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(IMPROPER IENTEGRAL)

بحث مقدم الى كلية التربية المقداد قسم الرياضيات
لنيل شهادة البكالوريوس في الرياضيات

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بسم الله الرحمن الرحيم

﴿هُوَ الَّذِي جَعَلَ الشَّمْسَ ضِيَاء وَالْقَمَرَ نُورًا وَقَدَّرَهُ مَنَازِلَ لِتَعْلَمُواْ عَدَدَ السِّنِينَ وَالْحِسَابَ مَا خَلَقَ اللّهُ ذَلِكَ إِلاَّ بِالْحَقِّ يُفَصِّلُ الآيَاتِ لِقَوْمٍ يَعْلَمُونَ﴾

[٥ يونس]

صدق الله العظيم

الاهداء

ألى من وضع المولى سبحانه وتعالى الجنة تحت قدميها ووقرها في كتابه العزيز

(أمي الحبيبة)

الى تلك الشيبة الطاهرة. الى ملجأ الأمان بعد الله الذي لم يتهاون يوم في توفير سبيل الخير

والسعادة لي (أبي الغالي)

أهدي اليكم بحثي هذا ...

الشكر والتقدير

الحمدالله رب العالمين والصلاة والسلام على أشرف الأنبياء والمرسلين

نبينا محمد وعلى آله وصحبه أجمعين

أما بعد

فأني أشكر الله وافر الشكرأن وفقني واعانني على اتمام هذا العمل ثم اوجه ايات الشكر والعرفان الجميل الى الاستاذة ((أيناس حسن)) المشرفة على هذا البحث والتي منحتني الكثير من وقتها وكان لرحابة صدرها وسموخلقها وأسلوبها المميز في متابعة البحث أكبر الاثر في المساعدة على اتمام هذا العمل وأسال الله العلي القدير أن يجازيها خير الجزاء

كما لا يفوتني ان اتقدم بجزيل الشكر إلى الساند الحقيقي بعد الله أبي اطال الله في عمره و كذا جميع الذين سهلوا مهمتي

في سبيل أتمام هذا العمل متمنيا لهم التوفيق في مسيرتهم الدراسية

وآخر دعوانا أن الحمد لله رب العالمين

CONTENTS

Introdiction:

The discussion of the definite integral in elementary calculus commonly starts from an area problem. Given a region under a function graph, how can its area be calculated? The sum of the areas of slender rectangles is a fairly natural approximation. A limit of such sums yields the exact area. When this process is stripped to its essentials the Riemann integral of a function over a given interval stands revealed.

Other geometric and physical quantities, such as volume and work, fit easily into the framework supplied by the concept of the Riemann integral. Moreover the link between the integral and the antiderivative is not hard to make. Thus students can be brought quickly to the evaluation of specific integrals in the context of interesting natural problems. These are among the reasons why the Riemann integral gets first attention when the integral concept is needed.

The Riemann integral has limitations, however. It applies only to bounded functions. The definition does not make sense on unbounded intervals either. Moreover a function which possesses a Riemann integral must exhibit a great deal of regularity. The need for regularity means that the convergence theorems for the Riemann integral are severely restricted. That is, the opportunity to integrate the limit of a sequence by calculating the limit of the sequence of integrals is scant. Improper integrals are an elementary way to allow for the integration of some unbounded functions and for integration over unbounded intervals.

Chapter one

1-1 Improper Integrals

1-2 Beta Function

1-3 Cauchy's Tests

Improper Integrals.1.1

The concept of Riemann integrals as developed in previous chapter requires that the range of integration is finite and the integrand remains bounded on that domain. if either (or both) of these assumptions is not satisfied it is necessary to attach a new interpretation to the integral

Definition1.1.1. In case the integrand f becomes infinite in the interval $a \leq x \leq b$,

That is f has points of infinite discontinuity (singular points) in $[a, b]$ or the limits of integration a or b (or both becomes infinite, the symbol $\int_a^b f dx$ is called an improper(or infinite or generalised) integral.

Thus,

$$\int_1^\infty \frac{dx}{x^2}, \int_{-\infty}^{\infty} \frac{dx}{1+x^2}. \int_0^1 \frac{dx}{x(1-x)}, \int_{-1}^{\infty} \frac{dx}{x(1-x)}$$

are examples of improper integrals.

The integrals which are not improper are called proper integral, thus

$\int_0^1 \frac{\sin x}{x} dx$ is a proper integral.

Integration of Unbounded Function with finite limits of integration.

Definition 1.1.2. Let a function f be defined in a interval $[a, b]$ everywhere except possible at finite number of points.

(i) Convergence at left-end. Let a be the only points of infinite discontinuity of f so that according to assumption made in the

last section, the integral

$\int_{a+\lambda}^{b} f dx$ exists $\forall \lambda, 0 < \lambda < b - a$.

The improper integral $\int_a^b f dx$ is defined as the

$\lim_{\lambda \to 0^+} \int_{a+\lambda}^{b} f dx$ so that,

$$\int_a^b f dx = \lim_{\lambda \to 0} \int_{a+\lambda}^{b} f dx$$

If this Limit exists and is finite, the improper integral $\int_a^b f dx$ is said to converge at (a) if otherwise, it is called divergent.

Note. For any $c, a < c < b$

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

Then,

$$\int_a^b f dx \text{ and } \int_a^c f dx$$

converges and diverges together and $\int_c^b f dx$ is proper.

(ii) Convergence at right-end. Let b be the only point of infinite discontinuity the improper integral is then defined by the relation

$$\int_a^b f dx = \lim_{\mu \to 0^+} \int_a^{b-\mu} f dx \; 0 < \mu < b - a.$$

If the limit exists, the improper integral is said to be convergent at b. Otherwise is called divergent.

Note : For the same reason as above,

$\int_c^b f dx$ and $\int_a^b f dx$ converges and diverges together $\forall c, a < c < b$.

(iii) Convergence at both the end points. If the end points a and b are the only points of infinite discontinuity of f, then for any point $c, a < c < b$,

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

If both the integrals are convergent as by case (i) and (ii), then

$\int_a^b f dx$ is convergent, otherwise it is divergent. The improper integral is also defined as:

$$\int_a^b f dx = \lim_{\substack{\lambda \to 0+ \\ \mu \to 0+}} \int_{a+\lambda}^{b-\mu} f dx.$$

The improper integral exists if the limit exists.

(iv) Convergence at Interior points. If an interior point c, $a < c < b$, is the only point of infinite discontinuity of f, we get

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

the improper integral $\int_a^b f dx$ exists of the both integral on R.H.S of (1) are exists.

Example1.1.1. Examine the convergence of:

(i) $\int_0^1 \frac{dx}{x^2}$

(ii) $\int_0^1 \frac{dx}{\sqrt{1-x}}$

(iii) $\int_6^2 \frac{dx}{2x-x^2}$.

(i) 0 is the point of infinite discontinuity of integrand [0,1].

Thus,

$$\int_0^1 \frac{dx}{x^2} = \lim_{\lambda \to 0+} \int_\lambda^1 \frac{dx}{x^2}$$

$$= \quad \lim_{\lambda \to 0+} \left(\frac{1}{\lambda} - 1\right) = \infty \quad 0 < \lambda < 1$$

Thus the proper integral is divergent.

(ii) Home Assignment.

Home Assignment

Comparison Tests for Convergence At ' a ' of $\int_a^b f dx$.

Theorem1.1.1. A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f dx$ at ' a ' where f is positive in $[a, b]$. This is, ∃ a psoitive number M, independent of λ, such that

$$\int_{a+\lambda}^b f dx < M,\ 0 < \lambda < b - a.$$

Proof. We know that the improper integral $\int_a^b f dx$ converges at ' a' if for 0 $0 < \lambda < b - a, \int_{a+\lambda}^b f dx$ tends to finite limit as $\lambda \to 0^+$.

Since f is positive in $[a + \lambda, b]$, the positive function of $\lambda, \int_{a+\lambda}^b f dx$ is monotonic incereasing as λ, decreases and will therefore tend to a finite limit iff it is bounded above, This is, ∃ a positive number M independent of λ, such that

$$\int_{a+\lambda}^b f dx < M,\ 0 < \lambda < b - a.$$

Hence the theorem is proved.

Note. If no such number M exists, the monotonic increasing function $\int_{a+\lambda}^{b} f dx$ is not bounded above and therefore tend to $+\infty$ as $\lambda \to 0^+$, and hence the improper integral $\int_a^b f dx$ diverges to $+\infty$.

Comparison Test.

Theorem 1.1.2: If f and g are two positive functions and ' a ' is only singular point of f and g on $[a. b]$, such that

$$f(x) \leq g(x), \text{ for all } x \in [a, b]$$

(i) $\int_a^b f dx$ converges, if $\int_a^b g dx$ converges.

(ii) $\int_a^b g dx$ diverges, if $\int_a^b f dx$ converges.

Proof. Since f and g are two positive functions on $[a, b]$ and ' a ' is only singular point of f and g. Therefore f and g are bound in $[a + \lambda, b]$, for all $0 < \lambda < b - a$.

Also Since, $f(x) \leq g(x)$, for all $x \in [a, b]$, implies

$$\int_{a+\lambda}^{b} f dx \leq \int_{a+\lambda}^{b} g dx$$

(1) Suppose $\int_a^b g dx$ be convergent, so that $\exists m > 0$, such that for all $\lambda, 0 < \lambda < b - a$,

$$\int_{a+\lambda}^{b} f dx < m.$$

From (i) we have

$$\int_{a+\lambda}^{b} f dx < m, \text{ for all } \lambda, 0 < \lambda < b - a.$$

Hence $\int_a^b f dx$ is convergent.

(2) Now suppose $\int_a^b f dx$ is divergent then the positive function $\int_{a+\lambda}^{b} f dx$ is not bounded above.

Therefore from (i) it follows that the positive function $\int_{a+\lambda}^{b} g dx$ is not bounded above.

Hence $\int_{a+\lambda}^{b} g dx$ is divergent. This completes the Theorem.

Comparison Test (limit form).

Theorem 1.1.3. If f and g are two positive functions $[a, b]$ and ' a ' is the only singular point of f and g in $[a, b]$, such that

$\lim_{x\to a^+} \frac{f(x)}{g(x)} = I$ where ' I' is a non – zero finite number.

Then, the two integrals $\int_a^b f dx$ and $\int_a^b g dx$ converges and diverges together at ' a '.

Proof. Evidently, $1 > 0$. Let ε be positive number such that $1 - \varepsilon > 0$. Since,

$\lim_{x\to a^+} \frac{f(x)}{g(x)} = 1.$

Therefore there exists a nbd of $]a, c[, a < c < b$, such that for all $x \in]a, c[$

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon$$

or $\quad (l - \varepsilon)g(x) < f(x) < (l + \varepsilon)g(x)$.

This implies that

$$(l - \varepsilon)g(x) < f(x)$$

and

$f(x) < (l + \varepsilon)g(x)$

$$\forall x \in [a, c]$$

If $\int_a^b f dx$ converges, then from (i)

$\int_a^b g(x) dx$ also converges at a.

If $\int_a^b f dx$ diverges, then from (ii)

$\int_a^b f dx$ diverges at a.

If in the above Theorem, $\text{Lim}_{x\to a^+} \frac{f(x)}{g(x)} \to 0$ and $\int_a^b g dx$ converges, then

$\int_a^b f dx$ converges and if

$\text{Lim}_{x\to a^+} \frac{f(x)}{g(x)} \to \infty$ and $\int_a^b g dx$ diverges, then, $\int_a^b f dx$ also diverges.

Useful Comparison Integral.

Theorem 1.1.4. The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ Converges if and only if $n < 1$.

Proof. It is proper integral if $n \leq 0$ and improper for all other values of $n, ' a'$ being only singular point of the integrand.

Now for $n \neq 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\lambda\to 0^+} \int_{a+\lambda}^b \frac{dx}{(x-a)^n},$$
$$= \lim_{\lambda\to 0^+} \frac{1}{-n+1}[(b-a)^{-n+1} - \lambda^{-n+1}]$$
$$= \begin{cases} \frac{1}{-n+1}[(b-a)^{-n+1}, & \text{if } n < 1 \\ \infty & \text{if } n > 1. \end{cases}$$

Also for $n = 1$

$$\mathrm{Lim}_{\lambda\to 0^+} \int_{a+\lambda}^b \frac{dx}{x-a} = \lim_{\lambda\to 0^+} [\log(b-a) - \log\lambda] = \infty.$$

Thus, $\int_a^b \frac{dx}{(x-a)^n}$ converges for $n < 1$.

Note. A similar result holds for convergence of $\int_a^b \frac{dx}{(b-x)} atb$.

Example1.1.2. Test the convergence of

(i) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(ii)

$\int_0^{\pi/2} \frac{\sin x}{x^p} dx$

Solution. Let $f(x) = \frac{1}{\sqrt{1-\mathrm{x}^3}}$

$$= \frac{1}{\sqrt{(1-x)(1+x+x^2)}}$$
$$= \frac{1}{(1+x+x^2)^{\frac{1}{2}}} \cdot \frac{1}{(1-x)^{\frac{1}{2}}}$$

Clearly, $\frac{1}{(1+x+x^2)^{\frac{1}{2}}}$ is a bounded function.

Let M be its upper bound, then,

$$\frac{1}{(1+x+x^2)^{\frac{1}{2}}} \cdot \frac{1}{(1-x)^{\frac{1}{2}}} \leq \frac{M}{(1-x)^{\frac{1}{2}}},\ x \in [1,0].$$

Also since $\int_0^1 \frac{mdx}{(1-x)^{\frac{1}{2}}}$ is convergent as $n = \frac{1}{2} < 1$.

Therefore, $\frac{1}{\sqrt{1-x^3}}$ is convergent.

(i) For $p \leq 1$, it is a proper integral for $p > 1$, it is an improper integral 0 being the point of infinite discontinuity

Now $\frac{\sin x}{x^p} = \frac{1}{x^{p-1}}\left(\frac{\sin x}{x}\right)$

The function $\frac{\sin x}{x}$ is bounded and $\frac{\sin x}{x} \leq 1$.

Therefore, $\frac{\sin x}{x^{\mathrm{p}}} \leq \frac{1}{x^{p-1}}$

Also $\int_0^{\pi/2} \frac{dx}{x^{p-1}}$ converges only if $p - 1 < 1$ or $p < 2$.

Therefore by comparison test $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^p}\ \mathrm{d}x$ converges for $p < 2$ and diverges for $p \geq 2$.

Note. If $\lim_{x\to 0+}[(x-a)^n f(x)]$ exists and is non-zero finite, then, the integral $\int_a^b f dx$ convergs iff $n < 1$.

Example1.1.3. Find the values of m and n for which the following

integrals converges.

(i) $\int_0^1 e^{-mx} x^n dx.$

(ii) $\int_0^1 \left(\log\frac{1}{x}\right)^m dx.$

Solution (i) Let k be positive number greater than 1,

Then, $e^{-mx}x^n \leq kx^n$, $\forall x \in [0,1]$ and m;

Also $\int_0^1 x^n = \int_0^1 \frac{dx}{x^{-n}}$ converges for $-n < 1$,

that is, $n > -1$ only.

Thus, $\int_0^1 e^{-mx} x^n dx$ converges only for $n > -1$ and $\forall m$.

(iii) Let $(x) = \left(\log\frac{1}{x}\right)^m$ converges at $x = 0$ and

$\int_0^{\frac{1}{2}} \left(\log\frac{1}{x}\right)^m dx$ is proper integral if $m \leq 0$. Also' $0'$ is the only singular point if $m > 0$.

For $m > 0$,

Take $g(x) = \frac{1}{x^{p'}}$ $0 < p < 1$, so that

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} x^p \left(\log\frac{1}{x}\right)^m$$
$$= 0, \text{ for } 0 < p < 1.$$

Therefore, $\int_0^{\frac{1}{2}} \left(\log \frac{1}{x}\right)^m dx$ converges for all m.

Convergence at $x = 1$

$\int_{\frac{1}{2}}^{1} \left(\log \frac{1}{x}\right)^m$ is proper integral for $m \geq 0$ and ' 1 ' is singular point, if $m < 0$.

For $m < 0$, take $g(x) = \frac{1}{(1-x)^{-m}}$, so that $\lim_{x \to 1} \left[\frac{\log \frac{1}{x}}{1-x}\right]^m = 1$.

Since $\int_{\frac{1}{2}}^{1} g\, dx$ converges for $-m < 1$ that is for $m > -1$.

Thus, $\int_0^1 \left(\log \frac{1}{x}\right)^m dx$ converges for > -1.

Hence $\int_0^1 \left(\log \frac{1}{x}\right)^m dx$ converges for $0 > m > -1$.

Example 1.1.4. Show that

(1) $\int_0^1 \frac{\log x}{\sqrt{x}} dx$ is convergent.

(2) $\int_1^2 \frac{\sqrt{x}}{\log x} dx$ is divergent.

Solution. (1) Since $\frac{\log x}{\sqrt{x}}$ is negative on [0,1].

Therefore we take $f(x) = -\frac{\log x}{\sqrt{x}}$

$$= \frac{\log x^{-1}}{\sqrt{x}} = \frac{\log 1/x}{\sqrt{x}},$$

' O ' is the only singular point.

Let

$$g(x) = \frac{1}{x^{\frac{3}{4}}},\ n = \frac{3}{4} < 1$$

We have

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x^{\frac{1}{4}} \log \frac{1}{x} = 0$$

Since $\int_0^1 g(x) dx$ converges.

Therefore, $\int_0^1 f(x) dx$ converges implies that $\int_0^1 \frac{\log x}{\sqrt{x}} dx$ converges.

(2) Let $f(x) = \int_1^2 \frac{\sqrt{x}}{\log x} dx$.

Here $x = 1$ is only singular point.

Take $g(x) = \frac{1}{x-1}$, then

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{(x-1)\sqrt{x}}{\log x}$$

$$= \lim_{x\to 1} \frac{x^{\frac{3}{2}} - x^{\frac{1}{2}}}{\log x}$$

$$= \lim_{x\to 1} \frac{\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{x}}$$

$$= \lim_{x\to 1} \frac{3}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{1}{2}}$$

$$= \frac{3}{2} - \frac{1}{2} = 1(\neq 0).$$

Thus, $\int_1^2 f dx$ and $\int_1^2 g dx$ behave same.

Since $\int_1^2 g dx$ is divergent.

Hence $\int_1^2 f dx$ is divergent.

Example1.1.5. Show that $\int_0^{\frac{\pi}{2}} \left(\frac{\sin^m x}{x^\pi}\right) dx$ exists iff $n < m + 1$

Solution. Let $f(x) = \left(\frac{\sin^m x}{x^n}\right)$

$$= \left(\frac{\sin x}{x}\right)^m \cdot \frac{1}{x^{n-m}}$$

Here as $x \to o+, f(x) \to 0$ if $n - m < 0$, and $f(x) \to \infty$ if $n - m > 0$

Thus it is proper integral if $n \leq m$ and improper if $n > m$.

$'0'$ being the only point of infinite discontinuity.

When $m > n$,

Let $g(x) = \frac{1}{x^{n-m}}$, so that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^m = 1.$$

Also, $\int_0^{\frac{\pi}{2}} g dx = \int_0^{\frac{\pi}{2}} \frac{1}{x^{n-m}} dx$ converges, Iff $n - m < 1$.

That is, $n < m + 1$.

Therefore $\int_0^{\frac{\pi}{2}} \left(\frac{\sin^m x}{x^n}\right) dx$ also converges iff $n < m + 1$

(Beta Function)1.2. Show that $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ exists iff m, n are both positive.

Proof. It is a proper integral for $m \geq 1, n \geq 1$,0 and 1 are the only points of infinite discontinuity; 0 when $m < 1$ and 1 .

When $n < 1$, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1} dx = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1} dx + \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1} dx$$

convergence at ' O', when $m < 1$.

Let $f(x) = x^{m-1}(1-x)^{n-1}$

$= \frac{(1-x)^{n-1}}{x^{1-m}}$.

Take $g(x) = \frac{1}{x^{1-m}}$,

Then $\text{Lim}_{x \to 0} \frac{f(x)}{g(x)} = 1$.

Since $\int_0^{\frac{1}{2}} g dx$ converges if and only if, $1 - m < 1$ or $m > 0$.

Thus, $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1} dx$ converges for $m > 0$.

Convergence at $x = 1$,

When $n < 1$,

Let $f(x) = x^{m-1}(1-x)^{n-1}$

$= \frac{(1-x)^{m-1}}{x^{1-n}}$

Take $g(x) = \frac{1}{x^{1-n}}$, then

$$\text{Lim}_{x \to 1} \frac{f(x)}{g(x)} = 1$$

Also, $\int_{\frac{1}{2}}^1 g dx = \int_{\frac{1}{2}}^1 \frac{1}{(1-x)^{1-n}} dx$ converges if and only if $1 - n < 1$ or $n > 0$.

Thus, $\int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1}dx$ converges if $n > 0$.

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ converges if $m > 0, n > 0$.

Example 1.2.1. For what values of m and n is the integral

$\int_0^1 x^{m-1}(1-x)^{n-1}\log x dx$ convergent.

Solution . The integrand is negative in [0,1], therefore we shall test for the convergence of

$\int_0^1 - x^{m-1}(1-x)^{n-1}\log x dx$

$= \int_0^1 x^{m-1}(1-x)\log\frac{1}{x}dx$

Since 0 and 1 are only possible singular points of integrand. We have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\log\frac{1}{x}dx$$

$$= \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\log\frac{1}{x}dx + \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\log\frac{1}{x}dx.$$

Convergence at 0.

It is proper integral for $m-1>0$ and improper for $m \leq 1.0'$ being the only point of infinite discontinuity.

Then, for $m \leq 1$

Let $f(x) = x^{m-1}(1-x)^{n-1}\log\frac{1}{x}$

$= (1-x)^{n-1}\log\frac{\frac{1}{x}}{x^{1-m}}$

Take $g(x) = \frac{1}{x^p}$

Also, $\lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+} x^{p+m-1}(1-x)^{n-1}\log\frac{1}{x}$

$$= 0$$

If $p+m-1>$ or $m > 1-p$.

Also $\int_0^{\frac{1}{2}}\frac{1}{x^p}dx$ converges for $1-p>0$.

Thus

$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\log\frac{1}{x}\,dx$ converges for $m > 1-p > 0$

converges at $x = 1$

For $n < 0$,

Let $f(x) = x^{m-1}(1-x)^{n-1}\log\frac{1}{x}$

$$= \frac{x^{m-1}\log\frac{1}{x}}{(1-x)^{-n+1}}$$

Take $g(x) = \frac{1}{(1-x)^q}$.

Therefore $\int_{\frac{1}{2}}^{1} g(x)$ converges for $q - 1 < 0$.

Also $\lim_{x\to 1_-}\frac{f(x)}{g(x)} = \lim_{x\to 1_-}\frac{x^{m-1}\log\frac{1}{x}}{(1-x)^{1-n-q}} = I$

where I is infinite if $1 - n - q \leq 1$.

That is if $n \geq -q > -1$.

Thus, $\int_{\frac{1}{2}}^{1} f\,dx$ converges if $n > -1$.

Hence the given integral is convergent when $m > 0, n > -1$.

Example 1.2.2. Show that $\int_0^{\frac{\pi}{2}}\log\sin x\,dx$ converges and also evaluate it.

Solution. Let $f(x) = \log\sin x$, then f is negative in $[0, \pi/2]$.

Therefore we consider $-f$ instead of f.

Clearly ' $0'$ is only point of infinite discontinuity.

Let $g(x) = \frac{1}{x^m}, m < 1$,

Then,

$$\lim_{x\to 0^+}\frac{-f(x)}{g(x)} = \lim_{x\to 0+} -x^m\log\sin x = 0, m < 1$$

Since $\int_0^{\frac{\pi}{2}}\frac{1}{x^m}dx$ converges for $m < 1$, thus

$\int_0^{\frac{\pi}{2}}\log\sin x\,dx$ converges.

Let $I = \int_0^{\frac{\pi}{2}}\log\sin x\,dx$.

We know that, $\sin 2x = 2\sin x \cos x$.

Therefore, $\log\sin 2x = \log 2 + \log\sin x + \log\cos x$.

This implies that

$$\int_0^{\frac{\pi}{2}} \log\sin 2x dx \quad = \int_0^{\frac{\pi}{2}} \log 2 dx + \int_0^{\frac{\pi}{2}} \log\sin x dx + \int_0^{\frac{\pi}{2}} \log\cos x dx$$
$$= \frac{\pi}{2}\log 2 + I + \int_0^{\frac{\pi}{2}} \log\cos x dx$$

Put $2x = t$.

In the Ist integral and $x = \frac{\pi}{2} - y$ in the last integral, therefoe we get

$$\frac{1}{2}\int_0^{\pi} \log\sin t dt = \frac{\pi}{2}\log 2 + I + \int_{\frac{\pi}{2}}^{0} \log\sin y(-dy)$$

$=\frac{\pi}{2} log2 + I + \int_0^{\frac{\pi}{2}} logsinxdx$.

$\therefore \ \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\sin x dx + \int_{\frac{\pi}{2}}^{\pi} \log\sin x dx = \frac{\pi}{2}\log 2 + 21.$

This implies $\frac{1}{2}\left[I + \int_0^{\frac{\pi}{2}} \log\sin\left(y + \frac{\pi}{2}\right) dy\right] = \frac{\pi}{2}\log 2 + 2I.$

Thus, $\frac{1}{2}\left[I + \int_0^{\frac{\pi}{2}} \log\cos x dx\right] = \frac{\pi}{2}\log 2 + 2I$

$$= \frac{1}{2}[I + I] = \frac{\pi}{2}\log 2 + 2I$$
$$= \frac{\pi}{2}\log 2 + 2I$$

$hhhhh \ \therefore \ I = \frac{\pi}{2}\log 2 + 2I$

This Implies that $I = -\frac{\pi}{2}\log 2$.

Hence $\int_0^{\frac{\pi}{2}} \log\sin x dx = \int_0^{\frac{\pi}{2}} \log\cos x dx = -\frac{\pi}{2}\log 2$.

Exercises.

(1) $\int_0^1 \frac{dx}{1-x} dx$

(2) $\int_0^1 \frac{x^n}{1+x} dx$

(3) $\int_0^1 \frac{\sin x}{x^{\frac{3}{2}}} dx$

(4) $\int_1^3 \frac{x^2+1}{x^2-4} dx$

(5) $\int_0^\pi \frac{\sqrt{x}}{\sin x} dx$

(6) $\int_0^1 \frac{x^n \log x}{(1+x)^2} dx$

Answer (1) divergent (2) convergent for $n > -1$ (3) convergent (4) divergent

(5) divergent (6) convergent for $n > -1$

General Test for Convergence. (Integrand May Change Sign).

We now discuss a general test for convergence of an improper integral (finite limits of integration, but discontinuous integrand) which holds whether or not integrand keeps the same sign.

Theorem 1.3 (Cauchy's Tests).

The improper integral $\int_a^b f dx$ converges at a iff to every $\varepsilon > 0$, there corresponds $\delta > 0$, such that

$$< \mu_1 < \mu_2 < \delta. \ \left|\int_{a+\mu_1}^{a+\mu_2} f dx\right| < \varepsilon$$

Proof. The improper integral $\int_a^b f dx$ is said to be exists.

When, $\lim_{\mu\to 0^+} \int_{a+\mu}^b f dx$ exists finitely.

Let $F(\mu) = \int_{a+\mu}^b f dx$.

So $F(\mu)$ is a function of μ.

According to Cauchy's Criterion for finite limits $F(\mu)$ tends to a finite limit as $\mu \to 0$. If and only if to every $\varepsilon > 0$, there corresponds $\delta > 0$, such that for all possible $\mu_1, \mu_2 < \delta$;

$$| F(\mu_1) - \mu_2) | < \varepsilon$$

$$\left|\int_{a+\mu_1}^{b} f dx - \int_{a+\mu_2}^{b} f dx\right| < \varepsilon$$

$$\left|\int_{a+\mu_1}^{a+\mu_2} f dx\right|$$

That is,

Absolute Convergence.

Definition 1.3.1 . The improper integral $\int_b^a f dx$ is said to be absolutely convergent if $\int_a^b |f| dx$ is convergent.

Theorem 3.6. Every absolutely convergent integral is convergent.

That is, $\int_a^b f dx$ exist if $\int_a^b |f| dx$ exist.

Proof. Since $\int_a^b |f| dx$ exist.

Therefore by Cauchy's test, to every $\varepsilon > 0 \; \exists \; \delta > 0$, such that

$$\left| \int_{a+\mu_1}^{a+\mu_2} \right| f|dx| < \varepsilon, \; 0 < \mu_1 < \mu_2 < \delta$$

(4)

Since $\left| \int_{a+\mu_1}^{a+\mu_2} f dx \right| \leq \int_{a+\mu_1}^{a+\mu_2} |f| dx$

and $\left| \int_{a+\mu_1}^{a+\mu_2} \right| f|dx| = \left| \int_{a+\mu_1}^{a+\mu_2} \right| f|dx|.$

Therefore (4) and (5) gives

$$\left| \int_{a+\mu_1}^{a+\mu_2} f dx \right| < \varepsilon, \; \forall \varepsilon > 0, 0 < \mu_1 < \mu_2 < \delta.$$

Thus, $\int_a^b f dx$ is convergent.

Alternative Method 3.6.

Since $f \leq |f|$ implies that $|f| - f \geq 0$.

Also, $|f| - f \leq 2|f|$

Thus, $|f| - f$ is a non- negative function on $[a, b]$ and satisfying (6).

Also $\int_a^b 2|f| dx$ is convergent. Therefore by (1) and comparison test, we get

$\int_a^b (f - |f|) dx$ is convergent.

This gives that $\int_b^a \{(f - |f|) + |f|\} \, dx$ is convergent

Hence $\int_a^b f dx$ is convergent.

Example 1.3.1. Show that $\int_0^1 \frac{\sin\frac{1}{x}}{x^p} dx, p > 0$ converges absolutely for $p < 1$.

Solution. Let $f(x) = \frac{\sin\frac{1}{x}}{x^p}, \; p > 0;$

' 0 ' is the only point of infinite discontinuity and f does not keeps the same sign in [0,1].

$$|f(x)| = \frac{\left|sin\frac{1}{x}\right|}{xp} < \frac{1}{xp}$$

Also, $\int_0^1 \frac{1}{x^p} dx$ converges for $p < 1$.

Thus $\int_0^1 \left|\frac{\sin\frac{1}{x}}{x^p}\right| dx$ converges if and only if $p > 0$.

Hence $\int_0^1 \left|\frac{\sin\frac{1}{x}}{x^p}\right| dx$ is absolutely convergent if and only if $p < 1$.

Infinite range of integration.

We shall now consider the convergence of improper integral of bounded integrable function with infinite range of integration (a or b both infinite).

Definition 1.3.2. (Convergence at ∞).

The symbol $\int_a^\infty f dx,\ x \geq a$

is defined as limit of $\int_a^x f dx$ when $X \to \infty$, so that

$$\int_a^\infty f dx = \lim_{x\to\infty} \int_a^x f dx$$

If the limit exists and is finite then the improper integral (8) is said to be divergent.

Note. For $a_1 > a,\ \int_a^X f dx = \int_a^{a_1} f dx + \int_{a_1}^X f dx$

which implies that the integrals $\int_a^\infty f dx$ and $\int_{a_1}^\infty f dx$ are either both convergent or both divergent.

Exercises.

(i) $\int_0^\infty \frac{xdx}{1+x^2}$

(ii) $\int_1^\infty \frac{dx}{\sqrt{x}}$

(iii) $\int_a^\infty \sin x dx$.

Solution. (i) For $x > 0$, we have

$$\int_0^X \frac{xdx}{1+x^2} = \frac{1}{2}\int_0^X \frac{2xdx}{1+x^2}$$

$$= \frac{1}{2}[\log(1+x^2)]_0^X$$

$$= \frac{1}{2}[\log(1+x^2)]$$

Clearly, $\lim_{x\to\infty}\int_0^X \frac{xdx}{1+x^2} = \infty$

Hence $\int_0^\infty \frac{xdx}{1+x^2}$ is divergent.

Solution (iii) We have

$$\int_a^X \sin x dx = \quad (-\cos x)_a^x \; X > a$$
$$= \cos a - \cos X$$

Clearly, $\lim_{X\to\infty}(\cos a - \cos X)$ exists finitely but not uniquely.

Thus, $\lim_{x\to\infty}\int_a^x \sin x dx$ does not exit.

Hence $\int_a^\infty \sin x dx$ diverges.

Convergence at $-\infty$.

$$\int_{-\infty}^b f dx,\ x \le b$$

is defined by equation

$$\int_{-\infty}^b f dx, \ = \lim_{x\to-\infty}\int_X^b f dx$$

If the limit exists and is finite the integral (9) converges otherwise it diverges.

Convergence at both ends.

$$\int_{-\infty}^\infty f dx, \forall x$$

Is understood to mean

$$\int_{-\infty}^c f dx + \int_c^\infty f dx$$

where c is any real number.

If both integrals in (12) converges according to definition (I) and (II), then, the integral $\int_{-\infty}^\infty f dx$ also converges, otherwise it diverges.

Exercises.

Examine for convergence the integrals

(I) $\int_0^\infty \sin x dx$

(II) $\int_{-\infty}^\infty \frac{dx}{1+x^2}$

(III) $\int_{2}^{\infty} \frac{2x^2dx}{x^4-1}$

(IV) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$

(V) $\int_{0}^{\infty} x^3e^{-x^2}dx$

Solution ☹I) try yourself (limit does not exist)

Solution☹II)

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} \quad = \lim_{\substack{X\to-\infty \\ Y\to-\infty}} \int_{Y}^{X} \frac{dx}{1+x^2}$$

$$= \lim_{\substack{x\to\infty \\ Y\to-\infty}} (\tan^{-1} x) \begin{matrix} x \\ y \end{matrix}$$

$$= \lim_{\substack{X\to\infty \\ Y\to-\infty}} (\tan^{-1} X - \tan^{-1} Y)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi.$$

Thus the integral converges and is equal to Π.

(III) $\int_{2}^{\infty} \frac{2x^2dx}{x^4-1} = \lim_{X\to\infty} \int_{2}^{X} 2x^2dx$

$$= \lim_{x\to\infty} \left[\tan^{-1} x - \tan^{-1} 2 + \frac{1}{2}\log\frac{x-1}{x+1} + \frac{1}{2}\log 3\right]$$

$$= \frac{\pi}{2} - \tan^{-1} 2 + \frac{1}{2}\log 3$$

Thus the integral converges.

(IV) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = 2\int_{0}^{\infty} \frac{dx}{(x^2+1)^2}$

$$= 2\lim_{x\to\infty} 2\int_{0}^{x} \frac{dx}{(x^2+1)^2}$$

$$= \lim_{x\to\infty} \left[\tan^{-1} x + \frac{x}{1+x^2}\right].$$

By Putting $x = \tan\theta$

$$= \Pi/2$$

(V) $\int_{0}^{\infty} x^3e^{-x^2}dx = 1/2$, converges.

Comparison test for convergence at ∞.

Theorem 1.3.2. A necessary and sufficient condition for the convergence of $\int_a^\infty f dx$, where f is positive in $[a, \infty)$, that there exists a positive number M, independent of X, such that

$$\int_a^X f dx < M, \forall X \geq a.$$

Proof. The integral $\int_a^x f dx$ is said to be convergent if $\int_a^x f dx$ tends to a finite limit as $X \to \infty$.

Since f is positive in $[a, x]$, $\forall X \geq a$ and $\int_a^X f dx$ is monotonic increasing function on X i.e. $\int_a^X f dx$ increases as X increases.

Also since $\int_a^X f dx < M$, for some $m > 0$ and $\forall X \geq a$.

That is, $\int_a^x f dx$ is bounded above.

Therefore, $\lim_{x \to 0} \int_a^x f dx$ exist finitely.

Conversely, suppose $\int_a^\infty f dx$ is convergent, then $\lim_{X \to \infty} \int_a^X f dx$ exists finitely.

Therefore, $\exists M > O$, such that $\forall x \geq a$

$$\int_A^x f dx < M$$

as $\int_A^X f dx$ increases as X increases.

Hence the theorem is proved completely.

Comparison Test I.

Theorem 1.3.3. If f and g are positive and $f(x) \leq g(x)$, for all $x \in [a, b]$.

Then, (I) $\int_a^\infty f dx$ converges if $\int_a^\infty g dx$ converges.

(II) $\int_a^\infty g dx$ diverges if $\int_a^\infty f dx$ diverges.

Proof. Suppose $\int_a^\infty g dx$ converges.

Therefore $\exists M > 0$ such that $\forall \mathrm{X} \geq a$,

$$\int_a^x g dx < M.$$

This gives $\int_a^x f dx < M$

Hence $\int_a^\infty f dx$ converges.

(II) Suppose $\int_a^\infty f dx$ diverges then $\exists X_1$, such that

$$\int_a^{x_1} f dx > M, \forall M > 0$$

This implies that $\int_a^{X_1} g dx > M, \forall M > 0$

This gives $\int_a^{\infty\infty} g dx$ diverges.

Note. Since f and g are bounded in $[a, \mathrm{X}]$.

Therefore, $f(x) \le g(x)$.

This implies that $\int_a^X f dx \le \int_a^X g dx \; \forall X \ge a$.

Comparison Test –II.

Theorem 3.9. If f and g are positive functions in $[a, \mathrm{X}]$ and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1 \; (\ne 0),$$

then two integrals converges or diverges together.

Also if $\lim_{x\to\infty} \frac{f}{g} = 0$ and $\int_a^\infty g dx$ converges, then $\int_a^\infty f dx$ converges and if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\int_a^\infty g dx$ diverges, then $\int_a^\infty f dx$ also diverges.

Proof. Evidently $l > 0$ choose > 0, such that $l - \varepsilon > 0$

Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = I$

Therefore $\forall \varepsilon > 0, \exists k > 0$ such that

$\left|\frac{f(x)}{g(x)} - l\right|$whenever $|x| > k$.

That is $l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon \; \forall \varepsilon > 0$, with $x > k$

$(l - \varepsilon) g(x) < f(x)$

$f(x) < (l + \varepsilon) g(x)$

for $x > k$ and $\forall \varepsilon > 0$.

Clearly $l - \varepsilon > 0$, by choosing ε so small.

Therefore by comparison test and (13) and (14) we get $\int_a^\infty g(x) dx$ diverges if $\int_a^\infty f dx$ converges.

Again

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$$

implies that $f(x)<g(x),\ \forall x>k$

Therefore if $\int_a^\infty f dx$ is divergent, then $\int_a^\infty g dx$ is convergent and if $\int_a^\infty g dx$ is convergent then $\int_a^\infty f dx$ is convergent.

Also if, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$

This implies $\frac{f(x)}{g(x)}>M, \forall x>k$

Therefore $f(x)>Mg(x),\ \forall \text{x} > \text{k}$

Hence if $\int_a^\infty g dx$ is divergent, then $\int_a^\infty f dx$ is divergent.

Useful Comparison Integral.

Theorem 1.3.4. Show that the improper integral $\int_a^\infty f dx=\int_a^\infty \frac{c}{x^n}dx, a>0$ where c is a positive constant, converges if and only if $n>1$.

Proof. We have

$$\int_a^\infty \frac{c}{x^n}dx=\begin{cases} c\log\frac{x}{a} & ,n=1 \\ & \frac{1}{1-n}\left[\frac{1}{x^{n-1}}-\frac{1}{a^{n-1}}\right], n\neq 1\end{cases}$$

$$\lim_{x\to\infty}\int_a^X \frac{c}{x^n}dx=\begin{cases}\infty & \text{if } n\leq 1 \\ \frac{c}{(n-1)^{n-1}} \quad \text{if} & n>1\end{cases}.$$

Thus, $\int_a^\infty \frac{c}{x^n}dx$ converges if and only if n $>$ 1.

From this useful integral and comparison test, the improper integral $\int_a^\infty f dx$ converges if there exists a positive number $n>1$ such that $f(x)\leq\frac{M}{x^n}$ for some $M>0$ and for some all $x\geq a$.

Also if, $\lim_{x\to\infty}x^n f(x)$ exists and is non-zero, then integral $\int_a^\infty f dx$ converges if and only if $n>1$.

Exercises.

(I) $\int_1^{\infty} \frac{dx}{x\sqrt{x^2}+1}$

(II) $\int_0^{\infty} \frac{x^2dx}{\sqrt{x^5}+1}$

(III) $\int_0^{\infty\infty} e^{-x^2}dx$

(IV) $\int_0^{\infty} \frac{\log x}{x^2}dx$

(V) $\int_1^{\infty} x^n e^{-x}dx$

Solution :- (I) Take $f(x) = \frac{dx}{x\sqrt{x^2}+1}$ and

$g(x) = \frac{1}{x^2}$

Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1(\neq 0)$

Thus

$$\int_1^{\infty} fdx = \int_1^{\infty} \frac{dx}{x\sqrt{x^2}+1}$$ converges.

(II) Let $f(x\ ) = \frac{x^2dx}{x^5+1}$

Take $g\left(x = \frac{1}{\sqrt{x}},\right.$

Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1\ (\neq 0)$

Thus $\int_0^{\infty} \frac{x^2dx}{\sqrt{x^5}+1}$ diverges.

(IV) Let $f(x) = \frac{\log x}{x^2}$

Take $g(x) = x^{\frac{3}{2}},$

Then $\lim_{x\to\infty} \frac{x^{\frac{3}{2}}\log}{x^2} = \lim_{x\to\infty} \frac{\log x}{x^{\frac{1}{2}}}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x - \frac{1}{2}}$$

$\mathrm{Lim}_{x\to\infty}\, 2\left(\frac{1}{x^{\frac{1}{2}}}\right) = 0$

Since $\int_1^\infty \frac{dx}{x^{\frac{3}{2}}}$ is convergent.

Therefore $\int_0^\infty \frac{\log x}{x^2} dx$ is convergent.

(V) Let $f(x) = x^n e^{-x} dx$

Take $g(x) = x^2$.

Then, $\lim_{x\to\infty} x^2 \cdot x^n e^{-x} = \lim_{x\to\infty} (n+2)!\, e^{-x} = 0$ and

$\int_1^\infty \frac{1}{x^2} dx$ is convergent.

Therefore $\int_1^{\infty\infty} x^n e^{-x} dx$ is convergent.

(III) Let $f(x) = \int_0^\infty e^{-x^2} dx$.

Clearly 0 is not point of infinite discontinuity, we may write

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx = I_1 + I_2.$$

Clearly I_1 is proper and I_2 is improper integral.

We test for $I_2 = \int_1^\infty e^{-x^2} dx$.

We have

$e^{-x^2} > x^2$

$\forall x \in R$

$$\frac{1}{e^{-x^2}} < \frac{1}{x^2}$$

$\forall x \in R$

This implies that $e^{-x^2} < \frac{1}{x^2}$

$\forall x \in R$.

Again, $\int_1^\infty \frac{1}{x^2} dx$ is convergent.

Therefore, $\int_1^\infty e^{-x^2} dx$ is convergent.

Hence $\int_0^\infty e^{-x^2} dx$ is convergent.

(VI) $\int_0^\infty \frac{\sin^2 x}{x^2} dx$ is convergent because $\sin^2 x \le 1,\ \forall x \in R$.

Exercises.

(I) $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{\frac{1}{3}}}dx$

(II) $\int_{e^2}^\infty \frac{dx}{x\log\log x}$

(III) $\int_0^\infty \left(\frac{1}{x}-\frac{1}{\sinh x}\right)$.

Solution (I). Let $f(x)=\frac{x\tan^{-1}x}{(1+x^4)^{1/3}}$ $\left(\sim x^{-\frac{1}{3}}\right)$

Take $g(x)=\frac{1}{x^{\frac{1}{3}}}$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\pi/2$.

Since $\int_0^\infty \frac{1}{x^{\frac{1}{3}}}dx$ is divergent.

Therefore $\int_0^\infty f dx$ is divergent.

(II) Put $\log x = t$, we get

$$\int_{\theta^2}^{\infty}\frac{dx}{\mathrm{xloglogx}}=\int_2^t\frac{dt}{\mathrm{logt}}$$

$$=\int_2^\infty\frac{dx}{\log x}$$

Let $f(x)=\frac{1}{\log x}$

Take $g(x)=\frac{1}{x^m}$, then

$$\lim_{x\to\infty}\frac{x^m}{\log x}=\lim_{x\to\infty}\frac{x}{\log x}\text{ by taking } m=1$$

Therefore $\lim_{x\to\infty}\frac{x}{\log x}=\lim_{x\to\infty}x=\infty$.

Since $\int_2^\infty \frac{dx}{x}$ is divergent, so that $\int_2^\infty \frac{dx}{\log x}$ is also divergent.

Hence $\int_{e^2}^\infty \frac{dx}{x\log\log x}$ is divergent.

(I) $f(x)=\left(\frac{1}{x}-\frac{1}{\sinh x}\right)/x$

Clearly 0 is not point of infinite discontinuity, because

$\lim_{x\to 0^+}f(x)=\frac{1}{6}$ (By L Hospital's rule)

We have $f(x) = \left(\frac{1}{x} - \frac{1}{\sin^h x}\right)\frac{1}{x}$

$$= \frac{1}{x^2} - \frac{1}{x\sinh x}$$

$$= \frac{1}{x^2} - \frac{1}{x}\left[\frac{1}{e^x - e^{-x}/2}\right]$$

$$= \frac{1}{x^2} - \frac{1}{x}\left[\frac{2}{e^x - e^{-x}}\right]$$

$$= \frac{1}{x^2} - \frac{1}{x}\left[\frac{2e^{-x}}{1 - e^{-2x}}\right].$$

Take $g(x) = \frac{1}{x^2}$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = I\ (\neq 0)$$

Thus, $\int_0^\infty f\,dx$ is convergent.

Note. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^2 \left[\frac{1}{x^2} - \frac{1}{x}\frac{2e^{-x}}{1-e^{-2x}}\right]$

$$= \lim_{x\to\infty} \left[1 - \frac{2xe^{-x}}{1 - e^{-2x}}\right].$$

We have $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$

$$= \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Therefore $\lim_{x\to\infty} \left[1 - \frac{2xe^{-x}}{1-e^{-2x}}\right] = 1 - \frac{0}{1-0} = 1(\neq 0).$

Chapter tow

2-1 Gamma Function

2-2 Abel's Test

. (Gamma Function)2.1.

The integral $\int_0^{\infty\infty} x^{m-1}e^{-x}dx$ is convergent if and only if $m > 0$.

Solution. Let $f(x) = x^{m-1}e^{-x}$.

If $m < 1$, the ' $0'$ infinite discontinuity.

So we must examine the convergence of above improper integral at both 0 and ∞.

$$\int_0^{\infty} x^{m-1}e^{-x}dx = \int_0^{1} x^{m-1}e^{-x}dx + \int_1^{\infty} x^{m-1}e^{-x}dx$$

Convergence at 0 for $m < 1$:

Let $g(x) = \frac{1}{x^{1-m}}$,

Then, $\lim_{x\to 0++} \frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-x} = 1(\neq 0)$.

Since $\int_0^1 \frac{1}{x^{1-m}} dx$ converges, if and only if m > 0.

Therefore $\int_0^1 x^{m-1}e^{-x}dx$ converges if and only if $m > 0$.

Converges at ∞.

Let $g(x) = \frac{1}{x^2}$, so that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{m+1}/e^x$$
$$= \lim_{x\to\infty} \frac{(m+1)!}{e^x} = 0.$$

Since $\int_1^{\infty} \frac{1}{x^2} dx$ is convergent.

Thus, $\int_1^{\infty} x^{m-1}e^{-x}dx$ is convergent $\forall m$.

Hence $\int_1^{\infty} x^{m-1}e^{-x}dx$ is convergent if and only if $m > 0$ and is denoted by $< m$).

Thus, $\Gamma(m) = \int_0^{\infty} x^{m-1}e^{-x}dx, m > 0$.

Thus $\Gamma(0), \Gamma(-1)$, etc. are not exists .

Example 2.1.1. Examine for the convergence of $\int_3^{\infty} \frac{dx}{x^2+x-2}$ and $G(x) = \frac{1}{x^2}$, then

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x^2+x-2}$

$$= \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}x^2}$$

$= 1.$

Thus, $\int_3^\infty \frac{dx}{x^2+x-2}$ is convergent.

Again let us decompose the integrand into partial fraction.

We have $\frac{1}{x^2+x-2} = \frac{1}{3(x-1)} - \frac{1}{3(x+2)}$.

It is obvious $\int_3^\infty \frac{1}{3(x-1)} dx$ and $\int_3^\infty \frac{1}{3(x+2)} dx$ are both divergent .

Thus, $\int_3^\infty \frac{dx}{x^2+x-2} = \int_3^\infty \frac{1}{3(x-1)} dx + \int_3^\infty \frac{-dx}{3(x+2)}$ is not correct.

Now we evaluate above improper integral.

We have $\int_3^\infty \frac{dx}{x^2+x-2} = lim_{x\to\infty} \int_3^x \frac{dx}{x^2+x-2}$

$$= \lim_{x\to\infty} \left[\int_3^x \frac{dx}{3(x-1)} - \int_3^x \frac{dx}{3(x+2)}\right]$$

$$= \lim_{x\to\infty} \left[\frac{1}{3}\{\log(x-1) - \log(x+2\}_3^x\right]$$

$$= \lim_{x\to\infty} \frac{1}{3}\left[\log\left(\frac{x-1}{x+2}\right)\right]_3^x$$

$$= \lim_{x\to\infty} \frac{1}{3}\left[\log\left(\frac{x-1}{x+2}\right)\right] - \log\left(\frac{2}{5}\right)$$

$$= \lim_{x\to\infty} \frac{1}{3}\left[\log\left[\frac{5(x-1)}{2(x+2)}\right]\right]$$

$$= \lim_{x\to\infty} \frac{1}{3}\left[\frac{\log\left[5\frac{5}{x}\right]}{2+\frac{4}{x}}\right]$$

$$= 1/3 \log 5/2.$$

General test for convergence at ∞ (Integrand may change sign).

Theorem 2.1.1. (Cauchy's Test).

The integral $\int_a^x f dx$ converges at ∞ if and only if to every $> 0, \exists X_0$, such that

$$\left|\int_{X_1}^{X_2} f dx\right| < \epsilon \ \forall X_1, X_2 > X_0.$$

Proof. The improper integrand $\int_a^\infty f dx$ exists if $\lim_{x\to\infty} \int_a^x f dx$ exists finitely Let $F(X) = \int_a^\infty f dx$, a function of X.

According to Cauchy's criterion for finite limits, $F(x)$ tends to a finite limits as $x \to \infty$ if to a finite $\epsilon > 0 \exists X_0$, such that $\forall X_1, X_2 > X_0$

$|F(X_1) - F(X_2)| < \epsilon$

That is $\left|\int_{X_1}^{X_2} f dx\right| < \epsilon.$

Example 2.1.1. Show that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent.

Solution. Since $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

Therefore ' 0 ' is not infinite discontinuity, we may put

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + \int_1^\infty \frac{\sin x}{x} dx.$$

We now test for the convergence of $\int_1^{\infty s} \frac{\sin}{x} dx$ as $\int_0^1 \frac{\sin x}{x} dx$ is proper integral. For any $\epsilon > 0$,

Let x_1, x_2 be two numbers both greater than $\frac{2}{\epsilon}$,

Now $\int_{x_1}^{x_2} \frac{\sin x}{x} dx = \left[-\frac{\cos x}{x}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\cos x}{x^2} dx$

so that, $\left|\int_{x_1}^{x_2 \sin x} \frac{x}{x} dx\right| \leq \left|\frac{\cos x_1}{x_1} - \frac{\cos x_2}{x_2}\right| + \left|\int_{x_1}^{x_2} \frac{\cos x}{x^2} dx\right|$

$$\leq \frac{1}{x_1} + \frac{1}{x_2} + \int_{x_1}^{x_2} \frac{dx}{x^2}$$
$$= 2 \cdot \frac{\epsilon}{2} = \epsilon$$

Therefore, by Cauchy's test the improper integral $\int_0^{\infty \sin x} \frac{x}{x} dx$ is convergent.

Absolute Convergence.

Definition 2.1.1. The improper integral $\int_a^\infty f dx$ is said to be absolutely convergent if $\int_a^\infty |f| dx$ is convergent.

Theorem 2.1.2.

Absolute convergences of $\int_a^\infty f dx$ implies convergence of $\int_a^\infty f dx$

i.e., $\int_a^\infty f dx$ exists if $\int_a^\infty |f| dx$ exist.

Proof. Suppose $\int_a^\infty |f| dx$ exists, then by Cauchy's Test, $\forall \varepsilon > 0, \exists$

X_0, such that

$$\left|\int_{x1}^{x2}\right| f|dx| < \epsilon\, , x_1, x_2, > x_0.$$

We have $\left|\int_{x_1}^{x_2} f dx\right| \leq \int_{x_1}^{x_2} |f| dx < \epsilon\ x_1, x_2 > x_0$.

Thus by Cauchy's test $\int_a^\infty f dx$ converges.

Example 2.1.2. Show that $\int_1^{\infty s} \frac{\sin x}{x^p} dx$ converges absolutely if $p > 1$

Solution. We have $\left|\frac{\sin x}{x^p}\right| = \frac{|\sin x|}{x^p} \leq \frac{1}{x^p}, \forall x \geq 1,$

and $\int_1^\infty \frac{1}{x^p} dx$ converges for $p > 1$.

Thus, $\int_1^\infty \left|\frac{\sin x}{x^p}\right| dx$ converges for $p > 1$.

Therefore, $\int_1^\infty \frac{\sin x}{x^p} dx$ converges absolutely for $p > 1$.

Integrand as a product of functions (convergent at ' ∞').

A test for absolutely convergence.

Theorem 2.1.3. If a function φ is bounded in $[a, \infty]$ and integrable in $[a, x], \forall x \geq a$.

Also if $\int_a^\infty f dx$ is absolutely convergent at ∞, then $\int_a^\infty f \varphi dx$ is also absolutely convergent at ∞.

Proof. Since f is bounded in $[a, \infty)$, therefore $\exists k > 0$, such that

$|\varphi(x)| \leq k, \forall x \in [a, \infty)$

Again since |f| is positive in $[a, \infty)$, and $\int_a^{\infty\infty} |f| dx$ is convergent.

Therefore we can find m, such that

$$\int_a^x |f| dx < m,\ \forall x \geq a$$

Using (1) we have

$$\begin{aligned} |f\varphi| &= |f||\varphi| \\ &\leq k|f|,\ \forall x \in [a, \infty). \end{aligned}$$

Therefore, $\int_a^x |f\varphi| dx \leq k \int_a^x |f| dx$

$< km\ \forall x \geq a$.

Thus, $\int_a^x |f\varphi| dx \leq km\ \forall x \geq a$.

Therefore, $\int_a^x |f\varphi| dx$ is convergent.
Hence $\int_a^x f\varphi dx$ is absolutely convergent.
Test for convergence.

Theorem (Abel's Test) 2.2. If φ is bounded and monotonic in $[a, \infty)$ and $\int_a^\infty f dx$ is convergent at ∞, then, $\int_a^\infty f\varphi dx$ is convergent at ∞.

Proof. Since φ is monotonic in $[a, \infty)$, then is integrable in $[a, x]$, $\forall X \geq a$.

Also since f is integrable in $[a, x]$, we have by 2[nd] mean value theorem

$\int_{X_1}^{X_2} f\varphi dx = \varphi(X_1)\int_{X_1}^{y} f dx + \varphi(X_2)\int_{y}^{X_2} f dx$ for $a < X_1 < Y \leq X_2$.

Let $\epsilon > 0$ be arbitrary.

Since φ is bounded in $[a, \infty)$, a positive number k exists, such that

$|\varphi(x)| \leq k,\ \forall X \geq a.$

In particular,

$|\varphi(x_1)| \leq k,\ |\varphi(x_2)| \leq\ k,$

Again since $\int_a^\infty f dx$ is convergent, therefore their exists X_0, such that

$$\left|\int_{x_1}^{x_2} f dx\right| < \frac{\varepsilon}{2k}\ ,\ \forall X_1, X_2 > X_0$$

Since, $X_1 \leq Y \leq X_2$.

Therefore, $\left|\int_{x1}^{y} f dx\right| < \frac{\epsilon}{2k}$ and

$$\left|\int_{y}^{X_2} f dx\right| < \frac{\epsilon}{2k}$$

Thus from $(17), (18), (19)$, and (20), we deduce that $\exists X_0$, such that for all $, \mathrm{X}_1, \mathrm{X}_2 > \mathrm{X}_0$ and $\epsilon > 0$

$$\left|\int_{X_1}^{X_2} f\varphi dx\right| \leq |\varphi(x1)| \left|\int_{X_1}^{y} f dx\right| + |\varphi(x2)| \left|\int_{y}^{X_2} f dx\right| < k\frac{\epsilon}{2k} + k\frac{\epsilon}{2k} = \epsilon.$$

Hence $\int_a^\infty f\varphi dx$ is convergent.

Theorem Drichlet's Test 2.15. If φ is bounded and monotonic in $[a, \infty)$ and tends to 0 as $x \to \infty$ and $\int_a^x f dx$ is bound for $X \geq a$, then $\int_a^\infty f\varphi dx$ convergent at ∞.

Proof. Since φ is bounded and integrable in $[a, x]$. Also since f is integrable in $[a, x]$, therefore by second mean value theorem:

$$\left|\int_a^x f dx\right| \leq k_{,} \forall X \geq a$$

Therefore,

$$\left|\int_{x_1}^{y} f dx\right| = \left|\int_{a}^{y} f dx - \int_{a}^{x_1} f dx\right|$$
$$\leq \left|\int_{a}^{y} f dx\right| + \left|\int_{a}^{x_1} f dx\right|$$
$$\leq 2k, \text{ for } X_1, \geq a$$

Similarly,

$$\left|\int_{y}^{x_2} f dx\right| \leq 2k, \text{ for } X_2 \geq a.$$

Let $\epsilon > 0$ be arbitrary.

Since $\varphi \to 0$ as $\to \infty$, there exists a positive X_0, such that $|\varphi(X_1)| < \frac{\epsilon}{4k}, |\varphi(X_2)| < \frac{\epsilon}{4k}$ where

$X_2 \geq X_1 \geq X_0$.

Let the numbers X_1, X_2 in (21) be $\geq X_0$, so that from (17), (18), (19) & (20), we get

$$\left|\int_{X_1}^{X_2} f\varphi dx\right| < \frac{\epsilon}{4k} 2k + \frac{\epsilon}{4k} 2k$$
$$= \epsilon \ \forall \ X_2 \geq X_1 \geq X_0$$

Hence by Cauchy's test $\int_a^\infty f\varphi dx$ is convergent at ∞.

Example 2.2.1. The improper integral $\int_1^{\infty osinx} \frac{x^p}{x^p} dx$ is divergent for p>0.

Solution. Take $\varphi(x) = \frac{1}{x^p}, p > 0$ and

$f(x) = \sin x$.

Then $\varphi(x)$ is monotonic decreasing and tends to 0 as $x \to \infty$.

Also,

$$\left|\int_{1}^{x} f dx\right| = \left|\int_{1}^{x} \sin x dx\right|$$
$$= |\cos 1 - \cos x|$$

$\leq |\cos 1| + |\cos x|$

$\leq 1 + 1 = 2, \ \forall X \geq 1.$

Thus, $\left|\int_1^x \sin x dx\right| \leq 2 \ \forall X \geq 2.$

Therefore $\int_1^x \sin x dx$ is bounded.

Hence by Drichlet's test $\int_1^\infty \sin x \frac{1}{x^p} = \int_1^\infty \frac{\sin x}{x^p} dx$ is convergent $p > 0$.

Also, we know that $\int_1^{\infty osinx}\frac{x^p}{x^p}dx$ is absolutely convergent if and only if $p > 1$

Thus, $\int_1^\infty \frac{\sin x}{x^p}$ dx is conditional convergent for $0 < p \leq 1$.

Conditionally Convergent.

An improper integral $\int_a^{\infty\infty} f dx$ is conditionally convergent at ∞ if $\int_a^\infty f dx$ is convergent at ∞, but $\int_a^\infty |f| dx$ is not convergent. That is the improper integral is said to be conditionally convergent if it is convergent but not absolutely.

Example 2.2.3. Show that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent, but not absolutely

Solution. We have $\int_0^{\infty \sin x}\frac{x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx$

Now, $\int_0^1 \frac{\sin x}{x}dx$ is proper integral.

To examine the convergence of $\int_1^\infty \frac{\sin x}{x}dx$ at ∞, we see that

$$\left|\int_1^x \sin x dx\right| = |\cos 1 - \cos X| \leq |\cos 1| + |\cos X| < 2, \text{ so that}$$

$\left|\int_1^x \sin x dx\right|$ is bounded above for all X ≥ 1.

Also, $1/x$ is a monotonic decreasing function tending to 0 $as\ x$→∞.

Therefore by Dirchlet's test $\int_0^\infty \frac{\sin x}{x}dx$ is convergent.

Hence $\int_0^\infty \frac{\sin x}{x}dx$ is convergent. follows:

$$\int_0^{n\pi}\left|\frac{\sin x}{x}\right|dx = \sum_{r=1}^{n}\int_{(r-1)\pi}^{r\pi}\frac{|\sin x|}{x}dx$$

Now, $\forall x \in [(r-1)\pi, r\pi]$

$$\int_{(r-1)\pi}^{r\pi}\frac{|\sin x|}{x}dx \geq \int_{(r-1)\pi}^{r\pi}\frac{|\sin x|}{r\pi}dx$$

Putting, $x = (r-1)\pi + y$

$$\int_{(r-1)\pi}^{r\pi}\frac{|\sin x|}{r\pi}dx = \int_0^\pi \frac{|\sin(r-1)\pi + y|dy}{r\pi}$$

$$= \frac{1}{\pi r}\int_0^\pi \sin y dy = \frac{2}{r\pi}.$$

But $\sum_{r=1}^{n}\frac{2}{r\pi}$ is a divergent series.

Therefore, $\lim_{n\to}\int_0^{n\pi}\frac{|\sin x|}{x}dx \geq \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{r\pi}$.

This implies that $\lim_{n\to\infty}\int_0^{n\pi}\frac{|\sin x|}{x}dx$ is infinite.

Now, let t be a real number, there exists positive integer n, such that

$$n\pi \leq t < (n+1)\pi.$$

We have, $\int_0^{t}\frac{|\sin x|}{x}dx \geq \int_0^{n\pi}\frac{|\sin x|}{x}dx$.

Let $t\to\infty$, so that $n\to\infty$, thus we see that

$$\int_0^{t}\frac{|\sin x|}{x}dx \to \infty$$

This implies $\int_0^{\infty}\frac{|\sin x|}{x}dx$ does not converge.

This example show that $\int_0^{\infty}\frac{\sin x}{x^p}dx, 0 < p \leq 1$, is convergent but not absolutely.

Example 2.2.4. Show that $\int_2^{\infty}\frac{\cos x}{\log x}dx$ is conditionally convergent.

Solution. Let $\varphi(x) = \frac{1}{\log x}, f(x) = \cos x$.

$$\left|\int_2^{x}\cos x dx\right| = |\sin X - \sin 2| \leq |\sin X| + |\sin 2| \leq 2, \text{ so that}$$

$\int_2^{x}\cos x dx$ is bounded for all X ≥ 2

Also, $\varphi(x) = \frac{1}{\log x}$ is monotonic decreasing function tending to 0 as $x\to\infty$.

Hence by Dirichlet's test $\int_2^{\infty}\frac{\cos x}{\log x}dx$ is convergent.

For absolute convergence consider

$$I = \int_2^{\infty}\left|\frac{\cos x}{\log x}\right|dx = \int_2^{\frac{3\pi}{2}}\frac{|\cos x|}{\log x}dx + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}}\frac{\cos x\ |}{\log x}dx + \frac{\frac{(2n+1)\pi}{2}\frac{|\cos x|}{2}}{\log x}dx + \cdots$$

Therefore,

$$I = \int_{\frac{\pi}{2}}^{2}\frac{|\cos x|}{\log x}dx + \int_2^{\frac{3\pi}{2}}\frac{|\cos x|}{\log x}dx + \cdots - \int_{\frac{(2n-1)\pi}{2}}^{\frac{(2\pi+1)\pi}{2}}\frac{|\cos x|}{\log x}dx + \cdots..$$

$$-\int_{\frac{\pi}{2}}^{2}\frac{|\cos x|}{\log x}dx$$

$= \sum_{r=1}^{n} \int_{\frac{(2n-1)\pi}{2}}^{\frac{(2n+1)\pi}{2}} \frac{|\cos x|}{\log x} dx - \int_{\frac{\pi}{2}}^{2} \frac{|\cos x|}{\log x} dx$

Now,

$$\int_{\frac{(2\pi-1)\pi}{2}}^{\frac{(2\pi+1)\pi}{2}} \frac{|\cos x|}{\log x} dx \geq \frac{1}{\log(2r+1)\pi/2} \left| \frac{\frac{(2r+1)\pi}{2}}{\frac{(2r-1)\pi}{2}} \cos x dx \right|$$

$$= \frac{1}{\log\left[\frac{(2r+1)\pi}{2}\right]} \left| \sin\left[(2r+1)\frac{\pi}{2}\right] - \sin\left[(2r-1)\frac{\pi}{2}\right] \right|$$

$$= \frac{|2(-1)r|}{\log\left[\frac{(2r+1)\pi}{2}\right]}$$

$$= \frac{2}{\log\frac{(2r+1)\pi}{2}}.$$

Therefore, $I \geq \sum_{r=1}^{\infty} \frac{2}{\log\frac{(2r+1)\pi}{2}} - \int_{\frac{\pi}{2}}^{2} \frac{|\cos x|}{\log x} dx.$

But $\sum_{r=2}^{\infty} \frac{1}{\log x}$ is divergent and $\int_{\frac{\pi}{2}}^{2} \frac{|\cos x|}{\log x} dx$ is proper integral.

Hence $I = \int_{0}^{\infty} \frac{|\cos x|}{\log x} dx$ is divergent and so $\int_{2}^{\infty} \frac{\cos x}{\log x} dx$ is conditionally convergent.

Example 2.2.5. Using $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{n}{2}$, show that

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Solution. To compute it let us integrate by parts, therefore

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx = \left[\frac{-\sin^2 x}{x}\right]_{0}^{\infty} + \int_{0}^{\infty} \frac{\sin 2x}{x} dx$$

Hence $\int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx = \int_{0}^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}.$

Example 2.2.6. The function f is defined on $[0, \infty[by f(x) = (-1)^{\mathrm{n}-1}$, for $n-1 \leq x < n, n \in N$, show that the integral $\int_{0}^{\infty} f(x) dx$ does not converge.

Solution. Consider

$$\int_0^{2n} f(x)dx = \int_0^1 (-1)^0 dx + \int_1^2 (-1)dx +$$

$$+\int_2^3 (-١)^{٢} dx + \cdots \int_{2n-1}^{2n} (-1)^{2n-1} dx$$

$$= 1 - 1 + 1 - 1 + 1 - 1 \ldots\ldots\ldots. +1 - 1$$

And

$\int_0^{2n+1} f(x)dx = \int_0^1 dx + \int_1^2 (-1)dx + \cdots \ \ldots\ldots. \int_{2n}^{2n+1} (-1)^{2n} dx$

$= 1 - 1 + 1 \ldots\ldots... -1 + 1 = 1$

$\therefore\ \lim_{n\to\infty} \int_0^{2n} f(x)dx = 0$

and $\lim_{n\to\infty} \int_0^{2n+1} f(x)dx = 1.$

Hence the integral does not exist and therefore it is not convergent.

Example 2.2.7. Test the convergence of

(I) $\int_0^{\infty} \frac{xdx}{1+x^4\cos^2 x}$

(II) $\int_0^{\infty} \frac{dx}{1+x^4\cos^2 x}$.

Solution. The integral is positive for positive value of x but the tests obtained for the convergence of positive integrands so far, are not applicable. In order to show the integral convergent we proceed as follows:

Consider $\int_0^{n\pi} \frac{xdx}{1+x^4\cos^2 x}$.

Therefore $\int_0^{n\pi} \frac{xdx}{1+x^4\cos^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{xdx}{1+x^4\cos^2 x}$.

Now, $\forall x \in [(r-1)\pi, r\pi]$.

We have

$$\frac{x}{1+x^4\cos^2 x} \geq \frac{(r-1)\pi}{1+r^4\cos^2 x}$$

Therefore $\int_{(r-1)\pi}^{r\pi} \frac{xdx}{1+x^4\cos^2 x} \geq \int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi dx}{1+x^4\cos^2 x}$

putting$\mathrm{x} = (\mathrm{r}-1)\pi + \mathrm{y}$ we see that

$$\int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi dx}{1+x^4\cos^2 x} = \int_0^{\pi} \frac{(r-1)\pi dy}{1+r^4\pi^4\cos^2((r-1)\pi+y)3}$$

$$= \int_0^{\pi} \frac{(r-1)\pi dy}{1+r^4\pi^4\cos^2 y}$$

$$= 2(r-1)\pi \int_0^{\frac{\pi}{2}} \frac{dy}{1+r^4\pi^4\cos^2 y}$$

$$= 2(r-1)\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 y dy}{1++\tan^2 y + r^4\pi^4}$$

$$= \frac{2(r-1)\pi}{\sqrt{1+r^4\pi^4}} \tan^{-1}\left(\frac{\tan y}{\sqrt{1+r^4\pi^4}}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$$

Therefore, $\sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{xdx}{1+x^4\cos^2 x} \geq \sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$.

Hence $\lim_{n\to\infty} \int_0^{n\pi} \frac{xdx}{1+x^4\cos^2 x} \geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$.

But $\sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$ is a divergent series $\left(\sim \sum_{r=1}^{n} \frac{1}{r}\right)$.

Therefore $\int_0^{\infty} \frac{xdx}{1+x^4\cos^2 x}$ is divergent.

(II) $\int_0^{\infty} \frac{dx}{1+x^4\cos^2 x}$ try yourself

Abstract:

The research can be summarized by my entry in this research by studying the unusual integrations that need to find integration of functions that are difficult to integrate to in the usual methods known. In addition to that we presented some characteristics and features for special functions that need a special .technology to find their integration, reinforced by theories and examples

توصيات:

١- يوصي الباحثان بضرورة الاهتمام لهكذا تكاملات لهذه الدوال الخاصة

٢- الاهتمام بتعزيز الامثلة والمبرهنات التطبيقية لهكذا دوال

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