Republic of Iraq

Ministry of Higher Education and

Diyala University

Miqdad College of Education

Mathematics department

Dynamical Systems

This research was presented as part of the requirements for obtaining a bachelor's degree in the Faculty of Education Miqdad in the Department of Mathematics at the University of Diyala

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1443 2022

بسم الله الرحمن الرحيم

{وَلَقَدْ نَعْلَمُ أَنَّكَ يَضِيقُ صَدْرُكَ بِمَا يَقُولُونَ (٩٧) فَسَبِّحْ بِحَمْدِ رَبِّكَ وَكُن مِّنَ السَّاجِدِينَ (٩٨) وَاعْبُدْ رَبَّكَ حَتَّى يَأْتِيَكَ الْيَقِينُ (٩٩)}

سوره الحجر

الأهدأ ء

بِسْمِ اللَّهِ الرحمن الرَّحِيمِ

الَّذِينَ آمَنُوا وَتَطْمَئِنُّ قُلُوبُهُم بِذِكْرِ اللَّهِ ۗ أَلَا بِذِكْرِ اللَّهِ تَطْمَئِنُّ الْقُلُوبُ

أولا: نحمدالله عز وجل الذي وفقنا في اتمام هذا البحث العلمي
......اهدي تخرجي
الى صاحب السيرة. العطرة الذي كان الفضل الأول في بلوغي التعليم العالي (والدي الحبيب أطال الله في عمره)
إلى من وضعتني في طريق الحياة (أمي الحبيبة)
الى كل من كان لهم بالغ الأثر في كثير من الصعاب الي جميع استاذتي الكرام ونتقدم بجزيل الشكر الدكتور المشرف خالد هادي على كل ما قدمه لنا من توجيهات ومعلومات ساهمت في اثراء
موضوع دراستنا
وكذلك نشكر الذين كإنو عونا ونور ا يضئ الظلمة التي كانت احيانا في طريقنا.

المحتويات

ملخص البحث

في هذا البحث تم دراسة الأنظمة الديناميكية التي تفسر الدوال التي تعتمد على متغير حقيقي له تأثير بدراسة شكل الدالة وخواصها.
حيث تم دراسة الأنظمة ذات البعد الواحد مع بعض امثلتها بالإضافة الى دراسة اشكالها بتوسيعها على خط الاعداد الحقيقة. ثم تم دراسة الأنظمة ذات البعدين وأكثر من خلال دراسة وإعطاء تعاريف خاصة لها بالإضافة الى بعض الأمثلة التوضيحية والاشكال المهمة لها. وفي الختام تم توضيح بعض الأمثلة التطبيقية لذلك.

The Introduction

In mathematics, a **dynamical system** is a system in which a function describes the time dependence of a point in an ambient space. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each springtime in a lake. The most general definition unifies several concepts in mathematics such as ordinary differential equations and ergodic theory by allowing different choices of the space and how time is measured. Time can be measured by integers, by real or complex numbers or can be a more general algebraic object, losing the memory of its physical origin, and the space may be a manifold or simply a set, without the need of a smooth space-time structure defined on it.
At any given time, a dynamical system has a state representing a point in an appropriate state space. This state is often given by a tuple of real numbers or by a vector in a geometrical manifold. The *evolution rule* of the dynamical system is a function that describes what future states follow from the current state. Often the function is deterministic, that is, for a given time interval only one future state follows from the current state.[1][2] However, some systems are stochastic, in that random events also affect the evolution of the state variables.

In physics, a **dynamical system** is described as a "particle or ensemble of particles whose state varies over time and thus obeys differential equations involving time derivatives".[3] In order to make a prediction about the system's future behavior, an analytical solution of such equations or their integration over time through computer simulation is realized.

The study of dynamical systems is the focus of dynamical systems theory, which has applications to a wide variety of fields such as mathematics, physics,[4][5] biology,[6] chemistry, engineering,[7] economics,[8] history, and medicine. Dynamical systems are a fundamental part of chaos theory, logistic map dynamics, bifurcation theory, the self-assembly and self-organization processes, and the edge of chaos concept.

The concept of a dynamical system has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of dynamical systems is an implicit relation that gives the state of the system for only a short time into the future. (The relation is either a differential equation, difference equation or other time scale.) To determine the state for all future times requires iterating the relation many times—each advancing time a small step. The iteration procedure is referred to as *solving the system* or *integrating the system*. If the system can be solved, given an initial point it is possible to determine all its future positions, a collection of points known as a *trajectory* or *orbit*.

Before the advent of computers, finding an orbit required sophisticated mathematical techniques and could be accomplished only for a small class of dynamical systems. Numerical methods implemented on electronic computing machines have simplified the task of determining the orbits of a dynamical system.

For simple dynamical systems, knowing the trajectory is often sufficient, but most dynamical systems are too complicated to be understood in terms of individual trajectories. The difficulties arise because:

- The systems studied may only be known approximately—the parameters of the system may not be known precisely or terms may be missing from the equations. The approximations used bring into question the validity or relevance of numerical solutions. To address these questions several notions of stability have been introduced in the study of dynamical systems, such as Lyapunov stability or structural stability. The stability of the dynamical system implies that there is a class of models or initial conditions for which the trajectories would be equivalent. The operation for comparing orbits to establish their equivalence changes with the different notions of stability.
- The type of trajectory may be more important than one particular trajectory. Some trajectories may be periodic, whereas others may wander through many different states of the system. Applications often require enumerating these classes or maintaining the system within one class. Classifying all possible trajectories has led to the qualitative study of dynamical systems, that is, properties that do not change under coordinate

changes. Linear dynamical systems and systems that have two numbers describing a state are examples of dynamical systems where the possible classes of orbits are understood.

- The behavior of trajectories as a function of a parameter may be what is needed for an application. As a parameter is varied, the dynamical systems may have bifurcation points where the qualitative behavior of the dynamical system changes. For example, it may go from having only periodic motions to apparently erratic behavior, as in the transition to turbulence of a fluid.
- The trajectories of the system may appear erratic, as if random. In these cases, it may be necessary to compute averages using one very long trajectory or many different trajectories. The averages are well defined for ergodic systems and a more detailed understanding has been worked out for hyperbolic systems. Understanding the probabilistic aspects of dynamical systems has helped establish the foundations of statistical mechanics and of chaos.

Chapter One

Dynamical Systems

A dynamical system is defined simply by:

$$\dot{x} = f(x),\ x, f \text{ vectors in } R^N \tag{1}$$

N is called the number of degrees of freedom of the system.

Here are some examples of dynamical systems.

-Normal form of a saddle-node bifurcation ($N = 1$)

$$\dot{x} = \mu - x^2 \tag{2}$$

–Lorenz model ($N = 3$)

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10(y-x) \\ -xz + rx - y \\ xy - 8z/3 \end{pmatrix} \tag{3}$$

–Navier-Stokes equations ($N \gg 1$)

$$\frac{d}{dt}\mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \frac{1}{Re}\Delta_{\mathbf{u}} \tag{4}$$

For the Navier-Stokes equations, $\mathbf{u}(\mathbf{x}) = (u\,(x, y, z), v\,(x, y, z), w\,(x, y, z))$ and $N = \infty$. $N = 3 \times 100^3 = 3\times10^6$. In a typical three-dimensional numerical simulation, one uses a spatial discretization of

$N_{XR} = N_y = N_{zo} = 100$, leading to $N = 3 \times 100^3 = 3 \times 10^6$.

Other systems can be re-written as dynamical systems, by writing additional variables, in particular, a *non-autonomous* system:

$$\dot{x} = f(x,t) \Longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} f(x,\theta) \\ 1 \end{pmatrix} \text{ with } \theta \equiv t \tag{5}$$

or a system of higher temporal order:

$$\ddot{x} = f(x,\dot{x}) \Longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f(x,y) \end{pmatrix} \text{ with } y \equiv \dot{x} \tag{6}$$

1.2 One-dimensional systems

1.2.1 Fixed points and linear stability

We begin with a dynamical system:

$$\dot{x} = f(x) \tag{7}$$

A fixed-point $\bar{x}$ is a solution to:

$$0 = f(\bar{x}) \tag{8}$$

Fixed points $\bar{x}$, also called steady states, are thus roots of the function f. The *linear stability* of $\bar{x}$ can be

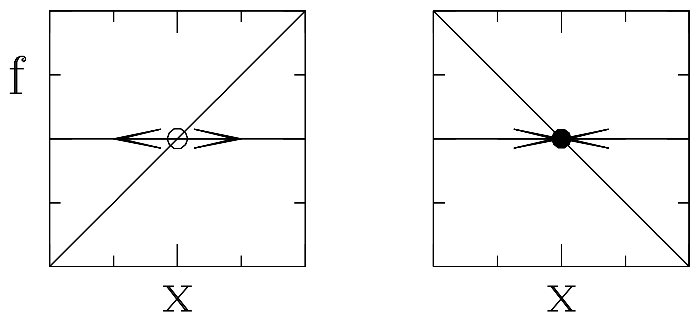


Figure 1: Unstable (left) and stable (right) fixed points. A fixed point of $\dot{x} = f(x)$ is a solution to $f(\bar{x}) = 0$. The dynamics causes x to increase (decrease) where f is positive (negative). If $f'(\bar{x}) > 0$, neighboring points evolve by leaving $\bar{x}$, which is thus unstable (left). If $f'(\bar{x}) < 0$, neighboring points evolve by approaching $\bar{x}$, which is thus stable (right).

studied by writing:

$$\begin{aligned} x(t) &= \bar{x} + \epsilon(t) \\ \frac{d}{dt}(\bar{x} + \epsilon) &= f(\bar{x} + \epsilon) \\ \cancel{\dot{\bar{x}}} + \dot{\epsilon} &= \cancel{f(\bar{x})} + f'(\bar{x})\epsilon + \frac{1}{2} f''(\bar{x})\epsilon^2 + \cdots \\ &\approx f'(\bar{x})\epsilon \\ \epsilon(t) &= e^{t f'(\bar{x})}\epsilon(0) \end{aligned} \tag{9a)(9b}$$

A perturbation ϱ will grow exponentially in time $f'(\bar{x}) > 0$, i.e., if $\bar{x}$ is unstable. In contrast, if $f'(\bar{x}) < 0$, then ϱ decreases exponentially in time and $\bar{x}$ is stable, as shown in figure 1.

In what follows, we will assume that f depends on a parameter μ, for example a Reynolds or Rayleigh number measuring a velocity or temperature gradient imposed on a fluid. A *steady bifurcation* is defined as a change in the number of fixed points (roots of f). We will see that this is closely connected to stability.

1.3 Saddle-node bifurcations

A linear function cannot change its number of roots. The simplest function that can change the number of its roots as μ is varied is a quadratic polynomial, like that shown in figure 2.

$$f(x, \mu) = c_{00} + c_{10}x + c_{01}\mu + c_{20}x^2 \tag{10}$$

is assumed to represent the first terms of a Taylor expansion of a general function. We re-write (10) as:

$$f(x, \mu) = c_{01}\mu + c_{00} - \frac{c_{10}^2}{4c_{20}} + c_{20}\left(x + \frac{c_{10}}{2c_{20}}\right)^2 \tag{11}$$

If $c_{20} < 0$ and $c_{01} > 0$, we can define

$$\tilde{\mu} \equiv c_{01}\mu + c_{00} - \frac{c_{10}^2}{4c_{20}} \qquad \tilde{x} \equiv \sqrt{-c_{20}}\left(x + \frac{c_{10}}{2c_{20}}\right) \tag{12}$$

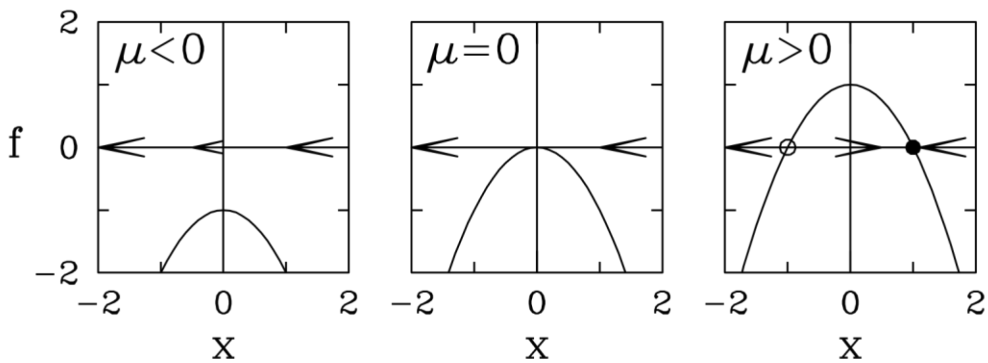


Figure 2: The function $f = \mu - x^2$ has 0, 1, or 2 roots, if $\mu < 0$, $\mu = 0$, or $\mu > 0$.

and write

$$f = \tilde{\mu} - \tilde{x}^2 \tag{13}$$

or, re-defining $\tilde{\mu} \to \mu$, $\tilde{x} \to x$,

$$f(x, \mu) = \mu - x^2 \tag{14}$$

We call (14) the *normal form* of the *saddle-node bifurcation*. Let us study the behavior of (14). The fixed points of (14) are

$$\bar{x}_\pm = \pm\sqrt{\mu} \tag{15}$$

which exist only for $\mu > 0$. Their stability is determined by $f'(\bar{x}_\pm) = -2\bar{x}_\pm = -2(\pm\sqrt{\mu}) = \mp 2\sqrt{\mu}$ (16)

By looking at the sign of $f'(\bar{x})$, we see that $\bar{x}_+ = \sqrt{\mu}$ is stable, whereas $\bar{x}_- = -\sqrt{\mu}$ is unstable.

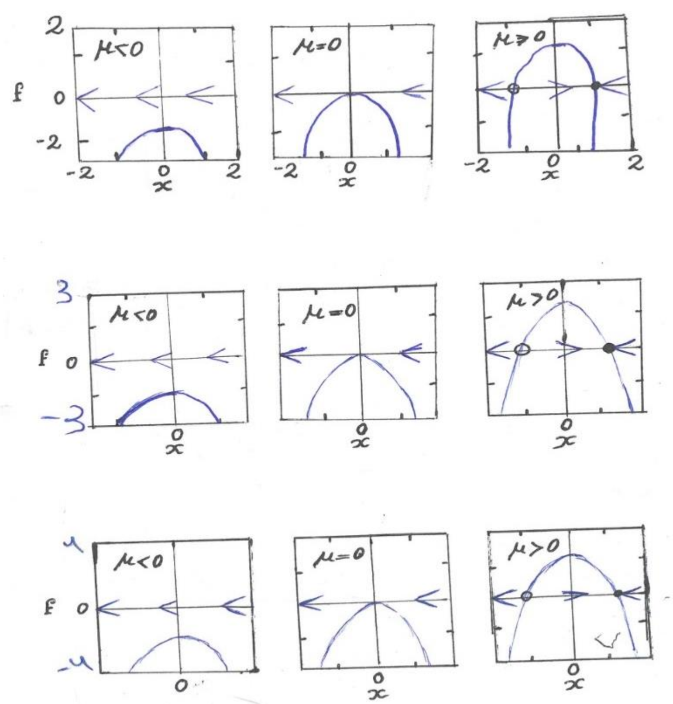
If $c_{20} > 0$ or $c_{01} < 0$, we deduce one of the following forms:

$$f(x,\mu) = -\mu + x^2 \quad (17a)$$

$$f(x,\mu) = \mu - x^2 \quad (17b)$$

$$f(x,\mu) = -\mu - x^2 \quad (17c)$$

In each case, there is a transition at $\mu = 0$ between no fixed points and two fixed points, one stable and one unstable. Figure 3 summarizes this information for each case on what is called its *bifurcation diagram*.



$f(x, m) = c_{00} + c_{10}x + c_{01}\mu + c_{20}x^2$

$f(-4,0) = 1+(-4) +(0) +(-4)^2$

$=13$

$f(-3,0) = 1+(-3) +(0) +(-3)^2$

$=7$

$f(-2,0) = 1+(-2) +(0) +(-2)^2$

$=3$

$f(-1,0) = 1+(-1) +(0) +(-1)^2$

$=1$

$f(0,0) = 1+(0) +(0) +(0)^2$

$=1$

$f(1,0) = 1+(1) +(0) +(1)^2$

$=3$

$f(2,0) = 1+(2) +(0) +(2)^2$

$=7$

$f(3,0) = 1+(3) +(0) +(9)^2$

$=13$

$f(4,0) = 1+(4) +(0) +(16)^2$

$=2$

μ=0

X= -4, -3, -2, -1

$f\,(x, m) = c_{00} + c_{10}x + c_{01}\mu + c_{20}x^2$

$f\,(0, -4) = 1+(0) +(-4) +(X)^2$

= -3

$f\,(0, -3) = 1+(0) +(-3) +(X)^2$

= -2

$f\,(0, -2) = 1+(0) +(-2) +(X)^2$

= -1

$f\,(0, -1) = 1+(0) +(-1) +(X)^2$

= 1

$f\,(0,0) = 1+(0) +(0) +(X)^2$

= 1

$f\,(0, 1) = 1+(0) +(1) +(X)^2$

= 2

$f\,(0,2) = 1+(0) +(2) +(X)^2$

= 3

$f\,(0,3) = 1+(0) +(3) +(X)^2$

= 4

$f\,(0,4) = 1+(0) +(4) +(X)^2$

= 5

1.4 Pitchfork bifurcations

For reasons of symmetry, to which we shall return later, it may be that $f(x)$ is restricted to be an odd function of x. (x is to be considered as a deviation from some special state, called a base state, rather than the distance from zero.) There is then no constant or quadratic term in x, and a cubic term must be included, as in figure 4, for a bifurcation to take place. We can reduce a cubic polynomial to four cases:

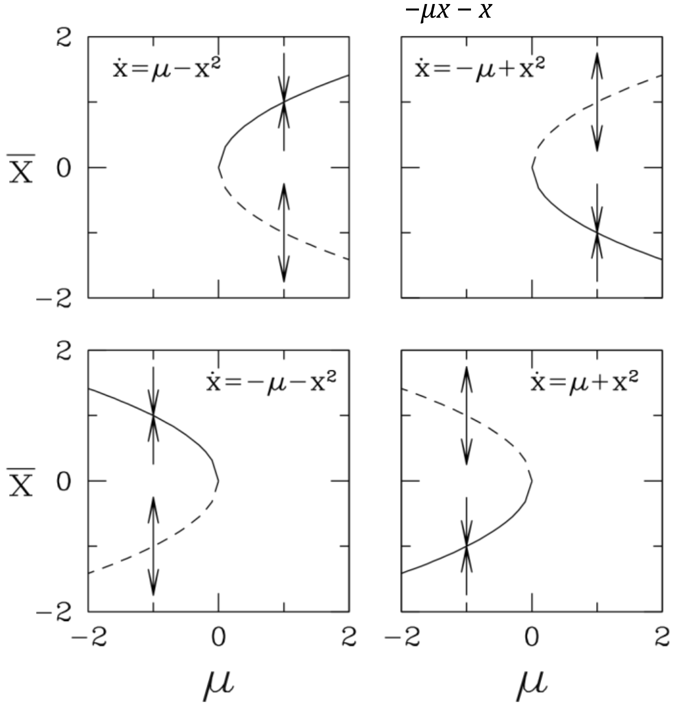
$$f(x,\mu) = \mu x - x^3 \quad (18a)$$

$$f(x,\mu) = \mu x + x^3 \quad (18b)$$

$$f(x,\mu) = -\mu x + x_3{}^3 \quad (18c)$$

$$f(x,\mu) = \quad (18d)$$

$$-\mu x - x$$



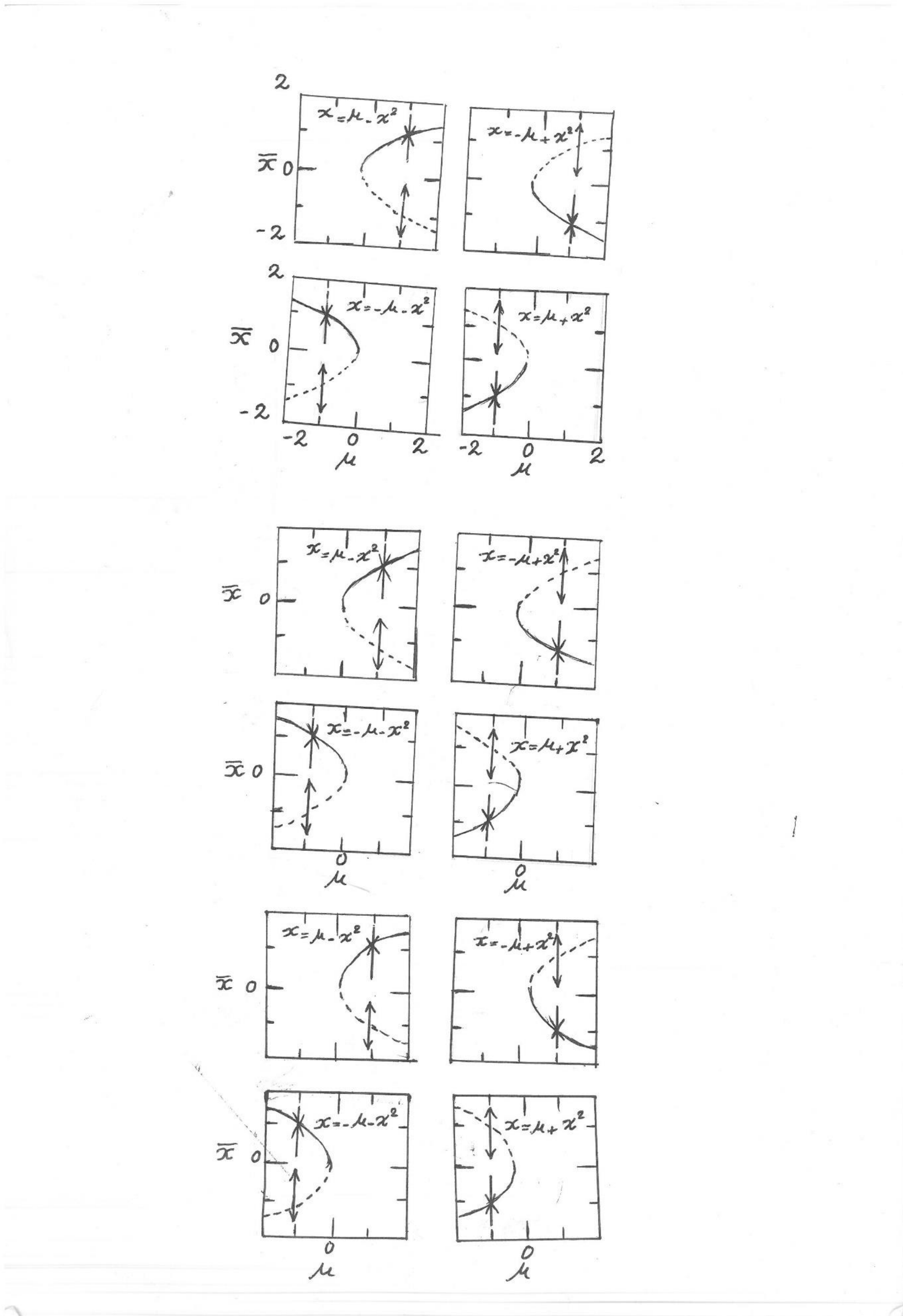


Figure 3: Saddle-node bifurcation diagrams. In each case, there exist two branches of fixed points, one stable and one unstable, on one side of $\mu = 0$, and no fixed points on the other side.

The four corresponding bifurcation diagrams are given in figure 1.3. Equation (18a) is called the normal form of a *supercritical pitchfork bifurcation*. Its fixed points are calculated by:

$$0 = \bar{x}(\mu - \bar{x}^2) \Rightarrow \begin{cases} \bar{x} = 0 & for\ all\ \mu \\ \bar{x} = \pm\sqrt{\mu} & for\ \mu > 0 \end{cases} \quad (19)$$

We now determine the stability of these fixed points.

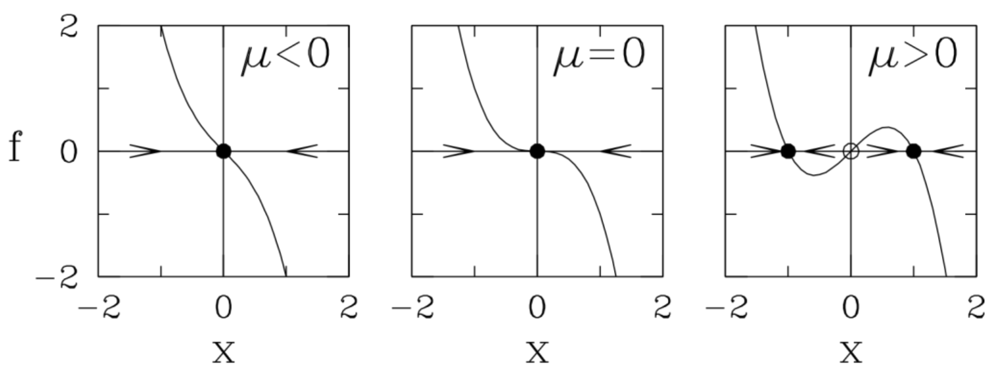
$$f'(\bar{x}) = \mu - 3\bar{x}^2 = \begin{cases} \mu & for\ \bar{x} = 0 \\ \mu - 3\mu = -2\mu & for\ \bar{x} = \pm\sqrt{\mu} \end{cases} \quad (20)$$

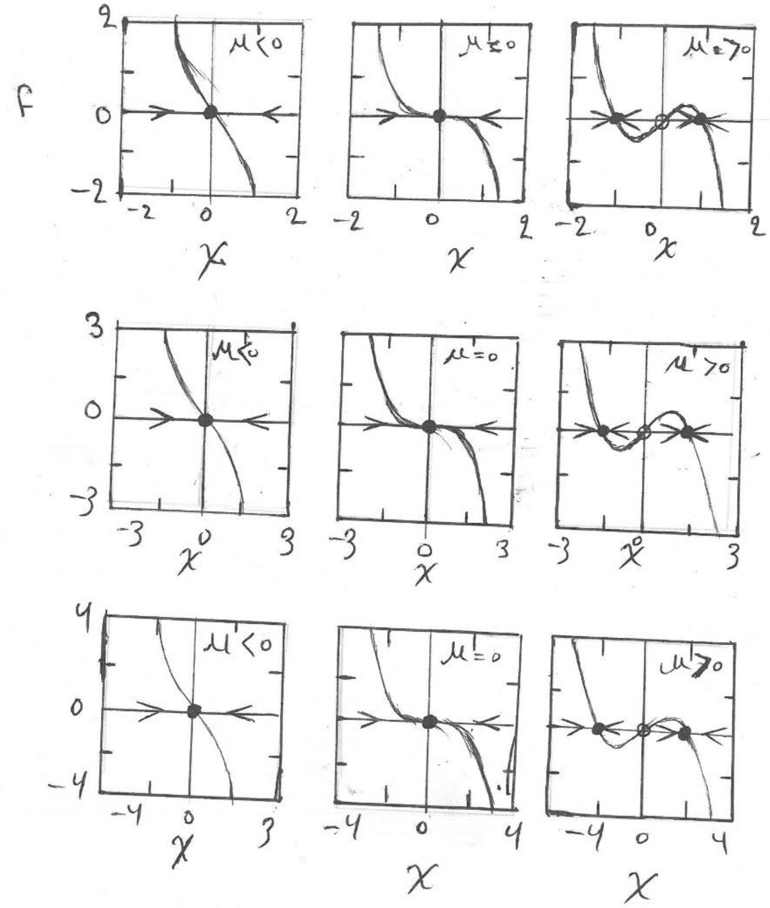
The fixed-point $\bar{x} = 0$ is therefore stable for $\mu < 0$ and becomes unstable at $\mu = 0$, where the new branches of fixed points $\bar{x} = \pm\sqrt{\mu}$ are created. These new fixed points are stable. We now repeat the calculation for (18b), which is the normal form of a *subcritical pitchfork bifurcation*.

$$0 = \bar{x}(\mu + \bar{x}^2) \Longrightarrow \begin{cases} \bar{x} = 0 & \text{for all } \mu \\ \bar{x} = \pm\sqrt{-\mu} & \text{for } \mu < 0 \end{cases} \quad (21)$$

$$f'(\bar{x}) = \mu + 3\bar{x}^2 = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ \mu + 3(-\mu) = -2\mu & \text{for } \bar{x} = \pm\sqrt{\mu} \end{cases} \quad (22)$$

As in the supercritical case, the fixed-point $\bar{x} = 0$ is stable for $\mu < 0$ and becomes unstable at $\mu = 0$. But, contrary to the supercritical case, the other fixed points $\pm\sqrt{\mu}$ exist in the region where´ $\bar{x} = 0$ is stable; the other fixed points are unstable.





f
2
0
-2
-2
0
2
x
-2
0
2
x
-2
0
x
2
3
0
-3
-3
x
0
3
-3
0
x
3
-3
x
0
3
4
0
-4
-4
0
x
3
-4
0
4
x
-4
0
4
x

Figure 4: The function $f = \mu x - x^2$ has 1 or 3 roots, according to whether $\mu < 0$, $\mu = 0$, or $\mu > 0$.

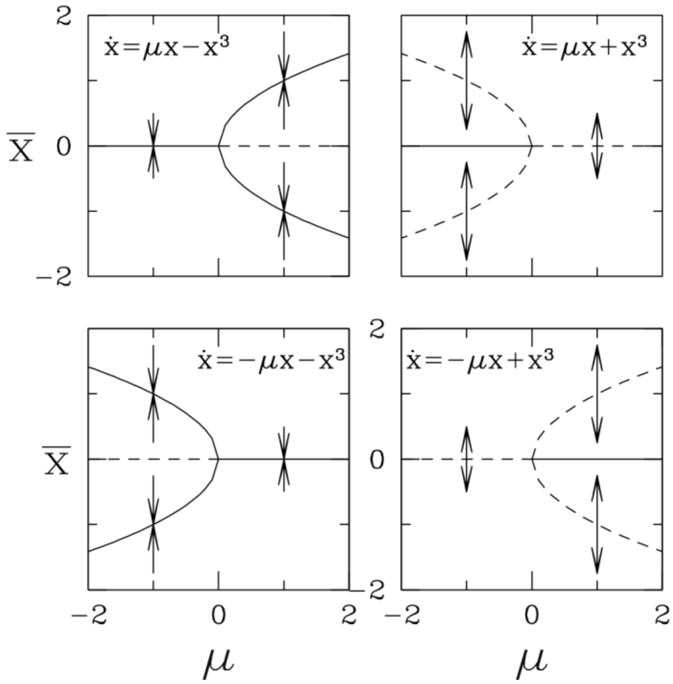


Figure 5: Pitchfork bifurcation diagrams. A branch of fixed points gives rise to two new branches when a critical value of μ is crossed. The bifurcation is called a supercritical (subcritical) pitchfork if the new branches are stable (unstable).

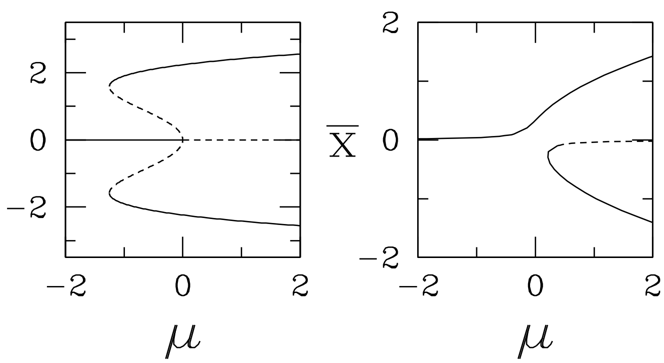
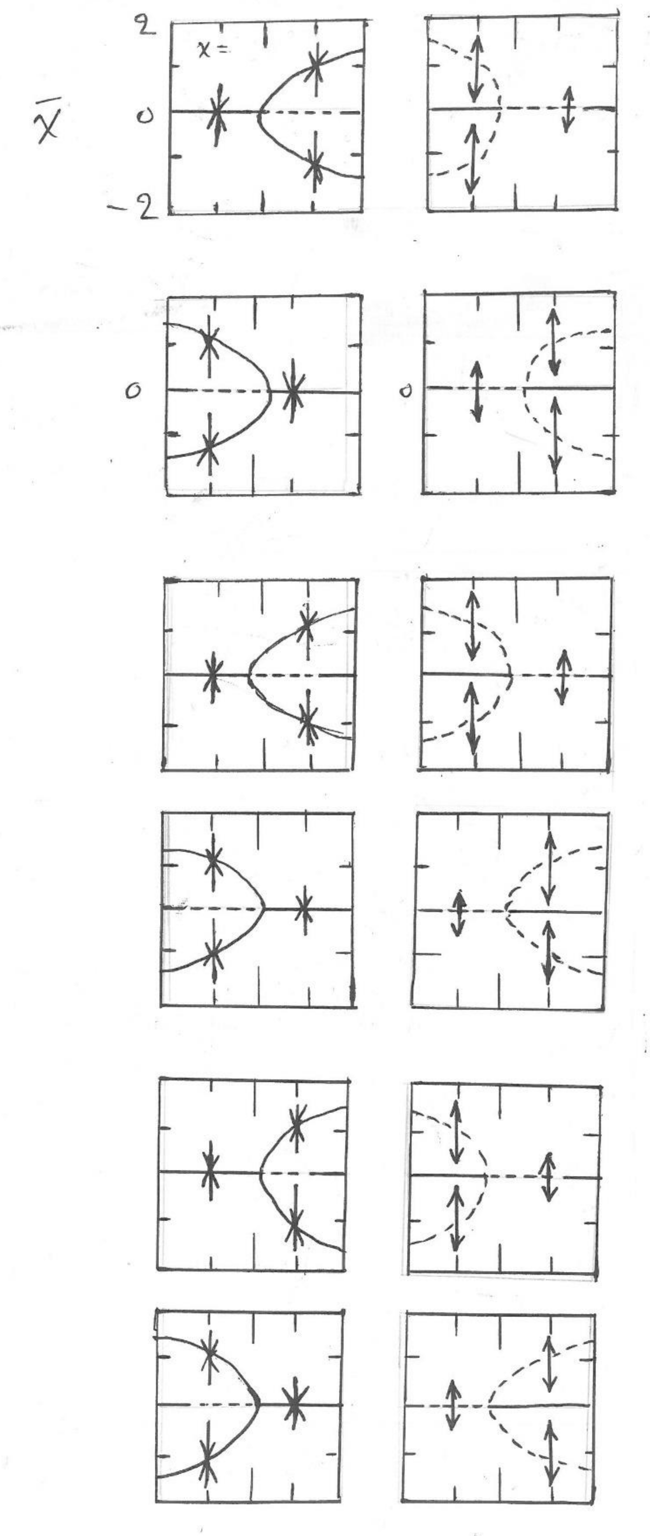


Figure 6: Left: bifurcation diagram for $\dot{x} = \mu x + x^3 - x^5/10$. The fifth-order term stabilizes the trajectories near a subcritical pitchfork bifurcation. This term causes two saddle-node bifurcations. As μ is increased, there is first one fixed point, then five, then finally three fixed points. Right: Diagram for an imperfect

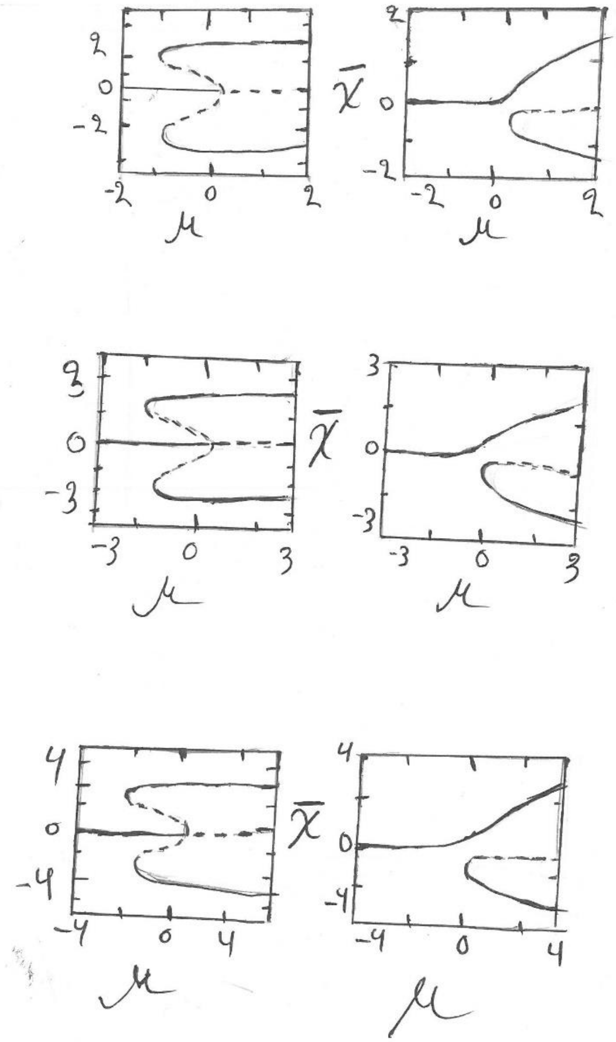
pitchfork bifurcation $\dot{x} = 1/27 + \mu x - x^3$. The constant term represents an imperfection, that causes the system to prefer one of the two branches. The pitchfork bifurcation has been transformed into a saddle-node bifurcation.

Note that most trajectories in the subcritical case evolve towards infinity. When we know that trajectories in a physical system do not behave in this way but remain bounded, we sometimes add to (18b) a stabilizing term of higher order, illustrated in figure 6.

The fixed points of (23) are:



2
x =
$\bar{x}$
0
−2
0
0



2
0
-2
μ
x̄
3
-3
4
-4

Chapter two

2 Systems with two or more dimensions

2.1 From one-to-many dimensions

As before, we consider a dynamical system:

$$\dot{x} = f(x), \qquad ft \in \mathbb{R}^N \tag{23}$$

whose fixed points are solutions of:

$$0 = f(\bar{x}) \tag{24}$$

To study the stability of $\bar{x}$, we perturb it by $\varrho(t) \in \mathbb{R}^N$.

$$\begin{aligned} \frac{d}{dt}(\bar{x} + \epsilon) &= f(\bar{x} + \epsilon) \\ \dot{\bar{x}} + \dot{\epsilon} &= f(\bar{x}) + Df(\bar{x})\,\epsilon + \epsilon\, D^2 f(\bar{x})\,\epsilon + \ldots \end{aligned} \tag{25a}$$

Since quadratic terms in ϱ are infinitesimally smaller than linear ones, (25a) is reduced to the linear differential system:

$$\dot{\varrho} = Df(\bar{x})\,\varrho \tag{26}$$

In (32a) -(33), $Df(\bar{x})$ is the *Jacobian* off, i.e., the matrix of partial derivatives, evaluated at the fixed-point $\bar{x}$. (When x is infinite dimensional rather than a vector in $\mathbb{R}^N$, the operator analogous to the Jacobian matrix is called the *Fréchet derivative´*.) To clarify the meaning of (25a) -(26), we rewrite these equations explicitly for each component:

$$\begin{aligned} \dot{\bar{x}}_i + \dot{\epsilon}_i &= f_i(\bar{x}) + Df(\bar{x})_{ij}\,\epsilon_j + \epsilon_j\,[D^2 f(\bar{x})]_{ijk}\,\epsilon_k + \ldots \\ &= f_i(\bar{x}) + \frac{\partial f_i}{\partial x_j}(\bar{x})\,\epsilon_j + \epsilon_j\,\frac{\partial^2 f_i}{\partial x_j \partial x_k}(\bar{x})\,\epsilon_k + \ldots \\ \dot{\epsilon}_i &= \frac{\partial f_i}{\partial x_j}(\bar{x})\,\epsilon_j \end{aligned} \tag{27a}$$

The solution to (26) is

$$\varrho(t) = d^{ef.(\bar{x})\,to}\,(0) \tag{28}$$

We define the exponential of a matrix via its Taylor series, as we can do for any analytic function $f(A)$:

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \ldots \tag{29}$$

The behavior of (29) depends on the spectrum of the matrix, i.e., its set of eigenvalues. Let A have the eigenvector-eigenvalue decomposition $A = V \Lambda V^{-1}$. According to (29),

$$
\begin{aligned}
e^{tA} &= VV^{-1} + tV\Lambda V^{-1} + \frac{t^2}{2} V\Lambda V^{-1}\, V\Lambda V^{-1} + \frac{t^3}{6} V\Lambda V^{-1}\, V\Lambda V^{-1}\, V\Lambda V^{-1} + \dots \\
&= V\left[I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \frac{t^3}{6}\Lambda^3 + \dots\right] V^{-1} \\
&= Ve^{\Lambda t}V^{-1}
\end{aligned} \tag{30a}
$$

Thus, we only need to know how to take exponentials of the matrix of ea.

For a matrix with real eigenvalues, we have, for the 2 × 2 case:

$$
\Lambda^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \tag{31a}
$$

$$
\Lambda^3 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix} \tag{31b}
$$

leading to

$$
e^{t\Lambda} = \begin{pmatrix} 1 + t\lambda_1 + \frac{1}{2}(t\lambda_1)^2 + \dots & 0 \\ 0 & 1 + t\lambda_2 + \frac{1}{2}(t\lambda_2)^2 + \dots \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \tag{32}
$$

The question "stable or unstable?" becomes "stable or unstable *in which directions*?" The fixed-point $\bar{x}$ is considered to be linearly stable if the real parts of *all* of the eigenvalues of $Df(\bar{x})$ are negative, and unstable if even *one* of the eigenvalues has a positive real part. The reasoning behind this is that initial random perturbations will contain components in all directions. If there is instability in one direction, then this component will grow and we will diverge away from $\bar{x}$, initially in the unstable direction.

In the simplest situation, we have $0 > \lambda_2 > \lambda_3 \dots$, and λ_1 changes sign at a bifurcation. By projecting onto the eigenvector v_1 corresponding to λ_1, i.e., by taking the scalar product with the adjoint eigenvector v_1^T satisfying:

$$
v_1^T Df(\bar{x}) = v_1^T \lambda_1 \tag{33}
$$

we obtain a one-dimensional equation. (In the other directions, the behavior is uninteresting: there is only contraction along these directions towards the fixed point.) The first terms in the Taylor series of this equation correspond to a saddle-node, pitchfork, or trans critical bifurcation. It is in this way that we obtain bifurcations in physical systems with a large number of degrees of freedom, such as thermal convection. We emphasize the correspondence between realistic physical systems and the simple polynomial equations that we have just written down:

• Complicated equations in $N \gg 1$ variables.

Calculate fixed points $\bar{x}$, their Jacobians $Df(\bar{x})$ and their spectra $\{\lambda_1, \lambda_2,\}$. Bifurcation if

the real part of one of them changes sign.

• Project onto the corresponding adjoint eigenvector =⇒ Function of one variable. • Taylor

expands about the fixed point.

Minimal truncation giving observed behavior =⇒ Normal form of the bifurcation.

2.2 Linear systems with complex eigenvalues

We have become familiar with the situations which are one-dimensional or reducible to one dimension. We now discuss the more complicated situations which can occur in two dimensions, in particular when λ_1 and λ_2 are part of a complex conjugate pair. We consider the 2 × 2 matrix corresponding to an imaginary pair of eigenvalues ±*in*, which is skew-symmetric anti-diagonal, i.e.

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix}$$

12

(34a) (34b)

(34c)

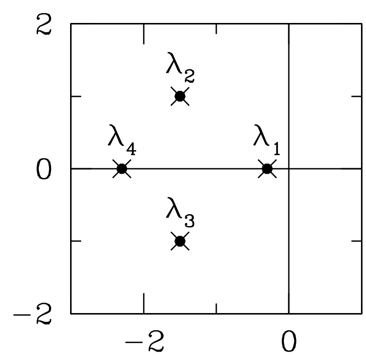


Figure 11: Eigenvalues of a Jacobian having two real eigenvalues and a pair of complex conjugate eigenvalues, with 0 > $Re(\lambda_1) > Re(\lambda_2) = Re(\lambda_3) > Re(\lambda_4)$.

Using

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \ldots \tag{35}$$

we have

$$e^{tA} = \begin{pmatrix} 1 - \frac{1}{2}(t\omega)^2 + \ldots & -t\omega + \frac{1}{6}(t\omega)^3 + \ldots \\ t\omega - \frac{1}{6}(t\omega)^3 + \ldots & 1 - \frac{1}{6}(t\omega)^2 + \ldots \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \tag{36}$$

We can combine (39) with (43) to obtain

$$\exp\left[t\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\right] = \begin{pmatrix} e^{\mu t}\cos(\omega t) & -e^{\mu t}\sin(\omega t) \\ e^{\mu t}\sin(\omega t) & e^{\mu t}\cos(\omega t) \end{pmatrix} \tag{37}$$

If a perturbation (o_x, *you*) of a fixed point with complex eigenvalues μ ± *in* is governed by:

$$\frac{d}{dt}\begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} \tag{38}$$

then it evolves according to:

$$\epsilon_x(t) = e^{\mu t}\left[\cos(\omega t)\epsilon_x(0) - \sin(\omega t)\epsilon_y(0)\right] \tag{39a}$$
$$\epsilon_y(t) = e^{\mu t}\left[\sin(\omega t)\epsilon_x(0) + \cos(\omega t)\epsilon_y(0)\right] \tag{39b}$$

More generally, for a mixture of real and complex eigenvalues, as in figure 11, we have:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \mu & -\omega & 0 \\ 0 & \omega & \mu & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \Longrightarrow \exp(t\Lambda) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\mu t}\cos(\omega t) & -e^{\mu t}\sin(\omega t) & 0 \\ 0 & e^{\mu t}\sin(\omega t) & e^{\mu t}\cos(\omega t) & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix} \tag{47}$$

2.3 Jordan blocks and transient growth

We now consider two 2 × 2 matrices which have only λ as an eigenvalue, but which behave quite differently. The matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{41}$$

is a multiple of the identity. We calculate its eigenvectors:

$$\lambda x_1 + 0x_2 = \lambda x_1 \quad \Longrightarrow \quad x_1 \text{ arbitrary} \tag{42a}$$
$$0x_1 + \lambda x_2 = \lambda x_2 \quad \Longrightarrow \quad x_2 \text{ arbitrary} \tag{42b}$$

All vectors $(x_1, x_2)^T$ are eigenvectors and the eigenspace corresponding to the double eigenvalue λ is two-dimensional. In contrast, the matrix

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{43}$$

is called a *Jordan block*. We calculate its eigenvectors:

$$\lambda x_1 + x_2 = \lambda x_1 \quad \Longrightarrow \quad x_2 = 0 \tag{44a}$$
$$0x_1 + \lambda x_2 = \lambda x_2 \quad \Longrightarrow \quad x_1 \text{ arbitrary} \tag{44b}$$

The eigenspace is thus one-dimensional and consists of all multiples of $(1,0)^T$, as shown on the left portion of figure 12. For most matrices A, whether or not the eigenvalues are multiple, there exist N linear independent eigenvectors, which are solutions to

$$(A - I)\,x = 0 \tag{45}$$

Any vector in R^N can thus be written as a sum of eigenvectors. This is not the case for a Jordan block. Where is the missing dimension? The Jordan block (43) also has a *generalized eigenvector*, which is a solution to

$$(A - I)\, v = x, \tag{46}$$

where x is an eigenvector. We calculate the generalized eigenvector of (43) as follows:

$$\lambda v_1 + 1v_2 - \lambda v_1 = c \qquad \Longrightarrow \qquad v_2 = c = 06 \tag{46a}$$

$$0v_1 + \lambda v_2 - \lambda v_2 = 0 \qquad \Longrightarrow \qquad v_1 \text{ arbitrary} \tag{46b}$$

Thus, any vector v satisfying $v_2 = 06$ is a generalized eigenvector of (43), as shown in the right portion of figure 12. A generalized eigenvector is determined, like an ordinary eigenvector, up to an arbitrary multiplicative constant, as is shown by (47a), but also up to an arbitrary additive constant, since we can add any multiple of the eigenvector, as is shown by (47b). This non-uniqueness can be eliminated using a scalar product, by requiring that the eigenvector be normalized, and that the scalar product of the generalized eigenvector with the eigenvector be zero. The eigenvector and generalized eigenvector of (43) selected by these criteria are then:

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{47}$$

The behavior of a system near a degenerate node is

$$x_1(t) \quad = \quad e^{at}(x_1(0) + x_2(0)t) \tag{48a}$$

$$x_2(t) \quad = \quad e^{\lambda t} x_2(0) \tag{48b}$$

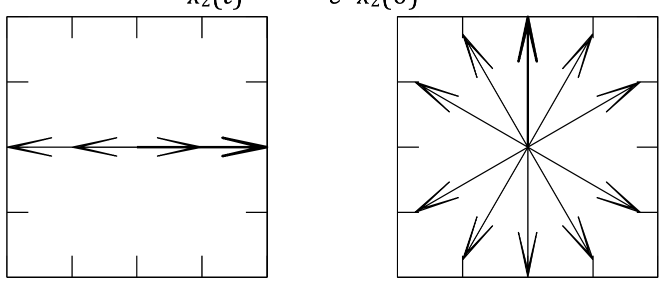


Figure 12: Left: the single eigenvector x of the 2×2 Jordan block (43) is directed along the x_1 axis and determined up to a multiplicative factor. Right: any vector v containing a non-zero x_2 component is a generalized eigenvector of (50). Wider arrows show x, v satisfying $||x|| = 1$, $||v|| = 1$ and h$x, vi = 0$.

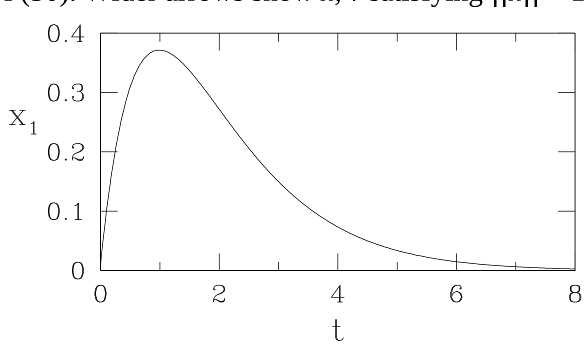
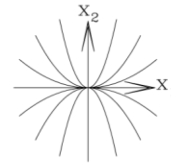


Figure 13: Transient growth. When (x_1, x_2) evolves according to a linear system which is a Jordan block, then $x_1(t)$ can start to grow, even if the negative eigenvalue of the matrix eventually leads to exponential decay. Here $\lambda = -1$, $x_1(0) = 0.01$ and $x_2(0) = 1$.

The linear term in (48a) can cause the system to display *transient growth*, even when the eigenvalue λ is negative, as shown in figure 13.

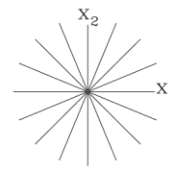
We classify all the possible linear behaviors of a fixed point of a two-dimensional system below:

Name and classification Matrix Behavior



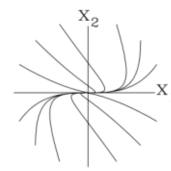
$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \begin{aligned} x_1 &= e^{\lambda_1 t} x_1(0) \\ x_2 &= e^{\lambda_2 t} x_2(0) \end{aligned}$$

Node: stable ($\lambda_2 < \lambda_1 < 0$)
unstable ($\lambda_2 > \lambda_1 > 0$)



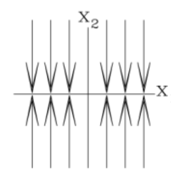
Star node:$x_1 = e^{\lambda t} x_1(0)$
stable ($\lambda < 0$) $\lambda t x_2(0)$
$x_2 = e$ unstable ($\lambda > 0$)

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$



$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

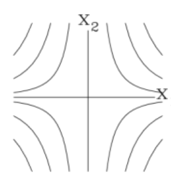
Degenerate node:$x_1 =$ $e^{at}(x_1(0) + t x_2(0))$ stable ($\lambda <$ 0) $x_2 = e^{\lambda t} x_2(0)$ unstable ($\lambda > 0$)



$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$$

Non-isolated fixed points:

stable ($\lambda < 0$) $xx_{21} == ex\lambda t1(0)x2(0)$ unstable ($\lambda > 0$)



$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \begin{aligned} x_1 &= e^{\lambda_1 t} x_1(0) \\ x_2 &= e^{\lambda_2 t} x_2(0) \end{aligned}$$

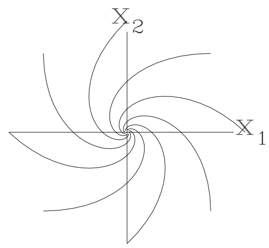
Saddle:
$\lambda_2 < 0 < \lambda_1$

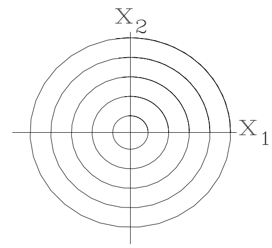
$$(\mu < 0) \quad \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x \end{pmatrix} = e^{\mu t} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

Spiral: stable
unstable ($\mu > 0$) 2

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

Center: \


X2
X1


X2
X1

التوصيات:

- **يرى الباحثان انه بالإمكان التوسعة في دراسة الأنظمة الديناميكية ذات ابعاد أكبر بالإضافة الى تطبيقات أوسع.**

- **دراسة الخواص الأنظمة الخطية للمصفوفات المربعة ذات قيم ذاتية الحقيقة.**

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