 2\\ 4 & 2 & 3-\lambda\end{bmatrix},$$

$$|\boldsymbol{A}-\lambda\boldsymbol{I}|=(3-\lambda)\{[(-\lambda)(3-\lambda)]-2(2)\}-2\{2(3-\lambda)-2(4)\}$$
$$+4\{2(2)-4(-\lambda)\}$$

$$=(3-\lambda)(\lambda^2-3\lambda-4\}-2\{6-2\lambda-8\}+4\{4+4\lambda\}$$
$$=-\lambda^3+6\lambda^2-5\lambda-12+4\lambda+4+16+16\lambda=-\lambda^3+6\lambda^2+15\lambda+8$$

To find the solution to $|A-\lambda I|=0$, i.e. to solve $(-\lambda^3+6\lambda^2+15\lambda+8=0\rightarrow$ $(\lambda^3-6\lambda^2-15\lambda-8=0)$,

$\lambda^3-6\lambda^2-15\lambda-8=(\lambda-8)(\lambda+1)^2=0\rightarrow\lambda=8,-1,-1$

To find eigenvectors, if $\lambda=8$, we solve the equation

$(\boldsymbol{A}-8\boldsymbol{I})v=0\rightarrow$ $$\left(\begin{bmatrix}3 & 2 & 4\\ 2 & 0 & 2\\ 4 & 2 & 3\end{bmatrix}-\begin{bmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{bmatrix}\right)\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},$$

$$\begin{bmatrix}-5 & 2 & 4\\ 2 & -8 & 2\\ 4 & 2 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},$$

$$\begin{bmatrix}-5x+2y+4z\\ 2x-8y+2z\\ 4x+2y-5z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},$$

$-5x+2y+4z=0, 2x-8y+2z=0, 4x+2y-5z=0,$

$x=2, y=1, z=2$. Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda=8$ are non-zero vectors of form $\begin{bmatrix}2\\ 1\\ 2\end{bmatrix}$.

To find eigenvectors, if $\lambda = -1$, we solve the equation :

$$(\boldsymbol{A}+\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$\begin{bmatrix}4&2&4\\2&1&2\\4&2&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$\begin{bmatrix}4x+2y+4z\\2x+y+2z\\4x+2y+4z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$4x+2y+4z=0, 2x+y+2z=0, 4x+2y+4z=0,$

This system reduces to single equation $(2x+y+2z=0)$ since the other two equations are twice this one. There are two parameters here (x and z), thus eigenvectors for $\lambda=-1$ must have the form $(y=-2x-2z)$ which corresponds to the vectors of form $\begin{bmatrix}s\\-2s-2t\\t\end{bmatrix}$. We must choose values of s and t that yield two orthogonal vectors (the third one is $\begin{bmatrix}2\\1\\2\end{bmatrix}$). First, choose anything, let $s=1$ and $t=0$, the eigenvector is $\begin{bmatrix}1\\-2\\0\end{bmatrix}$. Now find a vector $\begin{bmatrix}x\\-2x-2z\\z\end{bmatrix}$ such that: $0=\begin{bmatrix}1\\-2\\0\end{bmatrix}.\begin{bmatrix}x\\-2x-2z\\z\end{bmatrix}=x+4x+4z+0=5x+4z,$ we can choose $x=4$ and $z=-5$. Thus we have two orthogonal vectors $\begin{bmatrix}1\\-2\\0\end{bmatrix}$ and $\begin{bmatrix}4\\2\\-5\end{bmatrix}$ that corresponds to the two eigenvalue $\lambda=-1$.

Note that: since this matrix is symmetric we do indeed have three eigenvalues and a set of three orthogonal (and thus linearly independent) eigenvectors (one for each eigenvalue).

[illegible]

[illegible] following matrices.

[illegible] $\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}$, [illegible] $\begin{bmatrix}-4 & -6\\ -3 & [illegible]\end{bmatrix}$, $C = \begin{bmatrix}5 & 4\\ [illegible] & [illegible]\end{bmatrix}$, $D = \begin{bmatrix}-2 & 0\\ [illegible] & [illegible]\end{bmatrix}$

[illegible] $\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$, [illegible] $\begin{bmatrix}[illegible] & [illegible]\\ 0 & 0\end{bmatrix}$, $G = \begin{bmatrix}[illegible] & [illegible]\\ -1 & 2\end{bmatrix}$, $H = \begin{bmatrix}[illegible] & [illegible]\\ 1 & 2\end{bmatrix}$, $K = \begin{bmatrix}[illegible] & 1\\ 1 & 1\end{bmatrix}$,

$L = \begin{bmatrix}5 & [illegible]\\ 3 & -1\end{bmatrix}$, $M = \begin{bmatrix}5 & 4 & 1\\ [illegible] & [illegible] & 2\\ [illegible] & [illegible] & [illegible]\end{bmatrix}$, $N = \begin{bmatrix}[illegible] & [illegible] & [illegible]\\ -2 & 1 & 2\\ [illegible] & 0 & 3\end{bmatrix}$, $O = \begin{bmatrix}[illegible] & [illegible] & [illegible]\\ -4 & -3 & -4\\ -4 & -6 & -7\end{bmatrix}$.

Diagonalization of a Matrix with Distinct Eigenvalues

A square matrix $\boldsymbol{A}$ is said to be diagonalizable if there exists an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C}$ is a diagonal matrix.

Example9: Prove that the matrix $\boldsymbol{A} = \begin{bmatrix}\mathbf{7} & \mathbf{-10}\\ \mathbf{3} & \mathbf{-4}\end{bmatrix}$ is diagonalizable.

Solution: $\boldsymbol{\lambda_1} = 2$ and eigenvectors $\boldsymbol{v_1} = r_1\begin{bmatrix}2\\1\end{bmatrix}$,

$\lambda_2 = 1$ and eigenvectors $v_2 = r_2\begin{bmatrix}5\\3\end{bmatrix}$

There exists $\boldsymbol{C} = \begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$ such that $\boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C} = \begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}^{-1}\begin{bmatrix}7 & -10\\ 3 & -4\end{bmatrix}\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix} = \begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}\begin{bmatrix}7 & -10\\ 3 & -4\end{bmatrix}\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$ is a diagonal matrix.

Notes

- The product $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C}$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}$.
- $\boldsymbol{A}$ is a diagonalizable $\leftrightarrow$ it has linearly independent the eigenvectors.
- Matrix Powers: $\boldsymbol{A}$ is similar to a diagonal matrix $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C}$ then $\boldsymbol{A}^{k} = \boldsymbol{C}\boldsymbol{D}^{k}\boldsymbol{C}^{-1}$.
- If a matrix $\boldsymbol{A}$ with distinct eigenvalues then $\boldsymbol{A}$ is diagonalizable.
- The eigenvalues of $\boldsymbol{A}$ lies on the main diagonal of similar matrix
  $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C}$.
- If $\boldsymbol{A}$ is a symmetric matrix then eigenvectors that associated to distinct eigenvalues of $\boldsymbol{A}$ are orthogonal.

Example10: Let $\boldsymbol{A} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$, $\{\boldsymbol{B} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ $\boldsymbol{HW}\}$,

1. Prove that $\boldsymbol{A}$ is diagonalizable,

2. Find the diagonal matrix $\boldsymbol{D}$ similar to $\boldsymbol{A}$, and

3. Find $\boldsymbol{A}^{5}$.

Solution: $\boldsymbol{\lambda_1} = \boldsymbol{2}$ and eigenvectors $\boldsymbol{v_1} = \boldsymbol{r_1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ and eigenvectors $v_2 = r_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Since $\boldsymbol{A}$ has two distinct eigenvalues then $\boldsymbol{A}$ is diagonalizable.

- Select two linearly independent eigenvectors

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$$

- $\mathbf{C} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$, $\boldsymbol{C^{-1}AC} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} =$
  $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \boldsymbol{D},$
  The main diagonal of $\boldsymbol{D}$ has the distinct eigenvalues of $\boldsymbol{A}$.
- $\boldsymbol{D^5} = \begin{bmatrix} 2^5 & 0 \\ 0 & (-1)^5 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix},$
  $\boldsymbol{A^5} = \boldsymbol{CD^5C^{-1}} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} =$
  $\begin{bmatrix} -32 & 2 \\ 32 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -30 & -66 \\ 33 & 65 \end{bmatrix}.$

Example11: Prove that the matrix $\boldsymbol{A} = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ is not diagonalizable.

Solution: $\boldsymbol{\lambda} = \boldsymbol{2}$ (A repeated root) and eigenvector $\boldsymbol{v} = r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

a matrix $\boldsymbol{A}$ it does not have two distinct eigenvalues then $\boldsymbol{A}$ is not diagonalizable.

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_0 + b_0 - 1/2^n b_0 \\ 1/2^n\, b_0 \end{bmatrix}$$

Since $a_0 + b_0 = 1$, we have

$a_n = 1 - 1/2^n b_0$, $b_n = 1/2^n b_0$, $n = 1, 2, \ldots.$

as $n \to \infty$, we have $a_n \to 1$, $b_n \to 0$

The population in the limit contains only **AA**.

Exercises

1- Prove that the following matrices are diagonalizable, Find the diagonal matrix D similar to $\boldsymbol{A}$ and $\boldsymbol{A}^{23}$:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$$

2- In Example 11, suppose the plants are always fertilized with a plant of **Aa**. Derive formulas for the fractions of the plants of AA, Aa, and aa. Also find the limiting genotype distribution as **n** tends to ∞.

2.2.7 Orthogonally and Diagonalizable of a Matrix

A square matrix $\boldsymbol{A}$ is said to be orthogonally diagonalizable if there exists an orthogonal matrix $\boldsymbol{C}$ such that $\boldsymbol{D} = \boldsymbol{C}^T\boldsymbol{AC}$ is a diagonal matrix.

Notes:

- A square matrix $\boldsymbol{A}$ is said to be orthogonally diagonalizable $\leftrightarrow$ $\boldsymbol{A}$ is a symmetric matrix.
- A square matrix $\boldsymbol{C}$ is said to be orthogonal $\leftrightarrow$ the columns (rows) of $\boldsymbol{C}$ is an orthonormal set.
- The eigenvalues of a square matrix $\boldsymbol{A}$ lies on the main diagonal of $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{AC} = \boldsymbol{C}^T\boldsymbol{AC}$, where $\boldsymbol{C}$ is an orthogonal matrix.
- Norm of vector $v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is denoted and define as follows:

$$||v|| = \sqrt{{x_1}^2 + {x_2}^2 + .. + {x_n}^2}$$

Example14: Is $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ orthogonally diagonalizable.

Solution: $\lambda_1 = \mathbf{-2}$ and eigenvector $v_1 = r_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, take $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

$\lambda_2 = 4$ and eigenvector $v_2 = r_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$, take $v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$,

$\lambda_3 = -1$ and eigenvector $v_3 = r_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, take $v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$,

The set $\{v_1, v_2, v_3\}$ is linearly independent (**HW**) and orthogonal but not orthonormal. We can normalize these vectors as follows:

$v_{11} = \frac{v_1}{||v_1||}$, $v_{22} = \frac{v_2}{||v_2||}$, $v_{33} = \frac{v_3}{||v_3||}$

The set $\{v_{11}, v_{22}, v_{33}\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \right\}$ is an orthonormal set in R^3.

Let $\boldsymbol{C} = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ is orthogonal matrix because $\boldsymbol{C^{-1}} = \boldsymbol{C^T}$ (**HW**) (or by previous notes). Hence $\boldsymbol{C^{-1}AC} = \boldsymbol{C^TAC} \rightarrow \boldsymbol{C^TAC} =$

$\begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ which is diagonal matrix then $\boldsymbol{A}$ is orthogonally diagonalizable.

Exercises

Find orthogonally diagonalizable for the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}, \boldsymbol{E} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

Chapter Three

Bimatrices

Definition of Bimatrices

A bimatrix $\boldsymbol{A_B}$ is defined as the union of two rectangular away of numbers $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ arranged into rows and columns. It is written as follows $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ where $\boldsymbol{A_1} \neq \boldsymbol{A_2}$

$$\text{with } \boldsymbol{A_1} = \begin{bmatrix} a^1{}_{11} & a^1{}_{12} & \dots & a^1{}_{1n} \\ a^1{}_{21} & a^1{}_{22} & \dots & a^1{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ a^1{}_{m1} & a^1{}_{m2} & \dots & a^1{}_{mn} \end{bmatrix} \text{ and } \boldsymbol{A_2} = \begin{bmatrix} a^2{}_{11} & a^2{}_{12} & \dots & a^2{}_{1n} \\ a^2{}_{21} & a^2{}_{22} & \dots & a^2{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ a^2{}_{m1} & a^2{}_{m2} & \dots & a^2{}_{mn} \end{bmatrix}$$

$\cup$ is just the notational convenience (symbol) only. $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are called as the component matrices of the bimatrix $\boldsymbol{A_B}$.

The above array is called a m by n bimatrix (written as $\boldsymbol{B}(m \times n)$) since each of $\boldsymbol{A_i}$ (i = 1, 2) has m rows and n columns. It is to be noted a bimatrix has no numerical value associated with it. It is only a convenient way of representing a pair of arrays of numbers. The $\boldsymbol{A_1}$ and $\boldsymbol{A}_2$ be called the components matrices of the bimatrix $\boldsymbol{A_B}$.

Notes

- If $\boldsymbol{A_1} = \boldsymbol{A_2}$ then $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ is not a bimatrix. A bimatrix $\boldsymbol{A_B}$ is denoted by $(\boldsymbol{a^1}_{\boldsymbol{ij}}) \cup (\boldsymbol{a^2}_{\boldsymbol{ij}})$.
- If both $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are $(m \times n)$ matrices then the bimatrix $\boldsymbol{A_B}$ is called the $(m \times n)$ rectangular bimatrix.
- If both $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are square matrices then $\boldsymbol{A_B}$ is called the square bimatrix.
    - If one of the matrices in the bimatrix $\boldsymbol{A_B}$ is square and other is rectangular or both $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are rectangular matrices say $(m_1 \times n_1)$, $(m_2 \times n_2)$ with $m_1 \neq m_2$ or $n_1 \neq n_2$ then $\boldsymbol{A_B}$ is called the mixed bimatrix.

- A bimatrix whose all elements are zero is called zero (null) bimatrix and it is denoted by $\boldsymbol{O_B} = \boldsymbol{O_1} \cup \boldsymbol{O_2} = \boldsymbol{0_1} \cup \boldsymbol{0_2}$.
- We make an assumption the zero bimatrix is a union of two zero matrices even if $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are one and the same (i.e. $\boldsymbol{A_1} = \boldsymbol{A_2} = (\boldsymbol{0})$).
- The unit (identity) square bimatrix denoted by $\boldsymbol{I_B} = \boldsymbol{I^{m\times m}}_{\boldsymbol{1}} \cup \boldsymbol{I^{m\times m}}_{\boldsymbol{2}}$.
- Identity mixed square bimatrix denoted by $\boldsymbol{I_B} = \boldsymbol{I^{m\times m}}_{\boldsymbol{1}} \cup \boldsymbol{I^{n\times n}}_{\boldsymbol{2}}$.

- **<u>Example1</u>:** Classify each the following bimatrices

- a) $\boldsymbol{A_B} = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$, b) $\boldsymbol{A'_B} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cup \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$,

c) $\boldsymbol{A''_B} = \begin{bmatrix} 3 & -2 & 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \end{bmatrix}$,

d) $\boldsymbol{A^1}_{\boldsymbol{B}} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, e) $\boldsymbol{A^2}_{\boldsymbol{B}} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$,

f) $\boldsymbol{A^3}_{\boldsymbol{B}} = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$, and g) $\boldsymbol{A^4}_{\boldsymbol{B}} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \cup \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

<u>Solution</u>: a) (2×3) rectangular bimatrix, b) is a column bimatrix , c) is a row bimatrix, d) (3×3) square bimatrix, e) mixed square bimatrix ,f) mixed bimatrix and g) mixed rectangular bimatrix .

<u>Operations on Bimatrices</u>

Here the operations on Bimatrices are introduce

<u>Equal</u>

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two bimatrice, $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ are equal (i.e. $\boldsymbol{A_B} = \boldsymbol{C_B}$) $\leftrightarrow$ $\boldsymbol{A_1} = \boldsymbol{C_1}$ and $\boldsymbol{A_2} = \boldsymbol{C_2}$.
$\boldsymbol{A_B}$ is not equal to $\boldsymbol{C_B}$ (i.e. $\boldsymbol{A_B} \neq \boldsymbol{C_B}$) $\leftrightarrow$ $\boldsymbol{A_1} \neq \boldsymbol{C_1}$ or $\boldsymbol{A_2} \neq \boldsymbol{C_2}$.

<u>Example 2</u>: Let

1- $\boldsymbol{A_B} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, 2- $\boldsymbol{A_B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 0 & 4 & -2 \\ -3 & 0 & 0 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Solution: 1- and 2- $\boldsymbol{A_B} \neq \boldsymbol{C_B}$.

Multiply by a Constant (Scalar Multiplication)

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\lambda \in R$ be a scalar then

$$\lambda \boldsymbol{A_B} = \begin{bmatrix} \lambda a^1{}_{11} & \lambda a^1{}_{12} & \dots & \lambda a^1{}_{1n} \\ \lambda a^1{}_{21} & \lambda a^1{}_{22} & \dots & \lambda a^1{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ \lambda a^1{}_{m1} & \lambda a^1{}_{m2} & \dots & \lambda a^1{}_{mn} \end{bmatrix} \cup \begin{bmatrix} \lambda a^2{}_{11} & \lambda a^2{}_{12} & \dots & \lambda a^2{}_{1n} \\ \lambda a^2{}_{21} & \lambda a^2{}_{22} & \dots & \lambda a^2{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ \lambda a^2{}_{m1} & \lambda a^2{}_{m2} & \dots & \lambda a^2{}_{mn} \end{bmatrix},$$

The product $\lambda \boldsymbol{A_B}$ is another $(m \times n)$ bimatrix. If $\boldsymbol{A_B}$ is $(m \times n)$ bimatrix then $\lambda \boldsymbol{A_B} = [\lambda \boldsymbol{a^1}_{ij}] \cup [\lambda \boldsymbol{a^2}_{ij}] = [\boldsymbol{a^1}_{ij}\lambda] \cup [\boldsymbol{a^2}_{ij}\lambda] = \boldsymbol{A_B}\lambda$.

Example 3: 1- Let $\boldsymbol{A_B} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & -1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, find $\lambda \boldsymbol{A_B}$ when $\lambda = 3$,

2- Let $\boldsymbol{A_B} = [3 \quad 2 \quad 1 \quad -4] \cup [0 \quad 1 \quad -1 \quad 0]$, find $\lambda \boldsymbol{A_B}$ when $\lambda = -2$,

Solution

1- $3\boldsymbol{A_B} = \begin{bmatrix} 6 & 0 & 3 \\ 9 & 9 & -3 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & -3 \\ 6 & 3 & 0 \end{bmatrix}$,

2- $(-2)\boldsymbol{A_B} = [-6 \quad -4 \quad -2 \quad 8] \cup [0 \quad -2 \quad 2 \quad 0]$.

Addition

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two $(m \times n)$ bimatrices. The sum $\boldsymbol{D_B}$ of the bimatrices $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ is defined as follows:

$$\boldsymbol{D_B} = \boldsymbol{A_B} + \boldsymbol{C_B} = (\boldsymbol{A_1} \cup \boldsymbol{A_2}) + (\boldsymbol{C_1} \cup \boldsymbol{C_2}) = [\boldsymbol{A_1} + \boldsymbol{C_1}] \cup [\boldsymbol{A_2} + \boldsymbol{C_2}]$$

Where $\boldsymbol{A_1} + \boldsymbol{C_1}$ and $\boldsymbol{A_2} + \boldsymbol{C_2}$ are the usual addition on matrices (i.e. if

$\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2} = [\boldsymbol{a^1}_{ij}] \cup [\boldsymbol{a^2}_{ij}]$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2} = [\boldsymbol{c^1}_{ij}] \cup [\boldsymbol{c^2}_{ij}]$,

$$\boldsymbol{D_B} = \boldsymbol{A_B} + \boldsymbol{C_B} = [\boldsymbol{a^1}_{ij} + \boldsymbol{c^1}_{ij}] \cup [\boldsymbol{a^2}_{ij} + \boldsymbol{c^2}_{ij}] =$$

$$\begin{bmatrix} a^1{}_{11}+c^1{}_{11} & a^1{}_{12}+c^1{}_{12} & \dots & a^1{}_{1n}+c^1{}_{1n} \\ a^1{}_{21}+c^1{}_{21} & a^1{}_{22}+c^1{}_{22} & \dots & a^1{}_{2n}+c^1{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ a^1{}_{m1}+c^1{}_{m1} & a^1{}_{m2}+c^1{}_{m2} & \dots & a^1{}_{mn}+c^1{}_{mn} \end{bmatrix} \cup$$

$$\begin{bmatrix} a^2{}_{11}+c^2{}_{11} & a^2{}_{12}+c^{22}{}_{12} & \dots & a^2{}_{1n}+c^2{}_{1n} \\ a^2{}_{21}+c^2{}_{21} & a^2{}_{22}+c^2{}_{22} & \dots & a^2{}_{2n}+c^2{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ a^2{}_{m1}+c^2{}_{m1} & a^2{}_{m2}+c^2{}_{m2} & \dots & a^2{}_{mn}+c^2{}_{mn} \end{bmatrix}.$$

Notes

- The sum of two bimatrices is not in general bimatrix. For example, let $\boldsymbol{A_B} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$, and $\boldsymbol{C_B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix}$, $\boldsymbol{A_B} + \boldsymbol{C_B} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ is not bimatrix because $[\boldsymbol{A_1} + \boldsymbol{C_1}] = [\boldsymbol{A_2} + \boldsymbol{C_2}]$.

- If $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ be two mixed bimatrix then $(\boldsymbol{A_B} + \boldsymbol{C_B})$ is always mixed bimatrix.
- If $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ be two $(m \times n)$ bimatrices then $\boldsymbol{A_B} + \boldsymbol{C_B} = \boldsymbol{C_B} + \boldsymbol{A_B}$. Also if $\boldsymbol{A_B}$, $\boldsymbol{C_B}$ and $\boldsymbol{D_B}$ be three $(m \times n)$ bimatrices then $\boldsymbol{A_B} + (\boldsymbol{C_B} + \boldsymbol{D_B}) = (\boldsymbol{A_B} + \boldsymbol{C_B}) + \boldsymbol{D_B}$.

<u>Example4:</u> Let 1- $\boldsymbol{A_B} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & -1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, and

$\boldsymbol{C_B} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$,

2- $\boldsymbol{A_B} = [3 \quad 2 \quad 1 \quad -4 \quad 0] \cup [0 \quad 1 \quad -1 \quad 0 \quad 1]$,

$\boldsymbol{C_B} = [1 \quad 1 \quad 1 \quad 1 \quad 1] \cup [5 \quad -1 \quad 2 \quad 0 \quad 3]$, find $\boldsymbol{A_B} + \boldsymbol{C_B}$,

<u>Solution:</u> 1- $\boldsymbol{A_B} + \boldsymbol{C_B} = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 5 & -2 \end{bmatrix} \cup \begin{bmatrix} 3 & 4 & 0 \\ 2 & 3 & -1 \end{bmatrix}$,

2- $\boldsymbol{A_B} + \boldsymbol{C_B} = [4 \quad 3 \quad 2 \quad -3 \quad 1] \cup [5 \quad 0 \quad 1 \quad 0 \quad 4]$.

Example5: Let $A_B = \begin{bmatrix} 6 & -1 \\ 2 & 2 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}$, find $A_B + A_B$.

Solution: $A_B + A_B = \begin{bmatrix} 12 & -2 \\ 4 & 4 \\ 2 & -2 \end{bmatrix} \cup \begin{bmatrix} 6 & 2 \\ 0 & 4 \\ -2 & 6 \end{bmatrix} = 2\, A_B$.

Subtraction

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two ($m \times n$) bimatrices. The subtraction of the bimatrices A_B and C_B is defined as follows:

$$A_B - C_B = A_B + (-C_B) = (A_1 \cup A_2) + (-C_1 \cup -C_2)$$
$$= [A_1 - C_1] \cup [A_2 - C_2] = [A_1 + (-C_1)] \cup [A_2 + (-C_2)].$$

Where $A_1 + (-C_1)$ and $A_2 + (-C_2)$ are the usual addition on matrices.

Example6: Let $A_B = [1 \quad 2 \quad 3 \quad -1 \quad 2 \quad 1] \cup [3 \quad -1 \quad 2 \quad 0 \quad 3 \quad 1]$,

$C_B = [-1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0] \cup [2 \quad 0 \quad -2 \quad 0 \quad 3 \quad 0]$, find $A_B - C_B$.

Solution: $A_B - C_B = [2 \quad 1 \quad 2 \quad -2 \quad 1 \quad 1] \cup [1 \quad -1 \quad 4 \quad 0 \quad 0 \quad 1]$.

Notes

- The subtract of two bimatrices is not in general bimatrix. For example, let $A_B = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 2 \\ -1 & 1 & 2 \end{bmatrix}$, and $C_B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, $A_B - C_B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is not bimatrix because $[A_1 - C_1] = [A_2 - C_2]$.

- If A_B and C_B be two mixed bimatrix then $(A_B - C_B)$ is always mixed bimatrix.

Multiplication of Two Bimatrices

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two square bimatrices. The multiplication of the bimatrices A_B and C_B is defined as follows:

$$A_B . C_B = (A_1 \cup A_2).(C_1 \cup C_2) = [A_1 . C_1] \cup [A_2 . C_2].$$

Example7: a) Let $A_B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $C_B = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$,

b) $\boldsymbol{A_B} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$,

c) $\boldsymbol{A_B} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & -4 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ -1 & 2 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{bmatrix}$,

find $\boldsymbol{A_B}.\ \boldsymbol{C_B}$.

Solution: a) $\boldsymbol{A_B}.\ \boldsymbol{C_B} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} . \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} =$

$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$,

b) $\boldsymbol{A_B}.\ \boldsymbol{C_B} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 3 & 1 \\ 8 & -1 & -5 \\ 6 & 0 & -3 \end{bmatrix} \cup \begin{bmatrix} 5 & -1 \\ 1 & -1 \end{bmatrix}$,

c) $\boldsymbol{A_B}.\ \boldsymbol{C_B} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 0 \\ -1 & 2 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 5 \\ -1 & 2 \\ -1 & 3 \\ 13 & -6 \end{bmatrix} \cup \begin{bmatrix} 7 & 4 & 3 \\ 1 & -3 & -1 \end{bmatrix}$.

Notes:

- The multiply of two bimatrices is not in general bimatrix.
- If $\boldsymbol{A_B} = (\boldsymbol{A_1})^{m\times n} \cup (\boldsymbol{A_2})^{p\times q}$ be mixed rectangular bimatrix and $\boldsymbol{C_B}$ $=(\boldsymbol{C_1})^{n\times m} \cup (\boldsymbol{C_2})^{q\times p}$ be another mixed rectangular bimatrix then $(\boldsymbol{A_B}.\boldsymbol{C_B})$ is square bimatrix.

Transpose

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$,to transpose the bimatrix, swap the rows and columns of each matrix $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ (i.e. $\boldsymbol{A^T}_{\boldsymbol{B}} = \boldsymbol{A^T}_{\boldsymbol{1}} \cup \boldsymbol{A^T}_{\boldsymbol{2}}$).

- Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices and $D_B = A_B + C_B$ then $D^T{}_B = A^T{}_B + C^T{}_B$.
- If A_B and C_B be two bimatrices then $(A_B C_B)^T = C^T{}_B A^T{}_B$.
- If $A_B, C_B, \ldots, N_B$ be bimatrices such that their product $(A_B\, C_B \ldots N_B)$ is well defined then $(A_B C_B \ldots N_B)^T = N^T \ldots C^T{}_B A^T{}_B$.

Example8: Let $A_B = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$,

$C_B = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \cup \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 1 & -1 \\ -2 & 0 \end{bmatrix}$, Find $A^T{}_B, C^T{}_B$ and $(A_B C_B)^T$.

Solution: $A^T{}_B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 1 & 0 \\ 4 & 2 \end{bmatrix}$, $C^T{}_B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & -2 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 1 & -2 \\ 3 & 0 & -1 & 0 \end{bmatrix}$

$A_B C_B = \begin{bmatrix} 9 & 2 & 5 & 2 \\ 0 & 1 & 1 & -2 \\ 3 & 2 & 3 & -2 \end{bmatrix} \cup \begin{bmatrix} -1 & 5 \\ -4 & 0 \end{bmatrix}$, $(A_B C_B)^T = \begin{bmatrix} 9 & 0 & 3 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \\ 2 & -2 & -2 \end{bmatrix} \cup \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix}$

$$C^T{}_B A^T{}_B = \begin{bmatrix} 9 & 0 & 3 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \\ 2 & -2 & -2 \end{bmatrix} \cup \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix}$$

Then $(A_B C_B)^T = C^T{}_B A^T{}_B$.

Some Basic Properties of Bimatrices

- Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two $(m \times n)$ bimatrices. The sum $A_B + C_B = (A_1 \cup A_2) + (C_1 \cup C_2) = [A_1 + C_1] \cup [A_2 + C_2]$ is bimatrix $\leftrightarrow$ $[A_1 + C_1] \neq [A_2 + C_2]$.
- Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two $(m \times n)$ bimatrices. The $A_B - C_B = (A_1 \cup A_2) - (C_1 \cup C_2) = [A_1 - C_1] \cup [A_2 - C_2]$ is bimatrix $\leftrightarrow$ $[A_1 - C_1] \neq [A_2 - C_2]$.
- Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two square bimatrices. The $A_B . C_B = [A_1 . C_1] \cup [A_2 . C_2]$ is bimatrix $\leftrightarrow$ $[A_1 . C_1] \neq [A_2 . C_2]$.
- If A_B and C_B be two $(m \times m)$ square bimatrices. In general $A_B . C_B \neq C_B . A_B$.

Example9: Let $\boldsymbol{A_B} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$, Find $\boldsymbol{A_B}.\boldsymbol{C_B}$ and $\boldsymbol{C_B}.\boldsymbol{A_B}$.

Solution: $\boldsymbol{A_B}.\ \boldsymbol{C_B} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{C_B}.\boldsymbol{A_B} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}.\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boldsymbol{A_B}.\boldsymbol{C_B} \neq \boldsymbol{C_B}.\boldsymbol{A_B}$.

- ❖ In some cases for the bimatrices $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ only one type of product $\boldsymbol{A_B}.\boldsymbol{C_B}$ may defined and $\boldsymbol{C_B}.\boldsymbol{A_B}$ may not be even defined.

Example10: Let $\boldsymbol{A_B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 4 \end{bmatrix}$.

Solution: $\boldsymbol{A_B}.\ \boldsymbol{C_B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 0 \\ 3 & 0 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 8 & -1 & 6 \\ 9 & 0 & 3 \\ 5 & -2 & 9 \end{bmatrix},$$

But $\boldsymbol{C_B}.\ \boldsymbol{A_B}$ is not define.

- ❖ Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$, $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ and $\boldsymbol{D_B} = \boldsymbol{D_1} \cup \boldsymbol{D_2}$ be three square bimatrices:

  $$\boldsymbol{A_B}(\boldsymbol{C_B}\boldsymbol{D_B}) = (\boldsymbol{A_B}\boldsymbol{C_B})\ \boldsymbol{D_B} = \boldsymbol{A_B}\boldsymbol{C_B}\boldsymbol{D_B} \text{ (Associative law)}$$

  Where $\boldsymbol{A_B}(\boldsymbol{C_B}\boldsymbol{D_B}) = \boldsymbol{A_B}(\boldsymbol{C_1}\boldsymbol{D_1} \cup \boldsymbol{C_2}\boldsymbol{D_2}) = \boldsymbol{A_1}(\boldsymbol{C_1}\boldsymbol{D_1}) \cup \boldsymbol{A_2}(\boldsymbol{C_2}\boldsymbol{D_2}) = (\boldsymbol{A_1}\boldsymbol{C_1})\boldsymbol{D_1} \cup (\boldsymbol{A_2}\boldsymbol{C_2})\boldsymbol{D_2} = (\boldsymbol{A_1}\boldsymbol{C_1} \cup \boldsymbol{A_2}\boldsymbol{C_2})\boldsymbol{D_B} = (\boldsymbol{A_B}\boldsymbol{C_B})\ \boldsymbol{D_B}$.

- ❖ Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$, $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ and $\boldsymbol{D_B} = \boldsymbol{D_1} \cup \boldsymbol{D_2}$ be three square bimatrices:

  $$\boldsymbol{A_B}(\boldsymbol{C_B} + \boldsymbol{D_B}) = \boldsymbol{A_B}\boldsymbol{C_B} + \boldsymbol{A_B}\boldsymbol{D_B} \text{ (Distributive law)}$$

Where $A_B(C_B + D_B) = A_B\{(C_1 + D_1) \cup (C_2 + D_2)\} = (A_1(C_1 + D_1)) \cup (A_2(C_2+D_2)) = (A_1C_1 + A_1D_1) \cup (A_2C_2+A_2D_2) = (A_1C_1 \cup A_2C_2)+(A_1D_1 \cup A_2D_2) = A_BC_B + A_BD_B$.

- Let $A_B = A_1 \cup A_2$ be a $(m \times m)$ square bimatrix. A_B is called diagonal if each of A_1 and A_2 are $(m \times m)$ diagonal matrices. The identity bimatrix is diagonal bimatrix. But if A_B mixed square bimatrix, A_B is called mixed diagonal bimatrix if both A_1 and A_2 are diagonal matrices.
- Diagonal bimatrix cannot be defined in case of rectangular bimatrix or mixed bimatrix which is not mixed square bimatrix.
- For every bimatrix A_B there is exist a zero bimatrix O_B such that

$$A_B + O_B = O_B + A_B = A_B$$

- $O_BA_B = O_B$, this is true only in case of square bimatrix or mixed square bimatrix only.
- If A_B is square bimatrix or mixed square bimatrix then

$$A_BA_B = A^2{}_B \;, \quad A_BA_BA_B = A^3{}_B \text{ and so on.}$$

- This type of product does not exist in case of rectangular bimatrix or mixed rectangular bimatrix.
- For any scalar λ , the square bimatrix $A^{m\times m}{}_B$ is called a scalar bimatrix if

$$A^{m\times m}{}_B = \lambda I^{m\times m}{}_B.$$

- If $A_B = A^{m\times m}{}_1 \cup A^{n\times n}{}_2$ then scalar bimatrix of A_B is defined as:

$$A_B = \lambda I_B = \lambda I^{m\times m}{}_1 \cup \lambda I^{n\times n}{}_2$$

- Null the bimatrix can be got for any form of bimatrices A_B and C_B provided the product (A_BC_B) is defined and $A_BC_B = [0]$.

Example11: Let $A_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cup [0 \quad 0 \quad 0 \quad 1 \quad 0]$, $C_B = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 3 & 1 \end{bmatrix} \cup \begin{bmatrix} 5 \\ 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}$.

Find A_BC_B.

Solution: $A_B.\, C_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 4 & 3 & 1 \end{bmatrix} \cup [0 \quad 0 \quad 0 \quad 1 \quad 0]\begin{bmatrix} 5 \\ 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cup [0]$$

Symmetric and Skew Symmetric Bimatrices

A bimatrix is *symmetric* if it is equal to its own transpose, (i.e. for the bimatrix $\boldsymbol{A_B}$, the component matrices of $\boldsymbol{A_B}$ are also symmetric, $\boldsymbol{A_B} = \boldsymbol{A^T}_{\mathbf{B}}$). A symmetric bimatrix must be either a square bimatrix or a mixed square bimatrix. Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ be a $(m \times m)$ square bimatrix. This is an m^{th} order square bimatrix. This will not have $2m^2$ arbitrary elements since $a^1{}_{ij} = a^1{}_{ji}$ and $a^2{}_{ij} = a^2{}_{ji}$ where $\boldsymbol{A_1} = (a^1{}_{ij})$ and $\boldsymbol{A_2} = (a^2{}_{ij})$ both below and above the main diagonal. The number above the main diagonal of $\boldsymbol{A_B}$ is $(m^2 - m)$ and the diagonal elements are also arbitrary. Thus the total number of arbitrary elements in an m^{th} order square symmetric bimatrix is $(m^2 - m + 2m) = m(m+1)$. But if $\boldsymbol{A_B} = \boldsymbol{A}^{m\times m}{}_{\mathbf{1}} \cup \boldsymbol{A}^{n\times n}{}_{\mathbf{2}}$ be a mixed square bimatrix then it has a total number $(\frac{m(m+1)}{2} + \frac{n(n+1)}{2})$ arbitrary elements.

Example12: Let $\boldsymbol{A_B} = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & -5 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 2 \\ 1 & -5 & 3 \\ 2 & 3 & 0 \end{bmatrix}$,

$C_B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & 0 & -1 & 2 \\ 2 & -1 & 1 & -4 \\ 4 & 2 & -4 & 8 \end{bmatrix}$, find if A_B and C_B are symmetric bimatrices.

Solution: $\boldsymbol{A^T}_{\mathbf{B}} = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & -5 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 2 \\ 1 & -5 & 3 \\ 2 & 3 & 0 \end{bmatrix} = \boldsymbol{A_B}$, $\boldsymbol{A_B}$ has $(3(3+1) = 3(4) = 12)$ arbitrary elements.

$\boldsymbol{C^T}_{\mathbf{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & 0 & -1 & 2 \\ 2 & -1 & 1 & -4 \\ 4 & 2 & -4 & 8 \end{bmatrix} = \boldsymbol{C_B}$, $\boldsymbol{C_B}$ has $(\frac{2(2+1)}{2} + \frac{4(4+1)}{2} = \frac{2(3)}{2} + \frac{4(5)}{2} = \frac{6}{2} + \frac{20}{2} = 3 + 10 = 13)$ arbitrary elements.

$\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ are symmetric bimatrices.

- A ***skew-symmetric*** is a bimatrix $\boldsymbol{A_B}$ for which $(\boldsymbol{A_B} = -\boldsymbol{A^T}_{\mathbf{B}})$ where $(-\boldsymbol{A^T}_{\mathbf{B}} = -\boldsymbol{A^T}_{\mathbf{1}} \cup -\boldsymbol{A^T}_{\mathbf{2}})$ (i.e. the component matrices $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ of $\boldsymbol{A_B}$ are also skew-symmetric).

- If the m^{th} order skew-symmetric bimatrix have the diagonal elements of $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are zero (i.e. $a^1{}_{ii} = a^2{}_{ii} = 0$) then the number of arbitrary elements is $2m(m-1)$.
- If $\boldsymbol{A_B} = \boldsymbol{A}^{m\times m}{}_{\mathbf{1}} \cup \boldsymbol{A}^{n\times n}{}_{\mathbf{2}}$ is a mixed square bimatrix then $\boldsymbol{A_B}$ is called skew-symmetric $\boldsymbol{A_B} = -\boldsymbol{A^T}_{\mathbf{B}}$, i.e.,
  $(\boldsymbol{A_1}^{m\times m} = -(\boldsymbol{A}^{m\times m}{}_{\mathbf{1}})^T)$ and $(\boldsymbol{A_2}^{n\times n} = -(\boldsymbol{A}^{n\times n}{}_{\mathbf{2}})^T)$. This bimatrix has $(m(m-1) + n(n-1))$ arbitrary elements.

Example13: Let $\boldsymbol{A_B} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & 13 \\ -3 & 0 & -2 \\ -13 & 2 & 0 \end{bmatrix}$,

$\boldsymbol{C_B} = \begin{bmatrix} 0 & -1 & -2 & -4 \\ 1 & 0 & 1 & -2 \\ 2 & -1 & 0 & 4 \\ 4 & 2 & -4 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find if $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ are skew-symmetric bimatrices.

Solution: $-\boldsymbol{A^T}_{\mathbf{B}} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & 13 \\ -3 & 0 & -2 \\ -13 & 2 & 0 \end{bmatrix} = \boldsymbol{A_B}$, $\boldsymbol{A_B}$ has $2(3)(3-1) = 6(2) = 12$ arbitrary elements.

$-\boldsymbol{C^T}_{\mathbf{B}} = \begin{bmatrix} 0 & -1 & -2 & -4 \\ 1 & 0 & 1 & -2 \\ 2 & -1 & 0 & 4 \\ 4 & 2 & -4 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \boldsymbol{C_B}$, $\boldsymbol{C_B}$ has $4(4-1) + 2(2-1) = 4(3) + 2(1) = 12 + 2 = 14)$ arbitrary elements.

$\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ are skew-symmetric bimatrices.

Subbimatrix

Let $\boldsymbol{A_B} = \boldsymbol{A_1^{m\times n}} \cup \boldsymbol{A_2^{p\times q}}$ be a bimatrix. If we cross out all but $\boldsymbol{k_1}$ rows and $\boldsymbol{s_1}$ columns of $(m \times n)$ matrix $\boldsymbol{A_1}$ and cross out all but $\boldsymbol{k_2}$ rows and $\boldsymbol{s_2}$ columns of $(p\times q)$ matrix $\boldsymbol{A_2}$ the resulting $(\boldsymbol{k_1} \times \boldsymbol{s_1})$ and $(\boldsymbol{k_2} \times \boldsymbol{s_2})$ bimatrix is called a Subbimatrix of $\boldsymbol{A_B}$.

Example14: Let $A_B = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 6 & 0 & 1 & 2 \\ -1 & 6 & -1 & 0 \end{bmatrix} \cup \begin{bmatrix} 3 & 8 & 3 & 6 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & -1 & 3 \\ -1 & 4 & 0 & 0 & 2 \end{bmatrix}$.

Solution: $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 6 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 8 \\ 1 & 1 \\ 2 & 1 \\ -1 & 4 \end{bmatrix}$ is a subbimatrix of A_B.

Bideterminant

Let $A_B = A_1 \cup A_2$ be a square bimatrix. The bideterminant of a square bimatrix is an ordered pair (d_1, d_2) where $d_1 = |A_1|$ and $d_2 = |A_2|$.
$|A_B| = (d_1, d_2)$ where d_1 and d_2 are real maybe positive or negative or zero.

Example15: Let $A_B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 4 & 5 \\ -2 & 0 \end{bmatrix}$, find $|A_B|$.

Solution: $|A_B| = (0,10)$.

- If A_B and C_B be square bimatrices of order *n* then their product $D_B = A_B\, C_B$.

$$D_B = A_B C_B = (A_1 C_1) \cup (A_2 C_2)$$
$$|D_B| = |A_B C_B| = |A_1||C_1| \cup |A_2||C_2|$$

i.e., the determinant of the product is the product of the determinants.

Example16: Let $A_B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cup \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}$, and $C_B = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix} \cup \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$, find $|A_B\, C_B|$.

Solution: $|A_B\, C_B| = (-80, -39)$.

$$|A_1| = 5, |A_2| = -3, |C_1| = -16, |C_2| = 13,$$
$$|A_B C_B| = |A_1||C_1| \cup |A_2||C_2| = (-80,-39)$$

- If A_B and C_B be rectangular bimatrices then product
  $D_B = A_B\, C_B$, $A_B C_B = (A_1 C_1) \cup (A_2 C_2)$
  $|A_B C_B| = |A_1 C_1| \cup |A_2 C_2| = (d_1, d_2)$ where $d_1 = |A_1 C_1|$ and $d_2 = |A_2 C_2|$.

Example17: Let $A_B = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, and

$C_B = \begin{bmatrix} 3 & 0 \\ 9 & 2 \\ 1 & 7 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 5 & -1 \end{bmatrix}$, find $|\, A_B\, C_B|$.

Solution: $A_B\, C_B = \begin{bmatrix} \mathbf{44} & \mathbf{43} \\ \mathbf{9} & \mathbf{21} \end{bmatrix} \cup \begin{bmatrix} 5 & 1 \\ 7 & 1 \end{bmatrix}$

$|\, A_B\, C_B| = (537, -2).$

Biinverse of Bimatrix

Let $A_B = A_1 \cup A_2$ be a square bimatrix, if there exists a square bimatrix $A_B^{-1} = A_1^{-1} \cup A_2^{-1}$ which satisfied the following:

$A_B\, A_B^{-1} = A_B^{-1} A_B = A_1 A_1^{-1} \cup A_2 A_2^{-1} = I \cup I$, then A_B^{-1} is called the biinverse or bireciprocal of A_B.

- It is most important to note that even A_B be a mixed square bimatrix then also A_B^{-1} exists by $I_1 \cup I_2 = I_B$ will be such $I_1 \neq I_2$.
- It is most important to note that: $A_B^{-1} \neq \frac{1}{A_B}$ or $\frac{I}{A_B}$

Example18: Let $A_B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$, and

$C_B = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A_B^{-1} and $C_B^{-1}(HW)$.

Solution: $A_B^{-1} = \begin{bmatrix} 1 & 0 \\ -2/3 & 1/3 \end{bmatrix} \cup \begin{bmatrix} 1/2 & -1 \\ 1/2 & 0 \end{bmatrix} =$

$A_B A^{-1}{}_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Properties of biinverse

1- $(A_B C_B)^{-1} = C_B^{-1} A_B^{-1}$

2- $C_B^{-1} A_B^{-1} A_B C_B = C_B^{-1} C_B = I_B$

Example19: Let $A_B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, and $C_B = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 5 \\ 6 & 4 \end{bmatrix}$, find $(A_B C_B)^{-1}$.

Solution: $A_B^{-1} = \begin{bmatrix} 1 & 0 \\ -2/3 & 1/3 \end{bmatrix} \cup \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$$C_B^{-1} = \begin{bmatrix} -1/8 & 5/8 \\ 1/4 & -1/4 \end{bmatrix} \cup \begin{bmatrix} -37/30 & 7/15 \\ 3/5 & -1/5 \end{bmatrix}$$

$(A_B C_B)^{-1} = \left\{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\right\}^{-1} \cup \left\{\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 6 & 4 \end{bmatrix}\right\}^{-1} = \left\{\begin{bmatrix} 2 & 5 \\ 10 & 13 \end{bmatrix}\right\}^{-1} \cup \left\{\begin{bmatrix} 6 & 14 \\ 18 & 37 \end{bmatrix}\right\}^{-1} = \begin{bmatrix} -13/24 & 5/24 \\ 5/12 & -1/12 \end{bmatrix} \cup \begin{bmatrix} -37/30 & 7/15 \\ 3/5 & -1/5 \end{bmatrix} = C_B^{-1} A_B^{-1}$.

3- $(A_B^{-1})^{-1} = A_B$

Example20: Let $A_B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, find $(A_B^{-1})^{-1}$.

Solution: $A_B^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ -2/3 & 1/3 \end{bmatrix}$

$(A_B^{-1})^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = A_B$.

The square bimatrix A_B is non- bisingular if $|A_B| \neq (0,0)$. If $|A_B| = (0,0)$ then the bimatrix A_B is bisingular. Let $A_B = A_1 \cup A_2$, if one of A_1 or A_2 is non- singular matrix then the bimatrix A_B is called semi bisingular.

Example21: Let $A_B = \begin{bmatrix} 0 & 7 \\ 0 & 5 \end{bmatrix} \cup \begin{bmatrix} 3 & 8 \\ 6 & 16 \end{bmatrix}$, $C_B = \begin{bmatrix} 1 & 5 \\ 5 & 25 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $D_B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cup \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}$

Solution: A_B is bisingular since $|A_B| = (0,0)$,

C_B is semi bisingular since $\begin{vmatrix} 1 & 5 \\ 5 & 25 \end{vmatrix} = 0$ but $\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3$and D_B is non bisingular since $|D_B| = (5, -3)$.