وزارة التعليم العالي والبحث العلمي
جامعة ديالى - كلية التربية المقداد
قسم الرياضيات

Fourier Series

بحث مقدم الــى قسـم الرياضيات فــي كلية التربية المقداد وهو جزء من متطلبات نيل شهادة البكالوريوس فــي الرياضيات

من قبل الطالبان

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بإشراف

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١٤٤٣ هـ - ٢٠٢٢م

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

﴿ إِذْ قَالَ يُوسُفُ لِأَبِيهِ يَا أَبَتِ إِنِّي رَأَيْتُ أَحَدَ عَشَرَ كَوْكَبًا وَالشَّمْسَ وَالْقَمَرَ رَأَيْتُهُمْ لِي سَاجِدِينَ* ﴾

سورة يوسف / الآية : ٤)

ب

الإهـــــــداء

إلهي لا يطيب الليل إلا بشكرك ولا يطيب النهار الا بطاعتك...ولا تطيب اللحظات إلا بذكرك...ولا تطيب الاخرة إلا بعفوك ولا تطيب الجنة إلا برؤيتك (الله جل جلاله)

إلى من بلغ الرسالة وأدى الأمانة .. ونصح الأمة.. إلى نبي الرحمة ونور العالمين... سيدنا **(محمد صلى الله عليه وسلم)**

إلى من كلله الله بالهيبة والوقار .. إلى من علمني العطاء بدون انتظار .. إلى من أحمل اسمه بكل افتخار ... كلماتك نـجـوم أهتدي بها اليوم وفي الغد والى الابد.. (والدي العزيز)

إلى ملاكي في الحياة.. إلى معنى الحب وإلى معنى الحنان والتفاني ...إلى بسمة الحياة وسر الوجود ... إلى من كان دعائها سر نـجاحي وحنانها بلسم جراحي إلى اغلى الحبايب...

(أمي الحبيبة)

إلى منارة العلم والعلماء ... إلى الصرح الشامخ ... (كلية التربية المقداد)

إلى الذين حملوا أقدس رسالة في الحياة ... إلى الذين مهدوا لنا طريق العلم والمعرفة ... الاساتذة الافاضل

الباحث

شكر وتقدير

أن كان من شكر وتقدير فللواحد القدير على أنجاز هذا البحث والحمدلله رب العالمين والصلاة والسلام علي خير الخلق نبي الرحمة (محمد صلى الله عليه وسلم)

أطلاقاً من العرفان بالجميل فأنه ليسرنا أن نتقدم بالشكر والامتنان الى استاذنا ومشرف بحثنا (م.د خالد هادي حميد) الذي أمدنا من منابع علمه بالكثير والذي ما توان يوماً عن مديد المساعدة لنا وفي جميع المجالات وحمداً لله بأن يسره وعسى ان يطيل عمره ليبقى نبراساً متلألئاً في نور العلم والعلماء ..

لم ولن أنسى أتقدم بفائق الشكر والاحترام والتقدير الى احبائي (ابائنا، امهاتنا، إخوتنا) جميعاً الذين ساندونا ووقفوا بجانبنا منذ بداية مسيرتي العلمية ولغاية الان وجزاهم الله عني كل خير...

كما نتقدم بشكرنا وتقديرنا الى كل من أسدى لنا معروفاً او توجيه في انجاز هذا البحث وجزاهم الله تعالى عنا خير الجزاء

الباحثان

Introduction:

Fourier series introduced in 1807 by Jean-Baptiste Joseph Fourier (1768-1830) (after work by Euler and Daniel Bernoulli) was one of the most important developments in applied mathematics. It is very useful in the study of heat conduction, mechanics, concentrations of cheŵicals aŶd pollutaŶts, electrostatics, acoustics areas heard of such as computing and CAT scan(Computer Assisted Tomography). Fourier series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions. Fourier series is very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as non While TaLJlor's series edžpaŶsioŶ is |alid for functions are continuous and differentiable, Fourier series is possible not only for continuous functions but for periodic functions, functions discontinuous in their values and derivatives. Further, because of the periodic nature, Fourier series constructed for one period is valid for all values.

Fourier series, in mathematics, an infinite series used to solve special types of differential equations. It consists of an infinite sum of sines and cosines, and because it is periodic (i.e., its values repeat over fixed intervals), it is a useful tool in analyzing periodic functions.

A Fourier series can be defined as an expansion of a periodic function f(x) in terms of an infinite sum of sine functions and cosine functions. The fourier Series makes use of the orthogonality relationships of the sine functions and cosine functions.

The Fourier Transform is a tool that breaks a waveform (a function or signal) into an alternate representation, characterized by sine and cosines. The Fourier Transform shows that any waveform can be re-written as the sum of sinusoidal functions.

Fourier series is just a means to represent a periodic signal as an infinite sum of sine wave components. A periodic signal is just a signal that repeats its pattern at some period. The primary reason that we use Fourier series is that we can better analyze a signal in another domain rather in the original domain.

What is the function of a Fourier series?

A Fourier series is a way of representing a periodic function as a (possibly infinite) sum of sine and cosine functions. It is analogous to a Taylor series, which represents functions as possibly infinite sums of monomial terms. For functions that are not periodic, the Fourier series is replaced by the Fourier transform.

Fourier series make use of the orthogonality relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually,…

A Fourier series is a way to represent complex waves, such as sound, as a series of simple sine waves. The series breaks down a wave into a sum of sines and cosines. This means that elements of a wave can be isolated from each other.

Chapter One

Fourier Series

1.1 Fourier Series

A non-sinusoidal periodic function into a fundamental and its harmonies. A series of sines and cosines of an angle and its multiples of the form.

$$\frac{a_0}{2} + a_1 cos\, x + a_2 cos\, 2x + a_3 cos\, 3x + \cdots + a_n cos\, nx + \cdots$$

$$+ b_1 sin\, x + b_2 sin\, 2x + b_3 sin\, 3x + \cdots + b_n sin\, nx + \cdots$$

$$\frac{a_0}{2} + \sum_{n=1} a_0 \, cos\, nx + \sum_{n=1} b_0 \, sin\, nx$$

Is called the Fourier Series, where $a_1, a_2, \ldots, a_n, \ldots \; b_1, b_2, b_3, \ldots b_n$ are contrasts

1.2 Properties of Fourier series

The following integrals are useful Fourier series

(1) $\int_0^{2\pi} sin\, nx\, dx = 0$ (2) $\int_0^{2\pi} cos\, nx\, dx = 0$

(3) $\int_0^{2\pi} sin^2\, nx\, dx = \pi$ (4) $\int_0^{2\pi} cos^2\, nx\, dx = \pi$

(5) $\int_0^{2\pi} sin\, nx\, sin\, mx\, dx = 0$ (6) $\int_0^{2\pi} cos\, nx\, cos\, mx\, dx = 0$

(7) $\int_0^{2\pi} sin\, nx\, cos\, mx\, dx = 0$ (8) $\int_0^{2\pi} sin\, nx\, cos\, nx\, dx = 0$

(9) $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

$$where\ v_1 = \int v\, dx, v_2 = \int v_1 dx\ \ and\ so\ on\ u' = \frac{du}{dx}, u''$$

$$= \frac{d^2u}{dx^2}\ and\ so\ on\ and\ (x) sin\ n\ \pi = (-1)^n\ where\ n \in 1$$

1.3 Determination of Fourier coefficients (Euler's Formulae)

$$f(x) = \frac{a_0}{2} + a_1 cos\ x + \ a_2 cos\ 2x + \cdots + a_n cos\ nx + \cdots$$

$(i) To\ find\ a_0: Intergrate\ both\ sides\ of (1) from\ x = 0\ to\ x = 2\pi$

$\int_0^{2\pi} f(x)dx = \ \frac{a_0}{2}\int_0^{2\pi} dx + a_1 \int_0^{2\pi} cosxdx + a_2 \int_0^{2\pi} cos2xdx + \cdots +$
$a_n \int_0^{2\pi} cosnxdx + \cdots + b_1 \int_0^{2\pi} sin\ x\ dx + b_2 \int_0^{2\pi} sin2dx + \cdots +$
$b_n \int_0^{2\pi} sin\ nx\ dx + \cdots$

$$= \frac{a_0}{2}\int_0^{2\pi} dx\, , (other\ integrals = 0$$

$by\ formula\ (i) and\ (ii) of\ Art. 1.2)$

$$\int_0^{2\pi} f(x)dx = \frac{a_0}{2}\ 2\pi, \quad \Longrightarrow\ a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

(ii) To find a_0: Multiply each side of (1) by cos nx and integrate from x = 0 to x = 2π

$\int_0^{2\pi} f(x)\ cos\ nx\ dx = \frac{a_0}{2}\int_0^{2\pi} f(x)\ cos\ nx\ dx + a_1 \int_0^{2\pi} f(x)\ cos\ x\ cos\ nx\ dx +$
$\cdots + a_n \int_0^{2\pi} f(x)\ cos\ ^2nx$

$$b_1 \int_0^{2\pi} sin\ x\ cos\ nx\ dx + \cdots + b_2 \int_0^{2\pi} f(x)\ sin\ 2x\ cos\ nx\ dx + \cdots$$

$a_n \int_0^{2\pi} f(x) \cos^2 nx\, dx = a_n \pi$ ($Other\ integrals = 0$)

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

By taking n = 1 , 2 … we can find the values of a_1 , a_2 ….

(iii) To find b_n: Multiply each side of (1) by sin nx and integrate from x = 0 to x = 2π

$\int_0^{2\pi} f(x) \sin nx\, dx = \frac{a_0}{2} \int_0^{2\pi} \sin nx\, dx + a_1 \int_0^{2\pi} \cos x\ \sin nx\, dx + \cdots + a_n \int_0^{2\pi} \cos x\ \sin nx\, dx + \cdots + b_1 \int_0^{2\pi} \sin x \sin nx\, dx + \cdots + b_n \int_0^{2\pi} f(x) \sin^2 nx\, dx + \cdots$

$= b_n \int_0^{2\pi} f(x) \sin^2 nx\, dx$

$= b_n \pi$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

Note: To get similar formula a_0 , $\frac{1}{2}$ has been written with a_0 in Fourier series.

Example 1.1.1: Find the Fourier series representing

$f(x) = x, \qquad 0 < x < 2\pi$

and sketch its graph from $x = -4\pi \; to \; x = 4\pi$

Solution. Let

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots \ldots + b_1 \sin x + b_2 \sin 2x + \cdots \quad \ldots(1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$a_n = \& \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1 \cdot \left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1 \cdot \left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Example 1.1.2: Given that $f(x) = x + x^2 \; for \; -\pi < x < \pi$, find the Fourier expression of $f(x)$

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$

Solution. Let

$$x + x^2 = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots \quad (1.1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\frac{(\sin nx)}{n} - (2x)\frac{(-\cos nx)}{n^2} + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2\frac{\sin n\pi}{n} - 2\pi\left(\frac{-\cos n\pi}{n^2}\right) + 2\left(-\frac{\sin n\pi}{n^3}\right)\right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nxdx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nxdx \quad (x^2\,sin\,nx \text{ is an odd function})$$

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[-(\pi)\frac{\cos nx}{n} + 2\frac{\sin n\pi}{n^3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

Substituting the values of a_0, a_n, b_n in (1) we get

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right] +$$
$$2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right] \ldots (1.2)$$

Put $x = \pi$ in (2),

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

Put $x = -\pi$ **in** $(2), -\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$

Adding (3) and (4)

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Chapter Two

2.1 Function Defined Two or more Sub-Ranges

Example 2.1.1: Find the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ +1 & \text{for} \quad \frac{\pi}{2} < x < \pi \end{cases}$$

Solution. Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} (-1)dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1dx$$

$$= \frac{1}{\pi}[-x]_{-\pi}^{-\frac{\pi}{2}} + \frac{1}{\pi}[x]_{\frac{\pi}{2}}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2} - \pi - \frac{\pi}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx dx$$

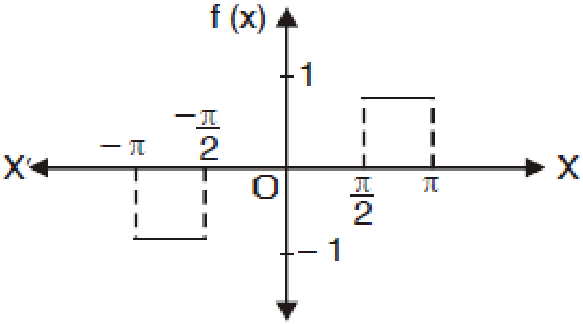
$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} (-1)\cos nx dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (0)\cos nx dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (1)\cos nx dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\frac{\pi}{2}} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\frac{\pi}{2}}^{\pi} = -\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n} + \frac{\sin n\pi}{n}\right] + \frac{1}{\pi}\left[\frac{\sin n\pi}{n} - \frac{\sin\frac{n\pi}{2}}{n}\right] = 0$$

$$b_n = \& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} (-1)\sin nx dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0)\sin nx dx$$

$$+\frac{1}{n}\int_{\frac{\pi}{2}}^{\pi}(1)\sin nx dx$$



$$= \pi\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$b_1 = \frac{2}{\pi},\ b_2 = -\frac{2}{\pi},\ b_3 = \frac{2}{3\pi}$$

Putting the values of a_0, a_n, b_n in (1) we get $f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \cdots\right]$ Ans.

Example 2.1.2. Find the Fourier series for the periodic function

$$f(x) = \begin{bmatrix} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{bmatrix}$$

$$f(x + 2\pi) = f(x)$$

Solution

Let

$f(x) = \frac{a_0}{2} + a_1\cos x + a_0\cos 2x + \cdots + v_1\sin x + b_2\sin 2x + \cdots$ **...(1)**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0.dx + \frac{1}{\pi}\int_{0}^{\pi} x dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx = \frac{1}{\pi}\left[x \cdot \frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2}\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = -\frac{2}{n^2\pi} \qquad \text{when } n \text{ is odd}$$

$= 0$, when n is even

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} =$$

$$\frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] = \frac{(-1)^{n+1}}{n}$$

Substituting the values of $a_0, a_1 a_2 \ldots b_1, b_{2'} \ldots$ in (1), we get

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \ldots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

2.2 Discontinuous Functions

At a point of discontinuity, Fourier series gives the value of $f(x)$ as the arithmetic mean of left and right limits.

At the point of discontinuity, $x = c$

At $x = c, f(x) = \frac{1}{2}[f(c-0) + f(c+0)]$

Example 2.2.1.

Find the Fourier series for $f(x)$, if $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ Deduce that

$$Dedice\ that \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Solution. Let

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots$$

$$+b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots \qquad \ldots (2.1)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Then

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) dx + \int_{0}^{\pi} x dx\right] = \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2};$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \cos nx dx + \int_{0}^{\pi} x \cos nx dx\right] = \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{n^2 \pi}[(-1)^n - 1] = \frac{-2}{n^2\pi}$$ **when n is odd**

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\sin nx dx + \int_{0}^{\pi} x\sin nx dx\right] = \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1-2\cos n\pi) = \frac{1}{n}(1-2(-1)^n$$

$$b_n = \frac{3}{n} \text{ when } n \text{ is odd}$$

$$= \frac{-1}{n} \text{ when } n \text{ is even}$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \quad \ldots(2.2)$$

Putting

$x = 0$ in (2), we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty\right) \ldots(2.3)$$

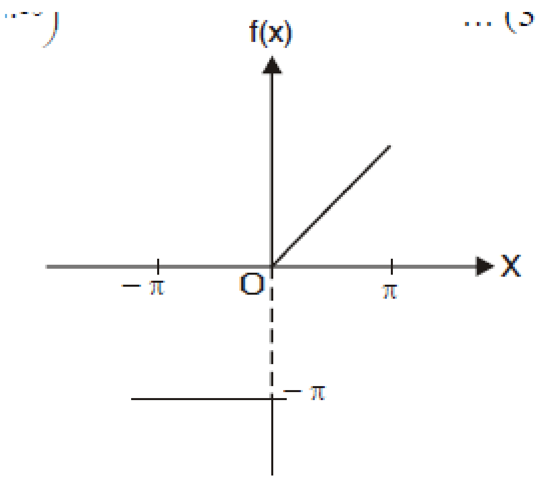
Now $f(x)$ is discontinuous at $x = 0$.

But $f(0-0) = -\pi$ and $f(0+0) = 0$

$$\therefore f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\frac{\pi}{2}$$

From(3), $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

or $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ ***Proved***



Example 2.1.2:

Find the Fourier series expansion of the periodic function of period 2π-, defined by

$$f(x) = \begin{cases} x & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - 1 & \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

Solution:
Let

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}} (\pi - x)dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\frac{3\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \frac{1}{\pi}\left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right) = \pi\left(\frac{3}{2} - \frac{9}{8} - \frac{1}{2} + \frac{1}{8}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\frac{3\pi}{2}}$$

$$=\frac{1}{\pi}\left[\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}+\frac{\cos\frac{n\pi}{2}}{n^2}-\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}-\frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$+\frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\sin\frac{3n\pi}{2}}{n}+\frac{\cos\frac{3n\pi}{2}}{n^2}-\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}+\frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2}+\sin\frac{n\pi}{2}\right)-\frac{1}{n^2}\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2}+\frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}x\sin nx dx+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\pi-x)\sin nx dx=\frac{2}{\pi}\int_{0}^{\pi/2}x\sin nx dx+$$
$$\frac{1}{\pi}\int_{\pi/2}^{\frac{3\pi}{2}}(\pi-x)\sin nx dx$$

$$=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}+\frac{1}{\pi}\left[(\pi-x)\left(-\frac{\cos nx}{n}\right)-\right.$$
$$\left.(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\frac{3\pi}{2}}$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]+\frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{3\sin\frac{n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos 3\frac{n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\pi+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]=$$
$$\frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2}-\sin\frac{3n\pi}{2}\right]$$

Substituting the values of $a_0, a_1, a_2 \dots b_1, b_2 \dots.$ we get

$$f(x)\frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\cdots\right]$$

Example 2.1.3:

Find the Fourier series of the function defined as

$$f(x)=\begin{cases} x+\pi & \text{for } 0<x<\pi \\ -x-\pi & -\pi<x<0 \end{cases} \text{ and } f(x+2\pi)=f(x).$$

Solution.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx-\frac{1}{\pi}\int_{0}^{\pi}f(x)\mathrm{dx}$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)dx=\frac{1}{\pi}\left(-\frac{x^2}{2}-\pi x\right)_{-\pi}^{0}+$$
$$\frac{1}{\pi}\left(\frac{x^2}{2}+\pi x\right)_{0}^{\pi}$$

$$=\frac{1}{\pi}\left(\frac{\pi^2}{2}-\pi^2\right)+\frac{1}{\pi}\left(\frac{\pi^2}{2}+\pi^2\right)=\pi\left(\frac{1}{2}-1\right)+\pi\left(\frac{1}{2}+1\right)=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cos nx dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\cos nx dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos nx dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos nx dx$$

$$=\frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n}-(-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{-x}^{0}$$
$$+\frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n}-(1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^x}{n^2}\right] + \frac{1}{\pi}\left[-\frac{(-1)^\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}[(-1)^\pi - 1]$$

$a_n = \frac{-4}{n^2\pi}$, **If n is odd**

and $a_n = 0$ if n is even.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin nx dx$$

$$= \frac{1}{\pi}\left[(-x-\pi)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right] + \frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n + \frac{\pi}{n}\right] = \frac{1}{n}[(1) - 2(-1)^n + (1)] = \frac{2}{n}[1-(-1)^n]$$

$= \frac{4}{n}$, **if n is odd.**

$= 0$, **if n is even.**

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x + \cdots$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right)$$

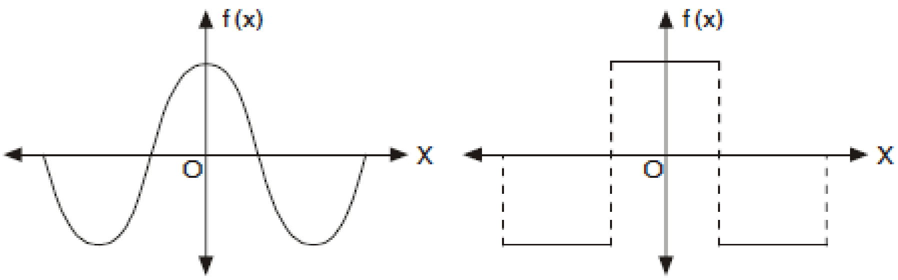
3.1 Even Function

A function *f* (*x*) is said to be even (or symmetric) function if,

$f(-x) = f(x)$

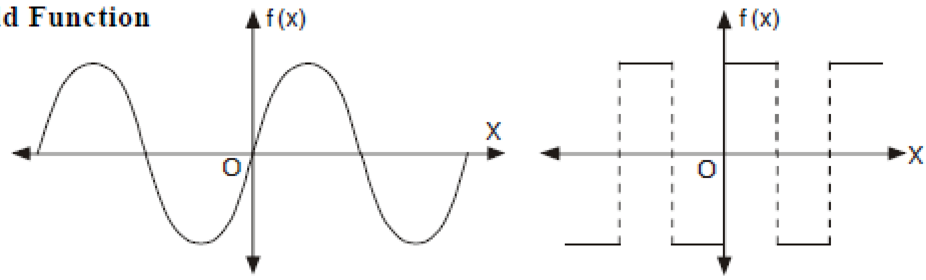
The graph of such a function is symmetric with respect to *y*-axis [*f*(*x*) axis]. Here *y*-axis is a

mirror for the reflection of the curve.



The area under such a curve from $-\pi$ to π is double the area from 0 to .

(b) Odd Function



A function $f(x)$ is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$

Here the area under the curve from $-\pi$ to π is zero.

$$\int_{-\pi}^{\pi} f(x)dx = 0$$

Expansion of an even function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

As $f(x)$ and $\cos nx$ are both even functions. $\therefore$ The product of $f(x)\cdot\cos nx$ is also an even function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$$

As $\sin nx$ is an odd function so $f(x)\cdot\sin nx$ is also an odd function. We need not to calculate b_n. It saves our labour a lot.

The series of the even function will contain only cosine terms.

Expansion of an odd function :

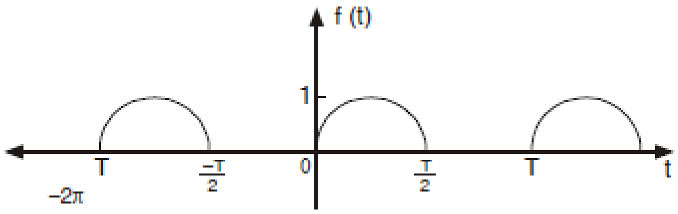
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0 \text{ [}f(x)\cdot\cos nx \text{ is odd function.]}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

[$f(x)$. sin nx is even function.]

The series of the odd function will contain only sine terms.



The function shown below is neither odd nor even so it contains both sine and cosine terms

Example 8

Find the Fourier series expansion of the periodic function of period 2

$$f(x) = x^2, -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Solution.

$$f(x) = x^2, -\pi \le x \le \pi$$

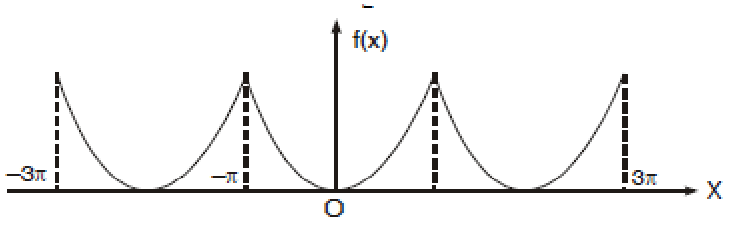
This is an even function. $\therefore\ b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$



Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots +$$

$$a_n \cos nx + \cdots$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^3} - \frac{\cos 4x}{4^2} + \cdots\right]$$

On putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots\right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12}$$

Example 3.1.1:

Obtain a Fourier expression for

$$f(x) = x^3 \text{ for } -\pi < x < \pi.$$

Solution. $f(x) = x^3$ **is an odd function.**

$\therefore\ a_0 = 0$ **and** $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx dx$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^3}\right] = 2\cdot(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$x^3 = 2\left[-\left(\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x \ldots\right]$$

الخاتمة

لقد وصلنا لنهاية هذا البحث، وفي النهاية لا يسعني سوى أن أشكركم على حسن متابعتكم لهذا البحث، وأنا قد عرضت بهذا البحث رائي المتواضع ببركة الله تعالى وكرمــه وتوفيقــة، وقــد أكرمنـــي الله بــأن أدلــوا بــدلوي تجــاه هــذا الموضـــوع (***Fourier Series***) ولعـل الله تعـالى قـد وفقنــي فـي هـذا البحـث فـي هـذا الموضوع، ولعل قلمي وفق في تقديم ما يدور بخلدي، وفـي نهايـة الأمـر فـإنني بشر أصيب وأخطـئ، وإننـي أتوجـه إلـى الله بالـدعاء علـى تـوفيقي فـي تقـديم هذا البحـث وعلـى حسـن قـراءتكم ومتـابعتكم لـهذا البحـث، ونشـكر لكـم سـعة صـدركم ونرجو أن ينال البحث إعجابكم، والحمد لله الذي هدانا إلى هذا.

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