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Some Basic Orthogonal Polynomials: Definitions And Properties

A Graduation project submitted to the Mathematics department in partial requirements for the degree bachelors in Mathematics fulfillment of the

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Some Basic Orthogonal Polynomials: Definitions And Properties

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بسم الله الرحمن الرحيم

﴿أَلَا إِنَّ أَوْلِيَاءَ اللَّهِ لَا خَوْفٌ عَلَيْهِمْ وَلَا هُمْ يَحْزَنُونَ﴾

[يونس:٦٢]

صدق الله العظيم

الإهداء

إلى الوالدين .... فلولاهما لما وجدت في هذه الحياة، ومهما تعلمت الصمود، مهما

كانت الصعوبات

إهداء

إلى أساتذتي الكرام.......... فعنهم استقيت الحروف وتعلمت كيف أنطق الكلمات، وأصوغ العبارات واحتكم إلى القواعد في مجال.....

إهداء

إلى الزملاء والزميلات الذين لم يدخروا جهدا في مدي بالمعلومات والبيانات

أهدي إليكم بحتى هذا

داعيا المولى سبحانه وتعالى أن يتكلل بالنجاح والقبول من جانب أعضاء

لجنة المنافسة المبجلين

الشكر والتقدير

لحمد لله رب العالمين والصلاة والسلام على سيد الأولين والآخرين وأشرف الخلق أجمعين محمد وعلى آله وصحبه وسلم تسليما كثيراً

اما بعد.......

يطيب لي أن أتقدم بجزيل الشكر والثناء إلى من لا أجد كلمة في سطور الكتب تستحق شرف الارتقاء لشكره ، إلى أستاذتي المشرف (م . م ساجد وليد عمران) الذي كان نعم العون لي ، لما أبداه من توجيهات وملاحظات علمية وما منحني إياه من وقت وجهد نورت طريق بحثي العلمي.

ومن واجب الاخلاص والعرفان أن أتقدم بالشكر والامتنان إلى الأستاذي الأفاضل (دكتور خالد هادي حميد)، لما قدمه لي من توجيهات وأراء سديدة خلال مدة الدراسة والبحث.

وأتوجه بالشكر الى كل من تكرم وسمح بتطبيق الدراسة عليه، و لما قدموه لي من خدمات جليلة لن أنساها

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والله ولي التوفيق

I. Introduction

Orthogonal polynomials are connected with many mathematical, physical, engineering, and computer sciences topics, such as trigonometry, hyper geometric series, special and elliptic functions, continued fractions, interpolation, quantum mechanics, partial differential equations. They are also be found in scattering theory, automatic control, signal analysis, potential theory, approximation theory, and numerical analysis.

Orthogonal polynomials are special polynomials that are orthogonal with respect to some special weights allowing them to satisfy some properties that are not generally fulfilled with other polynomials or functions. Such properties have made them well- known candidates to resolve enormous problems in physics, probability, statistics and other fields. Since their origin in the early 19th century, orthogonal polynomials have formed a somehow classical topic related to Legendre polynomials, Stieltjes' continued fractions, and the work of Gauss, Jacobi, and Christoffel, which has been generalized by Chebyshev, Heine, Szegö, Markov, and others. The most popular orthogonal polynomials are Jacobi, Laguerre, Hermite polynomials, and their special relatives, such as Gegenbauer, Chebyshev, and Legendre polynomials. An extending family has been developed from the work of Wilson, inducing a special set of orthogonal polynomi- als known by his name, which generalizes the Jacobi class. This new family has given rise to other previously unknown sets of orthogonal polynomials, including Meixner Pollaczek, Hahn, and A skey polynomials.

Orthogonal polynomials may also be classified according to the measure applied to define the orthogonality. In this context, we cite the class of discrete orthogonal polynomials that form a special case based on some discrete measure. The most common are Racah polynomials, Hahn polynomials, and their dual class, which in turn include Meixner, Krawtchouk, and Charlier polynomials.

Already with the classification of orthogonal polynomials, one can distinguish circular and generally spherical orthogonal polynomials, which consists of some special sets related to measures supported by the circle or the sphere. One well-known class is composed of Rogers–Szegö polynomials on the unit circle and Zernike polynomials, which are related to the unit disk.

Orthogonal polynomials, and especially classical ones, can generally be introduced by three principal methods. A first method is based on the Rodrigues formula which consists of introducing orthogonal polynomials as outputs of a derivation. The second method consists of introducing orthogonal polynomials as eigenvectors of Sturm–Liouville operators, or equivalently, solutions of second-order differential equations. The last method is based on a three-level recurrence formula.

Chapter one

Orthogonal polynomials

In this chapter, we reviews basic definitions as well as properties of orthogonal polynomials.

To do this, we first restrict ourselves to the field $\mathbb{R}$, and when it is necessary we recall that the development remains valid on the complex field $\mathbb{C}$.

1.1 Some Basic Definitions

Definition(1.1.1):

A Hilbert space is a vector space equipped with a scalar product, which makes it a complete space relative to the scalar product induced norm.

Definition (1.1.2):

A polynomial P of degree n on $\mathbb{R}$ is formally defined by the expression

$$P(X) = \sum_{k=0}^{n} a_k X^k$$

where X is the variable and $a_k s, 0 \leq k \leq n$, are elements of $\mathbb{R}$ called scalars and known as the polynomial coefficlent such that $a_n \neq 0$.

Remark.

The polynomial function associated with the polynomial P, which will also be denoted by P, is the functlon defined on the whole space $\mathbb{R}$ by $P(x) = \sum_{k=0}^{n} a_k x^k$. We denote by $\mathbb{R}[X]$ the set of all polynomials on $\mathbb{R}$. Of course, It is well known that $\mathbb{R}[X]$ is a vector space on $\mathbb{R}$ with infinite dimension and that for any $n \in \mathbb{N}$, the set $\mathbb{R}_n[X]$ of polynomials on $\mathbb{R}$ with degree at most n is a vector space with dimension $n + 1$ on $\mathbb{R}$.

Definition(1.1.3):

A set of polynomials $\mathcal{B} = (P_0, P_1, \dots, P_n, \dots)$ in $\mathbb{R}[X]$ is sald to be staggered with the degrees iff $\deg(P_t) = i, \forall i$.

The following result shows one important property of staggered degrees polynomials confirming the ability of such polynomials to be good candidates for polynomial spaces bases.

Proposition (1.1.1) :

Any finite set B = (P_0, P_1, . . . , P_n) of staggered degrees polynomials in $\mathbb{R}_n[X]$ is linearly independent.

Proof.

Let $(\alpha_0, \alpha_1, \dots, \alpha_n)$ be scalars in $\mathbb{R}$ such that $\sum_{i=0}^{n} \alpha_i P_t = 0$. This means that for all $x \in \mathbb{R}, \sum_{t=0}^{n} \alpha_l P_l(x) = 0$. By considering the n th-order derivative on x, we obtain $\alpha_n \frac{d^n P_n}{dx^n} = 0$. Consequently, $\alpha_n = 0$. Next, proceeding by induction on n, we prove that all the coefficlents α_i are null. Hence, $\mathfrak{B}$ is a free set in E. Observe next that the dimension of E (dim $E = n + 1$) coincides with the cardinality of $\mathcal{B}$. Therefore, $\mathcal{B}$ is a basis of E.

Theorem (1.1.1): (GRAM-SCHMIDT).

Let $\{f_n\}_{n\geq 0}$ be a countable system of linearly independent elements in a prehilbertian space. Then, there exists an orthonormal system $\{g_n\}_{n\geq 0}$ such that for any n, Vect $\{g_0, g_1, \dots, g_n\} =$ Vect$\{f_0, f_1, \dots, f_n\}$.

Proof. We proceed by induction to construct the system $\{g_n\}_{n\geq 0}$. Let $g_0 = f_0$. Then element g_1 will be defined by

$$g_1 = f_1 - \alpha g_0.$$

As we want g_0 and g_1 to be orthogonal, we obtain

$$\langle g_0, g_1 \rangle = \langle g_0, f_1 \rangle - \alpha \langle g_0, g_0 \rangle = 0.$$

So that, $\alpha = \frac{\langle g_0, f_1 \rangle}{\langle g_0, g_0 \rangle}$. Otherwise, we subtract from f_1 its orthogonal projection on g_0, 1.e.,

$$g_1 = f_1 - \frac{\langle f_1, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0$$

Hence, clearly we have Vect $\{0, g_1\} = \text{Vect}\{f_0, f_1\}$.

Next, g_2 is defined analogously by subtracting from f_2 its orthogonal projections on (g_0, g_1). In other words,

$$g_2 = f_2 - \frac{\langle f_2, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 - \frac{\langle f_2, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0.$$

It is straightforward that g_2 is orthogonal to g_0 and g_1. Assume next that g_n is well known. g_{n+1} will be obtained as follows:

$$g_{n+1} = f_{n+1} - \sum_{i=0}^{n} \frac{\langle f_{n+1}, g_i \rangle}{\langle g_i, g_t \rangle} g_t$$

We check easily that for all $k \leq n$,

$$\begin{aligned} \langle g_{n+1}, g_k \rangle \quad &= \langle f_{n+1}, g_k \rangle - \sum_{i=1}^{n} \frac{\langle f_{n+1}, g_i \rangle}{\langle g_i, g_i \rangle} \langle g_i, g_k \rangle \\ &= \langle f_{n+1}, g_k \rangle - \frac{\langle f_{n+1}, g_k \rangle}{\langle g_k, g_k \rangle} \langle g_k, g_k \rangle = 0. \end{aligned}$$

Obvlously, the elements g_n are not normalized. To do this, we divide each one by its norm. The equality Vect

$$\{g_0, g_1, \dots, g_n\} = \text{Vect}\{f_0, f_1, \dots, f_n\}$$

Is straightforward.

Definition (1.1.4):

Let I be an interval in $\mathbb{R}$ nonreduced to a point and let ω be a positive continuous function on $I. \omega$ is sald to be a weight function iff

$$\int_I |x|^d \omega(x) \mathrm{d}x < \infty, \ \forall d \in \mathbb{N}.$$

We denote by the next $\mathcal{C}_\omega(I)$ the vector space of continuous functions on the interval I, satisfying

$$\int_I |f(x)|^2 \omega(x) \mathrm{d}x < \infty \qquad (1.1)$$

It results from hypothesis 7 that the polynomials are elements of $\mathcal{C}_\omega(I)$. On this space of functions, a scalar product can be defined by

$$\langle f, g \rangle = \int_I f(x) g(x) \omega(x) \mathrm{d}x \qquad (1.2)$$

The integration interval I will be called the orthogonality interval.

Definition (1.1.5): A set of polynomials $(P_t)_{t\geq 0}$ is said to be orthogonal iff it satisfies

(1) Degree $(P_t) = i; \forall i \in \mathbb{N}$.

(2) $\langle P_1, P_j \rangle = 0; \forall (i,j) \in \mathbb{N}^2; i \neq j$.

The following result shows some generic propertles of orthogonal polynomials, as they are speclal cases of staggered degree polynomials and consequently they also form good candidates for polynomial spaces orthogonal bases.

Proposition (1.1.2):

Let $(P_t)_{t\geq 0}$ be a set of orthogonal polynomials. Then

(1) $\forall n \in \mathbb{N}; (P_0, P_1, \dots, P_n)$ is an orthogonal basis of $\mathbb{R}_n[X]$.

(2) $\forall (n,p) \in \mathbb{N}^2; n \geq p + 1 \Longrightarrow P_n \in \left(\mathbb{R}_p[X]\right)^\perp$.

Proof. The first assertion is a consequence of Proposition (1.1) and the orthogonality of the set $(P_0, P_1, \dots, P_n)$. (We can also use the second point in Definition (1.1) to prove the independence of the P_j s, $j = 0, \dots, n$). Next, as $\mathbb{R}_p[X]$ is generated by the set $(P_0, P_1, \dots, P_p)$ and $n \geq p + 1$, which means that $P_n \perp P_j$, for all $j = 0, \dots, p$, so it is orthogonal to $\mathbb{R}_p[X]$.

Remark.

Sometimes we need to use unitary orthogonal polynomials P_n. Thus, we need to multiply them by constants so that $\lambda_n P_n$ becomes unitary or not. So, in the following, we will not differentlate between the two notions and will use the notation $(P_n)_n$ and $\lambda_n P_n$ depending on the context.

Theorem(1.1.2):

The unitary orthogonal polynomials satisfy the following assertions:

(1) $P_0(x) = 1$.

(2) Degree $(P_n) = n, \forall n \in \mathbb{N}$.

(3) $\int_I P_n(x)Q(x)w(x)\mathrm{d}x = 0, \forall Q \in \mathbb{R}[X]$ such that Degree$(Q) < n$.

(4) $\mathbb{R}_n(X) = \mathrm{Vect}(P_0, \dots, P_n), \forall n \in \mathbb{N}$.

Proof. (1) P_0 is a unitary constant polynomial. So, it is equal to 1.

(2) It follows from the first assertion in Definition 1.2.

(3) As Degree(Q) $> n$ so $Q \in \mathbb{R}_n[X]^{\perp}$.

(4) Holds from proposition 1.1.

Lemma (1.1.1)

Let $(P_0, \dots, P_n)$ be a unitary orthogonal polynomial set. Hence,

(1) $(P_0, \dots, P_n)$ is a basis of $\mathbb{R}_n[X]$.

(2) P_n is orthogonal to $\mathbb{R}_{n-1}[X]$.

Indeed, Firstly, we know that $\dim \mathbb{R}_n[X] = n + 1 = \mathrm{card}(P_0, \dots, P_n)$. On the other hand, $(P_0, \dots, P_n)$ is orthogonal; hence, it is linearly independent. Thus, it consists of a basis in $\mathbb{R}_n[X]$.

The second point follows from the fact that P_n is orthogonal to $(P_0, \dots, P_{n-1})$, which means that it is orthogonal to $\mathbb{R}_{n-1}(X) = \text{Vect}(P_0, \dots, P_{n-1})$.

1.2 Orthogonal polynomials via a three-level recurrence

Theorem(1.2.1) (Recurrence rule).

Let $(P_t)_{t\geq 0}$ be a set of orthogonal polynomials. There exist scalars $(a_n)_n, (b_n)_n$, and $(c_n)_n$ such that

$$P_{n+1} = (a_n X + b_n)P_n + c_n P_{n-1};\ \forall n \in \mathbb{N}^*.$$

More precisely,

$$a_n = \frac{k_{n+1}}{k_n},\ b_n = -a_n \frac{\langle XP_n, P_n\rangle}{\|P_n\|^2} \text{ and } c_n = -\frac{a_n}{a_{n-1}} \frac{\langle P_n, P_n\rangle}{\langle P_{n-1}, P_{n-1}\rangle},$$

where k_n is the coefficient of X^n in $P_n(X)$.

Proof. Without loss of generality, we can assume that $(P_i)_{t\geq 0}$ is orthonormal. Let $\mathcal{B} = (XP_n, P_n, P_{n-1}, \dots, P_0)$ be a set of staggered degree polynomials in $\mathbb{R}_{n+1}[X]$. So, it is linearly independent in $\mathbb{R}_{n+1}[X]$. Consequently, it forms a basis of $\mathbb{R}_{n+1}[X]$. Consequently, there exist then scalars a_n, b_n, c_n and $\alpha_i, 0 \leq i \leq n-2$ such that

$$P_{n+1} = a_n XP_n + b_n P_n + c_n P_{n-1} + \sum_{t=0}^{n-2} \alpha_i P_t.$$

Next, using the orthogonality property of $(P_t)_{t\geq 0}$, we obtain

$$\langle P_{n+1}, P_t\rangle = a_n\langle XP_n, P_i\rangle + \alpha_i\|P_t\|^2 = 0,\ \forall 0 \leq i \leq n-2.$$

On the other hand,

$$\langle XP_n, P_t\rangle = \langle P_n, XP_t\rangle.$$

Since $XP_t \in \mathbb{R}_{n-1}[X]$, we obtain

$$\langle XP_n, P_l\rangle = 0$$

Consequently,

$$\alpha_i = 0,\ \forall 0 \leq i \leq n-2.$$

Hence,

$$P_{n+1} = (a_n X + b_n)P_n + c_n P_{n-1}\ . \qquad (2.3)$$

We now evaluate the coefficlents a_n, b_n, and c_n. Recall that P_n can be written as

$$P_n(X) = k_n X^n + k_{n-1} X^{n-1} + \cdots + k_0.$$

By identification of the higher degree monomials in (2.3), we obtain

$$a_n = \frac{k_{n+1}}{k_n}$$

Next, the inner product of (2.3) with P_n gives

$$\langle P_{n+1}, P_n \rangle = a_n \langle XP_n, P_n \rangle + b_n \langle P_n, P_n \rangle + c_n \langle P_{n-1}, P_n \rangle.$$

Using the orthogonality of the set, we get

$$a_n \langle XP_n, P_n \rangle + b_n \langle P_n, P_n \rangle = 0.$$

Hence,

$$b_n = -a_n \frac{\langle XP_n, P_n \rangle}{\langle P_n, P_n \rangle}.$$

Next, using the inner product with P_{n-1} and using again the orthogonality of the set, we obtain

$$a_n \langle XP_n, P_{n-1} \rangle + c_n \langle P_{n-1}, P_{n-1} \rangle = 0.$$

Hence,

$$c_n = -a_n \frac{\langle XP_n, P_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle} = -a_n \frac{\langle P_n, XP_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle}.$$

Next, denote $XP_{n-1} = \sum_{t=0}^{n} \alpha_t P_t$ as the decomposition of XP_{n-1} in the basis of polynomials $(P_t)_{0 \le 1 \le n}$. By observing the higher degree monomials in the decomposition, we get

$$X k_{n-1} X^{n-1} = \alpha_n k_n X^n \iff \alpha_n = \frac{k_{n-1}}{k_n} = \frac{1}{a_{n-1}}.$$

On the other hand,

$$\langle XP_n, P_{n-1} \rangle = \langle P_n, XP_{n-1} \rangle$$

$$= \alpha_n \langle P_n, P_n \rangle + \sum_{i=0}^{n-1} \alpha_i \langle P_n, P_t \rangle$$

$$= \alpha_n \langle P_n, P_n \rangle + \sum_{i=0}^{n-1} \alpha_i 0$$

$$= \alpha_n \langle P_n, P_n \rangle.$$

Consequently,

$$c_n = -a_n \frac{\langle P_n, XP_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle} = -a_n \frac{\alpha_n \langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle} = -\frac{a_n}{a_{n-1}} \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle}$$

Hence,

$$P_{n+1} = a_n X P_n + b_n P_n + c_n P_{n-1},$$

where

$$a_n = \frac{k_{n+1}}{k_n},\ b_n = -a_n \frac{\langle XP_n, P_n \rangle}{\langle P_n, P_n \rangle} \text{ and } c_n = -\frac{a_n}{a_{n-1}} \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle}.$$

In the case where $(P_0, \dots, P_n)$ is orthonormal, we obtain

$$a_n = \frac{k_{n+1}}{k_n},\ b_n = -a_n \langle XP_n, P_n \rangle \text{ and } c_n = -\frac{a_n}{a_{n-1}}.$$

Theorem (1.2.2): (Favard's theorem).

Let $\{c_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$ be sequences in $\mathbb{R}$, and $\{P_n\}_{n=0}^{\infty}$, a set of polynomials satisfying

$$P_n(x) = (x - c_n) P_{n-1}(x) - \lambda_n P_{n-2}(x),\ \forall n \in \mathbb{N}^*,$$

where $P_0(x) = 1$ and $P_1(x) = x - c_1$. Then, there exists a unique linear form φ on $\mathbb{R}_n(X)$ for which $\varphi(P_k P_m) = 0$ whenever $k \neq m$.

Proof. We proceed by steps.

Step 1. We claim that Degree $(P_n) = n, \forall n \in \mathbb{N}$. Indeed, for $n = 0, P_0(x) = 1$. Hence, Degree$(P_0) = 0$. For $n = 1, P_1(x) = x - c$, so it is of degree 1. Assume next that Degree$(P_n) = n$ and prove the same for P_{n+1}. The three-level relation above ylelds that

$$\text{Degree}(P_{n+1}) = \text{Degree}\big((x - c_{n+1})P_n(x)\big) = 1 + n.$$

Hence, we proved by recurrence on n that $\text{Degree}(P_n) = n, \forall n \in \mathbb{N}$.

Step 2. Consider the space $\mathbb{R}_n[X]$ of polynomials on $\mathbb{R}$ with degrees at most n. It results from Step 1 that the set $\mathcal{B}_n = (P_0, \dots, P_n)$, satisfying that the three-level relation is a degree-straggled set of polynomials. Henceforth, it is a basis of $\mathbb{R}_n[X]$. Let $\varphi: \mathbb{R}_n[X] \longrightarrow \mathbb{R}$ be the continuous linear form defined on such a basis by

$$\varphi(P_0) = 1, \varphi(P_1) = \cdots = \varphi(P_n) = 0.$$

It holds from the Riez-Fréchet theorem that there exists a function ω such that

$$\varphi(P) = \langle P, \omega \rangle = \int_{\mathbb{R}} P(x)\omega(x)dx.$$

We now prove that

$$\varphi(P_k P_m) = 0,\ \forall 0 \le k \ne m \le n.$$

For $k < m$, denote $P_k(x) = \sum_{s=0}^{k} \alpha_s(k) x^s$. We get

$$\varphi(P_k P_m) = \sum_{s=0}^{k} \alpha_S(k)\varphi(x^s P_m).$$

On the other hand, $x^s P_m$ can be written as

$$x^s P_m = \sum_{t=m-s}^{m+s} d_i P_t.$$

Hence,

$$\varphi(x^s P_m) = \sum_{i=m-s}^{m+s} d_i \varphi(P_t) = 0.$$

Example (1.2.1) We set here some examples of induction relations for the most known orthogonal polynomials.

(1) Legendre polynomials

$$P_{n+1} = \frac{2n+1}{n+1} X P_n - \frac{n}{n+1} P_{n-1},\ \forall n \in \mathbb{N}^*$$

(2) Chebyshev polynomials

$$P_{n+1} = 2XP_n - P_{n-1},\ \forall n \in \mathbb{N}^*.$$

(3) Hermite polynomials

$$P_{n+1} = 2XP_n - 2nP_{n-1},\ \forall n \in \mathbb{N}^*.$$

1.3 Orthogonal polynomials via Rodrigues rule

A literature review of orthogonal polynomials reveals that there are many methods to obtain such polynomials. One is explicit and based on the Rodrigues rule, which applies derivation. Let

$$P_n(x) = \frac{1}{k_n \omega(x)} \frac{d^n}{dx^n} [\omega(x) S^n],$$

where S is a polynomial in x, ω is a weight function, and k_n is a constant. We have precisely the following result.

Theorem 1.3.1

Let $I = [a, b]$ and ω is a weight function on I and $(\phi_n)_{n\in\mathbb{N}}$ be a set of real functions on I satisfying

(1) ϕ_n is C^n on $]a, b[$ for all n.

(2) $\phi_n^{(k)}(a^+) = \phi_n^{(k)}(b^-) = 0$ for all $k, 0 \leq k \leq n-1$.

(3) $T_n = \frac{1}{k_n \omega} (\omega \phi_n)^{(n)}$ is a polynomial of degree n, (k_n is a normalization constant). Then, $(T_n)_{n\in\mathbb{N}}$ is orthogonal. The converse is true iff ω is C^∞.

Proof. It suffices to prove the orthogonality. For $n < m$, we have

$$
\begin{aligned}
\langle T_n, T_m \rangle &= \int_a^b T_n(x) T_m(x) \omega(x) dx \\
&= \int_a^b T_n(x) \frac{1}{k_m \omega(x)} (\omega \phi_m)^{(m)} \omega(x) dx \\
&= \int_a^b T_n(x) \frac{1}{k_m} (\omega \phi_m)^{(m)} dx \\
&= (-1)^m \int_a^b (T_n(x))^{(m)} \frac{\omega \phi_m}{k_m} dx \\
&= 0
\end{aligned}
$$

The fourth equality is a consequence of Hypothesis (2) and the integration by the parts rule. The last equality is a consequence of Hypothesis (3).

1.4 Orthogonal polynomials via differential equations

A large class of orthogonal polynomials is obtained from first-order linear differential equations of the type

$$a(x)y'' + b(x)y' - \lambda_n y = 0, \qquad (2.6$$

Whe0re a is a polynomial of degree 2 , and b is a polynomial of degree 1 , where both are independent of the integer parameter n, and finally, λ_n are scalars. y is the unknown function. By introducing the operator $T: \mathbb{R}[X] \longrightarrow \mathbb{R}[X]$ such that $T(y) = ay'' + by'$, the solution y appears as an eigenvector of T associated with the eigenvalue λ_n. We introduce next a resolvent function $\omega > 0$, which permits us to express the operator T on the form $T(y) = \frac{1}{w}(awy')'$. The equality $T(y) = ay'' + a'y' + \frac{aw'}{W}y'$ shows that ω is a solution of the differential equation $a\omega' + (a' - b)\omega = 0$. So, it is of the form $\omega = e^A$, where A is a primitive of $\frac{b-a'}{a}$. Recall now that

$$\langle T(f), g\rangle = \int_I (a\omega f')'(x) g(x)\omega(x)dx$$

$$= [a\omega f' g]_I - \int_I a(x) f'(x) g'(x)\omega(x)dx.$$

Iff the weight ω vanishes on the frontler of the integration interval I, we obtain

$$\langle T(f), g\rangle = -\int_I a(x) f'(x) g'(x)\omega(x)dx = \langle f, T(g)\rangle.$$

This means that the operator T is symmetric.

Denote for the next $T_n: \mathbb{R}_n[X] \longrightarrow \mathbb{R}_n[X]$ the restriction of T on $\mathbb{R}_n[X]$. It is straightforward that $\mathbb{R}_n[X]$ is invariant under the action of T_n since the degrees of a

and b are less than 2 and 1 , respectively. So, we can arrange the pairs (λ, y) into a sequence (λ_k, y_k), where we re-obtain the eigenpairs of the operator T_n for $k = 0, \dots, n$. Next, observing that $a(x) = a_2X^2 + a_1X + a_0$ and $b(x) = b_1X + b_0$, it results that T_n is an endomorphism on $\mathbb{R}_n[X]$. Thus, there exists a T-eigenvector's orthonormal basis of such a space. In particular, there exists at least an elgenvector P_n of degree n, which may be assumed to be unitary and satisfying

$$aP_n'' + bP_n' = \lambda_n P_n.$$

This means that for $n \neq m$, we obtain $\lambda_n \neq \lambda_m$ and thus the polynomials P_n are orthogonal.

Chapter Two

Some Orthogonal Polynomials

2.1 Some classical orthogonal polynomials

In the previous chapter, we reviewed the three most well-known schemes to obtain orthogonal polynomials. The first one is based on the explicit Rodrigues derivation rule, which states that the nth element of the set of orthogonal polynomials, which is also of degree n, is obtained by

$$P_n(x) = \frac{1}{k_n\omega(x)}\frac{d^n}{dx^n}[\omega(x)S^n],$$

where S is a suitable polynomial in x.

The next method is based on an induction rule eventually necessitates that the first and the second elements of the desired set of orthogonal polynomials be known. It states that

$$P_{n+1} = (a_nX + B_n)P_n + c_nP_{n-1}, \qquad (2.1)$$

where a_n, b_n, and c_n are known scalars.

Finally, the last scheme consists of introducing orthogonal polynomials as the solutions of ordinary differential equations (ODEs) of the form

$$a(x)y'' + b(x)y - \lambda_n y = 0$$

where a is a 2-degree polynomial and b is a polynomial with degree 1 and λ_n are scalars. The idea consists of developing polynomial solutions of the ODEs. According to the coefficlents of each equation, we obtain the desired class of polynomials, such as Legendre and Laguerre.

In this section, we propose to revisit some classical classes of orthogonal polynomials and show their construction with the three schemes.

2.1.1 Legendre polynomials

From Rodrigues rule:

Legendre polynomials consist of polynomials defined on the orthogonality interval $I = [-1,1]$ relative to the weight functlon $\omega \equiv 1$, the polynomial $S(x) = (x^2 - 1)$, and the constant $k_n = 2^n n!$. The nth Legendre polynomial, usually denoted in the literature by L_n, is obtained by

$$L_n(x) = \frac{d^n}{dx^n}\left[\frac{(x^2-1)^n}{2^n n!}\right]$$

Using the Leibniz rule of derivation, $L_n(x)$ can be explicitly computed. We have

$$L_n(x) = \frac{1}{2^n n!}\sum_{k=0}^{n} C_n^k((x-1)^n)^{(k)}((x+1)^n)^{(n-k)} =$$

$$\frac{1}{2^n}\sum_{k=0}^{n}(C_n^k)^2(x-1)^{n-k}(x+1)^k$$

For example,

$$\begin{aligned} L_0(x) &= 1, & L_1(x) &= x, \\ L_2(x) &= \frac{1}{2}(3x^2-1), & L_3(x) &= \frac{1}{2}(5x^3-3x), \\ L_4(x) &= \frac{1}{8}(35x^4-30x^2+3), & L_5(x) &= \frac{1}{8}(63x^5-70x^3+15x). \end{aligned}$$

From the induction rule

Legendre polynomials can also be introduced via the induction rule

$$L_{n+1} = \frac{2n+1}{n+1}XL_n - \frac{n}{n+1}L_{n-1},\ \forall n \in \mathbb{N}^*$$

with initial data $L_0(x) = 1$ and $L_1(x) = x$. It ylelds, for $n = 1$, that

$$L_2(x) = \frac{3}{2}xL_1(x) - \frac{1}{2}L_0(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

For $n = 2$, it ylelds that

$$L_3(x) = \frac{5}{3}xL_2(x) - \frac{2}{3}L_1(x) = \frac{5}{3}x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{2}{3}x = \frac{5}{2}x^3 - \frac{3}{2}x.$$

Applying the same procedure, we obtain

$$L_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \text{ and } L_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

From ODEs

Legendre polynomials are obtained as the polynomial solutions of the following ODE:

$$(1 - \chi^2)y'' - 2xy' + n(n+1)y = 0, x \in I$$
$$= [-1,1]. \qquad (2.2)$$

Using the notations of Section 2.2, this means that $a(x) = 1 - \chi^2, b(x) = -2\chi$ and $\lambda_n = -n(n+1)$. In the sense of the linear operator T, the polynomials L_n can be introduced vla the operator

$$T(y) = (1 - x^2)y'' - 2xy' = \left((1 - \chi^2)y'\right)',$$

which corresponds to the weight function $\omega(x) = 1$ and $a(x)\omega(x) = 1 - \chi^2$. Note that $a\omega$ vanishes at the frontiers ± 1 of the orthogonality interval I. Furthermore, in terms of elgenvalues as in equation (2.7), If we suppose that the same eigenvalue λ_n is assoclated with at least two elgenvectors P_n and P_m, we obtain $(n-m)(n+m-1) = 0$, which has no integer solutions except $n = m$. This confirms that the eigenvalues and elgenvectors are one to one, which means that the elgenvectors (polynomials) are orthogonal. Figure 2.1 lllustrates the graphs of the first Legendre polynomials.

For clarity and convenience, we will develop the polynomial solutions. So, denote $P(x) = a_p x^p + a_{p-1}x^{p-1} + \cdots + a_1 x + a_0$ as a polynomial solution of degree p of equation (2.8). We obtain the following system:

$$\begin{cases} 2a_2 + n(n+1)a_0 = 0 \\ 6a_3 + (n^2 + n - 2)a_1 = 0 \\ \left[(n(n+1) - (p-1)(p+1)\right]a_{p-1} = 0 \\ \left[(n(n+1) - p(p+1)\right]a_p = 0, \\ (k+1)(k+2)a_{k+2}[(n(n+1) - k(k+1)]a_k = 0,\ 2 \le k \le p-2 \end{cases}$$

Hence, $p = n$ and

$$\begin{cases} 2a_2 + n(n+1)a_0 = 0, \\ 6a_3 + (n^2 + n - 2)a_1 = 0, \\ [(n(n+1) - n(n-1)]a_{p-1} = 0, \\ (k+1)(k+2)a_{k+2}[(n(n+1) - k(k+1)]a_k = 0,\ 2 \le k \le p-2. \end{cases}$$

For example, for $n = 0$, we obtain

$$P(x) = a_0$$

For $n = 1$, we get

$$P(x) = a_1\chi$$

For $n = 2$, we obtain

$$P(x) = -a_0(3x^2 - 1)$$

For $n = 3$,

$$P(x) = -a_1\left(\frac{5}{3}x^3 - x\right)$$

For $n = 4$, we have

$P(x) = -a_0\left(\frac{35}{3}x^4 - 10x^2 + 1\right)$

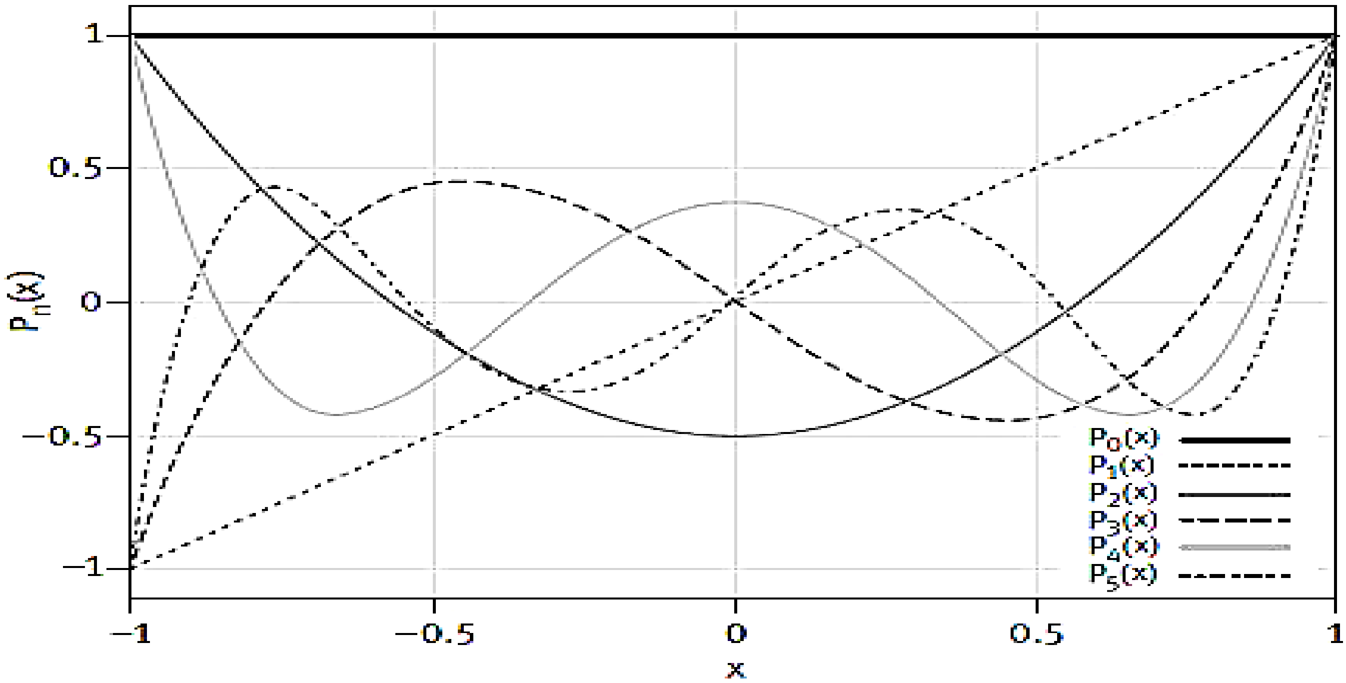

Fig. 2.1: Legendre polynomlals

Next, for $n = 5$, we obtain

$$P(x) = a_1\left(\frac{21}{5}\chi^5 - \frac{14}{3}x^3 + x\right)$$

Now, using the orthogonality of these polynomials on $[-1,1]$, we obtain the same polynomials.

One important question is how to choose the polynomial S in Rodrigues rule to be equivalent with the same outputs of the recurrence rule and the ODE scheme.

Firstly, the degree of S is fixed in an obvlous way as $\deg P_n = n, \forall n$. Hence, for example, in the Legendre case, S should be of degree 2 , that is,

$$S(x) = a + bx + cx^2,\ c \neq 0.$$

Thus,

$$L_n(x) = e_n \frac{d^n}{dx^n}(S^n(x)),\ e_n = \frac{1}{2^n n!}.$$

Consequently, from the induction rule of Legendre polynomials we obtain, for $n = 2$,

$$\begin{aligned} L_2 &= \frac{3}{2}\chi L_1 - \frac{1}{2}L_0 \\ e_2(S^2(x))'' &= \frac{3}{2}e_1\chi(S'(x)) - \frac{1}{2}e_0 \end{aligned}$$

As a result,

$$\begin{cases} b^2 + 2ac + 2 = 0 \\ 6bc - 3b = 0 \\ 6c^2 - 6c = 0 \end{cases}$$

Hence, we obtain

$$c = 1,\ b = 0,\ a = -1$$

Or equivalently,

$$S(x) = x^2 - 1$$

2.1.2 Laguerre polynomials

From the Rodrigues rule:

These polynomials are obtained via the Rodrigues rule with $\omega(x) = e^{-x}, S(x) = x$ and the constant $k_n = n\,!$ by

$$\mathcal{L}_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}\chi^n) = \sum_{k=0}^{n}\frac{C_n^k}{k!}(-\chi)^k.$$

The first polynomials are then

$\mathcal{L}_0(x) = 1,\ \mathcal{L}_1(x) = 1 - x,$

$\mathcal{L}_2(x) = \frac{1}{2}(x^2 - 4x + 2),\ \mathcal{L}_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6),$

$\mathcal{L}_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24),$

$\mathcal{L}_5(x) = \dfrac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120).$

It holds clearly from simple calculus that these polynomials are orthogonal in the interval $[0,\infty[$ relative to the weight function $\omega(x) = e^{-x}$.

From the induction rule

Laguerre polynomials are solutions of the following recurrent relation:

$$(n+1)\mathcal{L}_{n+1}(x) + (x - 2n - 1)\mathcal{L}_n(x) + n\mathcal{L}_{n-1}(x) = 0$$

with the first and second elements $\mathcal{L}_0(x) = 1$ and $\mathcal{L}_1(x) = 1 - x$. For $n = 1$, we get

$$2\mathcal{L}_2(x) + (x - 3)\mathcal{L}_1(x) + \mathcal{L}_0(x) = 0,$$

which implies that

$$\mathcal{L}_2(x) = \frac{1}{2}(3 - x)(1 - x) + \frac{1}{2} = \frac{1}{2}(x^2 - 4x + 2).$$

Next, for $n = 2$, we obtain

$$3\mathcal{L}_3(x) + (x - 5)\mathcal{L}_2(x) + 2\mathcal{L}_1(x) = 0,$$

which means that

$$\mathcal{L}_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6).$$

Similarly, we can obtain

$$\mathcal{L}_4(x) = \frac{1}{24}(-x^4 - 16\chi^3 + 72x^2 - 96x + 24)$$

and

$$\mathcal{L}_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120).$$

From ODEs

To apply the ODE procedure, we set $I =]0, \infty[$ as the orthogonality interval, $a(x) = x$, $b(x) = 1 - x, \omega(x) = e^{-x}$ and consequently, the operator T will be $T(y) = xy'' - (1 - x)y'$. We observe immediately that $a(x)\omega(x) = xe^{-x}$ is null at 0 and has the limit 0 at $+\infty$. Furthermore, $P(x)\omega(x)$ is integrable on I for all polynomial P. In the present case, equation (2.7) becomes $(n + m)(n - m) = 0$ and the eigenvalues are $\lambda_n = -n$. The assoclated ODE is

$$xy'' - (1 - x)y' + ny = 0$$

It is straightforward that for all n, the polynomial $\mathcal{L}_n$ is a solution of this differential equation. Figure 2.2 illustrates the graphs of some examples of Laguerre polynomials.

2.1.3 Hermite polynomials

From Rodrigues rule:

Hermite polynomials are related to the orthogonality interval $I = \mathbb{R}$ with the weight function $\omega(x) = e^{-x^2}$. Denote H_n as the nth element, 1.e., a Hermite polynomial of degree $n. H_n$ is explicitly expressed via Rodrigues rule as follows:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

As examples, we get

$$H_0(x) = 1,\ H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x, \ \text{and} \quad H_4(x) = 16x^4 - 48x^2 + 12$$

From the induction rule

Hermite polynomials H_n can be obtained by means of the induction rule

$$H_{n+1} = 2XH_n - 2nH_{n-1},\ \forall n \in \mathbb{N}^*,$$

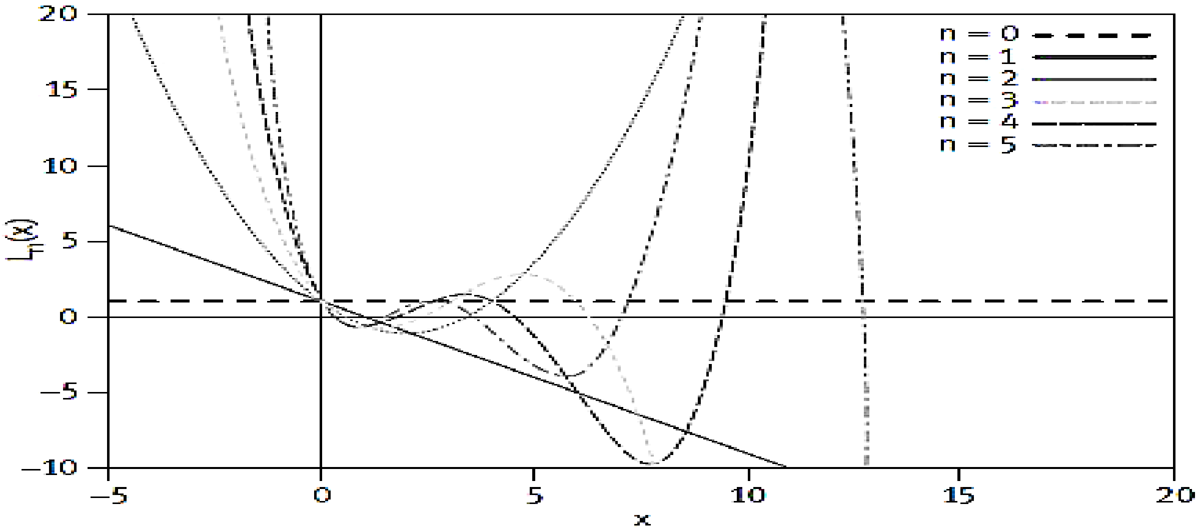

Fig. 2.2: Laguerre polynomlals.

with the initial data $H_0(X) = 1$ and $H_1(X) = 2X$. So, for $n = 1,2,3,4,5$ we obtain as examples

$$H_2(X) = 4X^2 - 2,\ H_3(X) = 8X^3 - 12X,\ H_4(X) = 16X^4 - 48X^2 + 12$$
$$H_5(X) = X^5 - 10X^3 + 15X,\ H_6(X) = X^6 - 15X^4 + 45X^2 - 15$$

From ODEs

Hermite polynomials are also solutions of a second-order ODE in the interval $I = \mathbb{R}$. this means that $a(x) = 1, b(x) = -2x$, and $\omega(x) = e^{-x^2}$ as a weight function. It is immediate that $a(x)\omega(x) = e^{-x^2}$, which has 0 limits at the boundaries of the interval I. Furthermore, $P(x)\omega(x)$ is integrable on I for all polynomials P. By means of the eigenvalues of the linear operator T, Hermite polynomials are eigenvectors of $T(y) = y'' - 2xy'$ assoclated with eigenvalues $\lambda_n = -2n$. The corresponding ODE is

$$y'' - 2xy' + 2ny = 0$$

Some examples of Hermite polynomials are illustrated in figure 2.3.

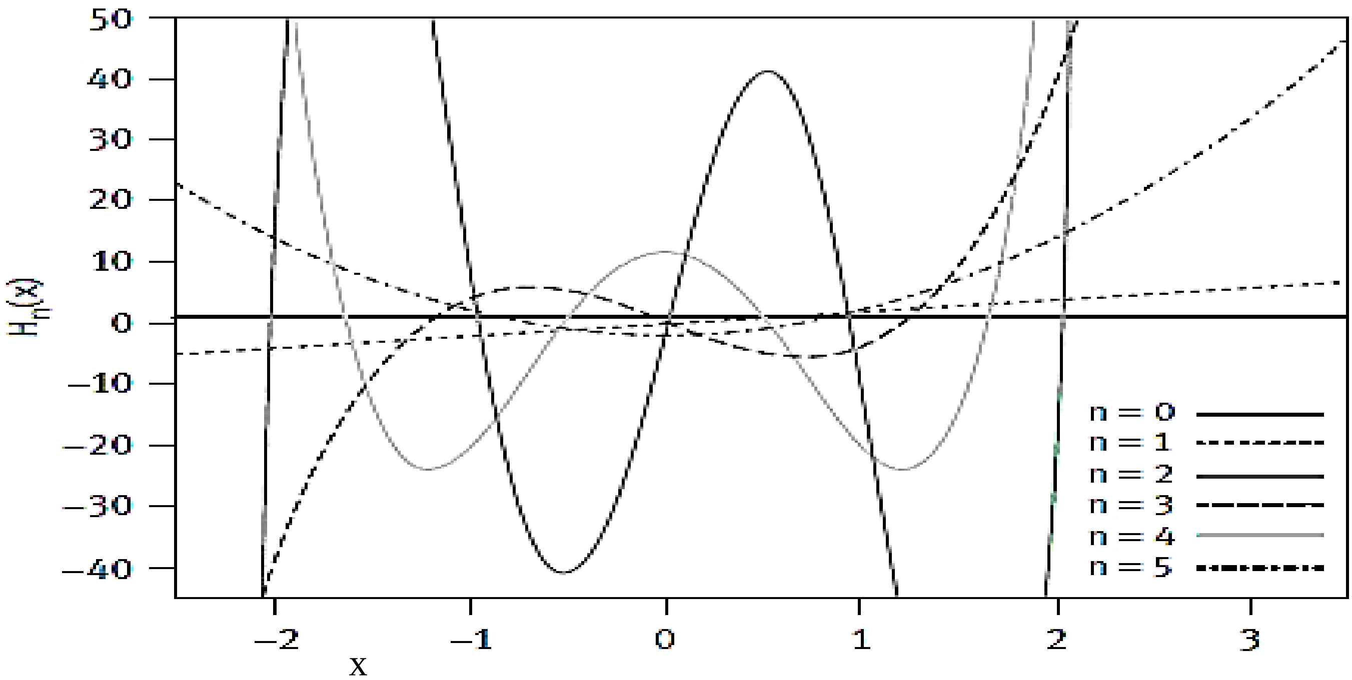

Fig. 2.3: Hermite polynomlals.

2.1.4 Chebyshev polynomials

From Rodrigues rule

Chebyshev polynomials are related to the orthogonality interval $I =]-1,1[$ and the weight function $\omega(x) = (1-\chi^2)^{-1/2}$. Denoted usually by T_n for the Chebyshev polynomial of degree n, these are explicitly expressed via the Rodrigues rule as

$$T_n(x) = \frac{(-1)^n (1-\chi^2)^{\frac{1}{2}} \sqrt{\pi}}{2^n \Gamma\left(n+\frac{1}{2}\right)} \frac{d^n}{dx^n}\left((1-x^2)^{n-\frac{1}{2}}\right),$$

where $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ is Euler's well-known functlon. It is immediately seen (by recurrence for example) that

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!}, \ \forall n \in \mathbb{N},$$

and hence, the first Chebyshev polynomials can be obtained as

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x,\ T_4(x) = 8x^4 - 8x^2 + 1,\ T_5(x) = 16x^5 - 20x^3 + 5x.$$

From the induction rule

Chebyshev polynomials are solutions of the induction formula

$$T_{n+1} = 2xT_n - T_{n-1},\ \forall n \in \mathbb{N}^*,$$

with initlal data, $T_0(x) = 1$ and $T_1(x) = x$. Let, $T_n(x) = \sum_{k=0}^{n} a_k^n x^k$, then

$$\sum_{k=0}^{n+1} a_k^{n+1} \chi^k = 2\chi \sum_{k=0}^{n} a_k^n \chi^k - \sum_{k=0}^{n-1} a_k^{n-1} \chi^k$$
$$\sum_{k=0}^{n+1} a_k^{n+1} x^k + a_{n+1}^{n+1} x^{n+1} + a_0^{n+1} = \sum_{k=0}^{n} 2a_k^n x^{k+1} - \sum_{k=0}^{n-1} a_k^{n-1} x^k$$
$$= \sum_{k=1}^{n+1} 2a_{k-1}^n x^k - \sum_{k=0}^{n-1} a_k^{n-1} x^k$$
$$\sum_{k=1}^{n-1} (a_k^{n+1} - 2a_{k-1}^n + a_k^{n-1}) x^k + a_0^{n+1} + a_0^{n-1}$$
$$+(a_n^{n+1} - 2a_{n-1}^n) x^n + (a_{n+1}^{n+1} - 2a_n^n) x^{n+1} = 0.$$

We obtain the following system:

$$\begin{cases} a_0^{n+1} + a_0^{n-1} = 0 \\ a_n^{n+1} = 2a_{n-1}^n \\ a_{n+1}^{n+1} = 2a_n^n \\ a_k^{n+1} = 2a_{k-1}^n - a_k^{n-1},\ 1 \le k \le n-1. \end{cases}$$

We have

$$T_0(x) = 1 \Leftrightarrow a_0^0 = 1$$

and

$$T_1(\chi) = \chi \Leftrightarrow a_0^1 = 0, a_1^1 = 1.$$

Hence,

$$T_2(\chi) = a_2^2 \chi^2 + a_1^2 \chi + a_0^2$$

From the above system, we obtain

$$\begin{cases} a_0^2 = -a_0^0 = -1 \\ a_1^2 = 2a_0^1 = 0 \\ a_2^2 = 2a_1^1 - a_2^0 = 2 \end{cases}$$

which means that

$$T_2(\chi) = 2\chi^2 - 1$$

Now, replacing n by 2 in the system we obtain

$$\begin{cases} a_0^3 = -a_0^1 = 0. \\ a_2^3 = 2a_1^2 = 0. \\ a_1^3 = 2a_0^2 - a_1^1 = -3. \\ a_3^3 = 2a_2^2 - a_3^1 = 4. \end{cases}$$

Therefore,

$$T_3(x) = 4x^3 - 3x.$$

Next, for $n = 3$, the system becomes

$$\begin{cases} a_0^4 = -a_0^2 = 1. \\ a_3^4 = 2a_2^3 = 0. \\ a_1^4 = 2a_0^3 - a_1^2 = 0. \\ a_2^4 = 2a_1^3 - a_2^2 = -8. \\ a_4^4 = 2a_3^3 - a_4^2 = 8. \end{cases}$$

Thus,

$$T_4(x) = 8x^4 - 8x^2 + 1.$$

Now, by replacing n with 4 , the system ylelds

$$\begin{cases} a_0^5 = -a_0^3 = 0. \\ a_4^5 = 2a_3^4 = 0. \\ a_1^5 = 2a_0^4 - a_1^3 = 5. \\ a_2^5 = 2a_1^4 - a_2^3 = 0. \\ a_3^5 = 2a_2^4 - a_3^3 = -20. \\ a_5^5 = 2a_4^4 - a_5^3 = 16. \end{cases}$$

Hence,

$$T_5(x) = 16x^5 - 20x^3 + 5x.$$

So, we obtain the same Techebychev polynomials as for the Rodrigues and ODE rules.

From ODEs

We set $I =]-1,1[$, $a(x) = 1 - x^2, b(x) = -x$ and $\omega(x) = (1 - x^2)^{-1/2}$. The linear operator T is then given by

$$T(y) = (1 - x^2)y'' - xy'.$$

It is straightforward that $a(x)\omega(x) = \sqrt{1-x^2}$ vanishes at ± 1 and the eigenvalues $\lambda_n = -n^2$ give rise to elgenvectors (polynomials), T_n s. This ylelds that the T_n s are the corresponding solutions of the ODE

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

Remark 24. Chebyshev polynomials T_n can be explicitly defined on $[-1,1]$ by

$$T_n(x) = \cos(n\mathrm{Arccos}(x)).$$

Indeed, by considering Molvre's rule $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, and by setting for $\theta \in [0,\pi], x = \cos\theta$, we obtain $\sin\theta\sqrt{1-x^2}$. This implies that

$$\cos(n\theta) = \cos(n\mathrm{arccos}(x)) = \sum_{m=0}^{\left[\frac{n}{2}\right]} C_n^{2m}(-1)^m \chi^{n-2m}(1-x^2)^m,\ n \in \mathbb{N}.$$

Next, we observe that

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta).$$

Henceforth, we obtain explicit T_n s as above. Figure 2.4 illustrates the graphs of the first Chebyshev polynomials.

Remark 25. It holds that a second kind of Chebyshev polynomial already exists. It is defined by means of the Rodrigues rule as

$$U_n(x) = \frac{(-1)^n(n+1)\sqrt{\pi}}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)(1-x^2)^{\frac{1}{2}}}\frac{d^n}{dx^n}\left((1-x^2)^{n+\frac{1}{2}}\right),\ x \in [-1,1]$$

or by means of trigonometric functions as

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},\ \forall n \in \mathbb{N}^*.$$

These polynomials satisfy the same induction rule as the previous but with different initial data $U_0(x) = 1$ and $U_1(x) = 2x$. Finally, similar to other classes of orthogonal polynomials, they satisfy the ODE

$$\forall x \in \mathbb{R},\ (1 - x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0.$$

2.2.5 Gegenbauer polynomials

From Rodrigues rule:

Gegenbauer polynomials, also called ultraspherical polynomials, are defined relative to the weight function $\omega(x) = (1 - x^2)^{p-1/2}$, where p is a real parameter, and to the orthogonality interval $I = [-1,1]$. From the Rodrigues rule, these are defined as

$$G_m^p(x) = \frac{(-1)^m \Gamma\left(p + \frac{1}{2}\right)\Gamma(n + 2p)}{2^m m!\, \Gamma(2p)\Gamma\left(p + m + \frac{1}{2}\right)}(1 - \chi^2)^{\frac{1}{2}-p} \frac{d^m}{dx^m}\left((1 - \chi^2)^{p+m-\frac{1}{2}}\right). \quad (2.9)$$

Hence, by applying the Leibniz derivation rule, we obtain

$$G_m^p(x) = C_m^p[x^m - a_{m-2}x^{m-2} + a_{m-4}x^{m-4} + \cdots]$$

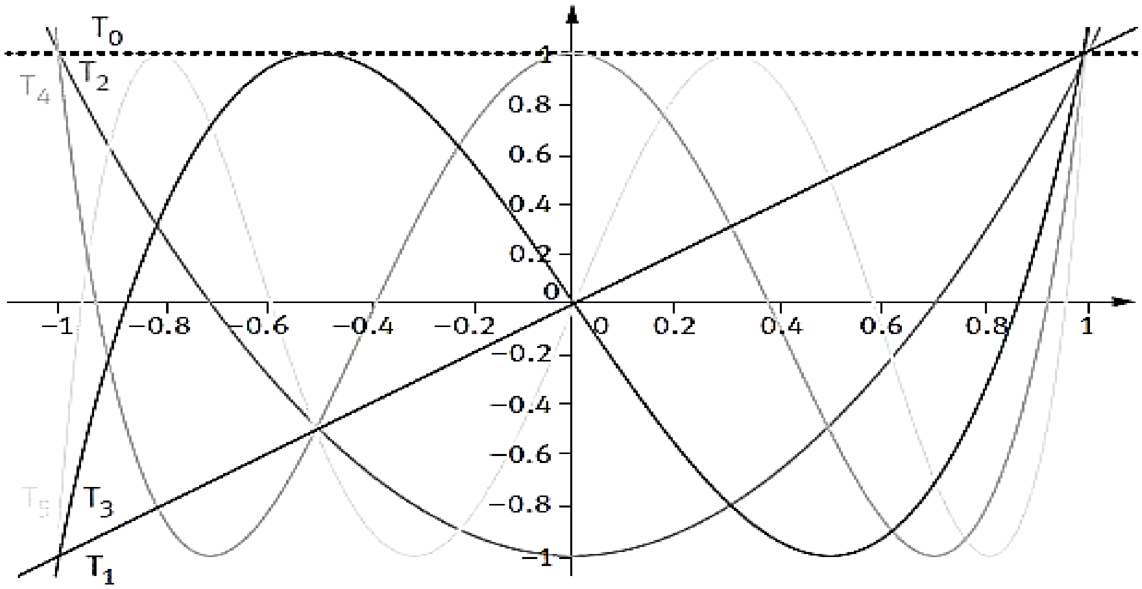

Fig. 2.4: Chebyshev polynomials.

where

$$C_m^p = \frac{2^m \Gamma(p+m)}{m!\,\Gamma(p)},$$

$$a_{m-2} = \frac{m(m-1)}{2^2(p+m-1)},\ a_{m-4} = \frac{m(m-1)(m-2)(m-3)}{2^4(p+m-1)(p+m-2)}, \dots$$

From the induction rule

Gegenbauer polynomials G_m^p can also be introduced vla the induction rule stated for $p \geq \frac{-1}{2}$ by

$$mG_m^p(x) = 2\chi(m+p-1)G_{m-1}^p(x) - (m+2p-2)G_{m-2}^p(x) \qquad (2.4)$$

already with

$$G_0^p(x) = 1 \text{ and } G_1^p(x) = 2p(1-x).$$

This gives, for example,

$$G_2^p(x) = 2p(p+1)\left[\chi^2 - \frac{1}{2p+2}\right]$$

and

$$G_3^p(x) = \frac{4}{3}p(p+1)(p+2)\left[\chi^3 - \frac{3}{2p+4}\chi\right]$$

Furthermore, we notice that G_m^p is composed of monomials having the same parity of the index m.

Remark. The following assertions hold:

$G_m^p(-\chi) = (-1)^m G_m^p(\chi)$. •

$G_{2m+1}^p(0) = 0$. •

$- G_{2m}^p(0) = \dfrac{(-1)^m \Gamma(p+m)}{T(p)I(m+1)}$.

From ODEs

Gegenbauer polynomials G_m^p are solutions of the ODE

$$(1-x^2)y'' - (2p+1)xy' + m(m+2p)y = 0,$$

In the interval $I =]-1,1[$ with coefficlents $a(x) = 1-\chi^2, b(x) = -(2p+1)x$ and $c(\chi) = m(m+2p)$. These polynomials can be introduced as the eigenvectors of the linear Sturm-Llouvllle-type operator T defined by

$$T(y) = (1-\chi^2)y'' - (2p+1)xy'.$$

By choosing $\omega(x) = (1-\chi^2)^{p-\frac{1}{2}}$, we observe that $a(x)\omega(x) = (1-\chi^2)^{p+\frac{1}{2}}$ vanishes at the boundary points ± 1. The eigenvalues are $\lambda_m = -m(m+2p)$.

Conclusion:

In this work, we outlined the concepts and the main properties and characteristics of orthogonal polynomials. Some basic notions concerning orthogonal polynomials are recalled that are related to weight functions, integration theory, linear algebra theory of vector spaces, their basis, and orthogonal systems and their relation to orthogonal polynomials. Next, the three main methods for introducing orthogonal polynomials were reviewed. The first method uses Rodrigues formula and it yields orthogonal polynomials as outputs of a higher order derivatives of some special functions. The second is based on recurrence relations, which yield orthogonal polynomials as sequences of functions defined by a three-level induction rule. We recalled and redeveloped Favard's results on orthogonal polynomials as well as its reciprocals. The last method consists of orthogonal polynomials as solutions to ordinary differential equations or equivalently as eigen functions of Sturm–Liouville operators. Some concluding and illustrating examples are provided to enlighten theoretical developments.

References:

1-Abramowitz, M. and I. S. Stegun, Handbook of Mathematical Functions, Dover Publications, Inc., New York, NY, 1965.

2-Andrews, L. C., Special Functions for Engineers and Applied Mathematicians, MacMillan Publishing Co. New York, NY. 1985.

3-Hochstadt, H., The Functions of Mathematical Physics, Dover Publications Inc., New York, NY, 1986.

4-McLachlan, N. W., Bessel Functions for Engineers, 2nd Edition, Oxford University Press, London, 1961. Sansone, G., Orthogonal Functions, Interscience Publishers, New York, NY., 1959.

5-Nazem A. R., Hesam S. ,haghbin A. ,“A fast numerical method for Solving calculus of variation problems”, Advanced Modeling and Optimization v.15, N.2, (2013) 133-149.