وزارة التعليم العالي والبحث والعلمي
جامعة ديالى
كلية تربية المقداد / قسم الرياضيات

محاضرات مادة

الإحصاء والاحتمالية

للعام الدراسي (2023-2024)

المرحلة الثالثة

Chapter two

مدرس المادة: م.م هند إبراهيم محمد

Probability and Statistics

الاحتماليـــــة والاحصـــاء

Chapter Two : Introduction to Probability

<u>Def :-</u> A random (Statistical) experiment is an experiment with :-

1. All out comes (results) of the experiment are known in advance
2. Any performance of the experiment results in an out comes is not known in advance .
3. The experimant can be repeated under the identical conditions .

ويمكن تعريف التجربة العشوائية بأنها تلك التجربة التي ينتج عنها مجموعة من الاحـــداث كل حدث منها مستقل عن الاخر وان وقوع ذلك الحدث يرجع الى عامل الصدفة وحــــده (لا يمكن التنبؤ بحدوثه)

ملاحظة: ستطرق الى جملة من التجارب كأمثلة تساعدنا في تفسير بعض المفاهيم مثل :

1. Tossing a coin	تجربة رمي العملة
2. Rolling adice	تجربة رمي الزار
3.Playing cards	تجربة ورق اللعب

وهكذا

Def: (Sample Space)

A sample space of an experiment is a set of all possible outcomes denoted by (S)

فضاء العينة : هو كل النتائج المحتملة من تجربة عشوائية معينة.

Def: (Events)

Any events is a (proper) subset of a sample space

الحوادث: هي مجموعه جزئيه من(S) ويكون الحادث بسيطاً اذا تكون من عنصر واحد فقط او مركب اذا تكون من اكثر من عنصر ومستحيلاً اذا لم يحوي على اي عنصر واكيداً اذا احتوى على عناصر(S) جميعا

ie: If A is an Event, then $A \subseteq S$

ex: Toss a coin once

sol. $S=\{H,T\}$

S has (2) elts since acoin has two faces

Let A: to get H

$A=\{H\} \subseteq S$

Let B: to get T

$B=\{T\} \subseteq S$

$\therefore$ A and B are Events

ex: Roll adice once

a dice has (6) faces

each face has adots

$S=\{1,2,3,4,5,6\}$

$= \{d : 1 \le d \le 6\}$

A: To get one odd no.

$A = \{1,3,5\} \subseteq S$

A is an Event

B: To get one even no.

$B = \{2,4,6\} \subseteq S$

$\therefore$ B is an event

C: $d \le 3$

$C = \{1,2,3\} \subseteq S$

$\therefore$ C is an Event

ملاحظة : سيكون فضاء العينة في هذا الفصل من النوع المنتهي والقابل للعد .

Def:- Empty set (Φ) الحادثة التي لا تحدث

Φ is an Impossible event

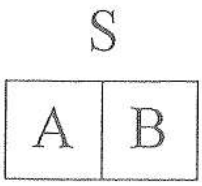
ex Toss a dice once

Let A: to get 7

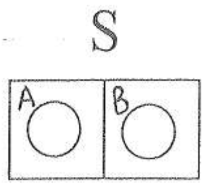
$\therefore A = \Phi$

Def :- **(Disjoint events) الحوادث المنفصلة**

If A and B are events, then A and B are disjoint iff $A \cap B = \Phi$



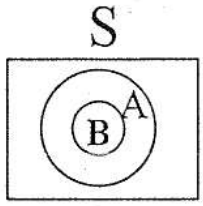
$AB = \Phi$



$AB = \Phi$

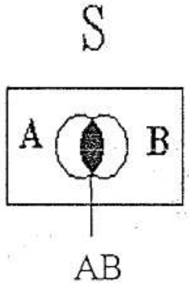
<u>Def.</u> (Joint events) الحوادث المتصلة

If A and B are event , then A and B are Joint iff $A \cap B \neq \Phi$



$B \subset A$

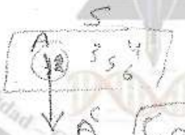
$AB = B$



Note :- We shall use the symbol (AB) to denote of $A \cap B$

Def :- If A and B are events , then $(A \cup B)$ & $(A \cap B)$, $(A - B)$, (B/A) $A^C \cdots etc$ are also events .

$A^c = S \backslash A = S - A$
$= \{ x ; x \notin A \}$



Complement

ex. Roll a dice once .

Let

$$A = \{d : d \geq 2\} = \{2,3,4,5,6\}$$
$$B = \{d : d \leq 3\} = \{1,2,3\}$$
$$C = \{d : d \leq 1\} = \{1\}$$

Find $A \cup B$, $A \cap B$, $A \cup C, \cdots$

A^c , B^c , AB^c , BA^c, ...

Ex.

$AB = \{ x ; x \in A \wedge x \in B \} \subseteq S$
$A \cup B = \{ x ; x \in A \vee x \in B \} \subseteq S$
$A - B = \{ x ; x \in A \wedge x \notin B \} \subseteq S$
$B - A = \{ x ; x \in B \wedge x \notin A \} \subseteq S$

$A \cup B - AB$

$A \cup B = \{1,2,3,4,5,6\}$

$A \cap B = \{2,3\}$

$A^c \cup C = \{1,2,3,4,5,6\}$

$A^c = \{1\}$

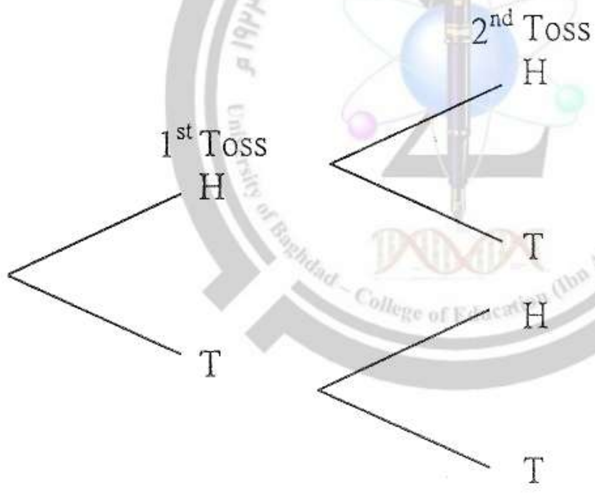
$B^c = \{4,5,6\}$

$AB^c = \{4,5,6\}$

$BA^c = \{1\}$

Toss a coin twice (2- time)

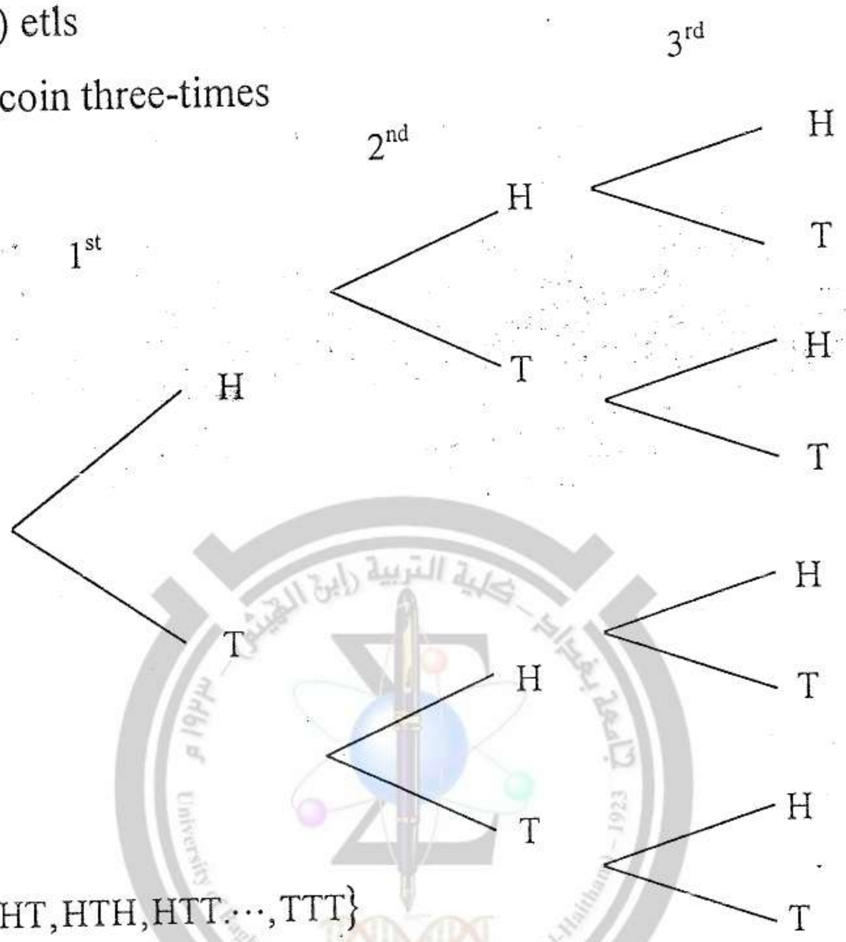
When toss a coin twice (2-time)



$S = \{HH, HT, TH, TT\}$

S has $(2^2 = 4)$ etls

When toss a coin three-times



$S = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \cdots, \text{TTT}\}$

$\therefore$ S has $(2^3{=}8)$ elts

when toss a coin n-times , then the sample space (S) has 2^n elts

ex. Roll a dice twice (2-times)

$S = \{(d_1, d_2); 1 \le d_1 \le 6; 1 \le d_2 \le 6\}$

S= has $(6^2{=}36)$ elts

When roll a dice n-times , then the sample space (S) has (6^n) elts

Simple Probability الاحتمالية البسيطة

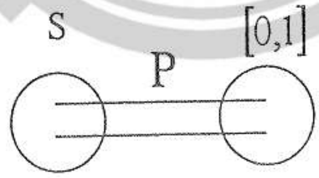
<u>Def:-</u> If A be an events , P(A)= probability of event A = Pr.(A) that is mean (pr) that event A happence . If S has (n) elts & A has (m) elts , then

$$p(A) = \frac{\text{no. of elts of event A}}{\text{no. of elts of S}} = \frac{|A|}{|S|} = \frac{m}{n}$$

<u>ملاحظة :</u> عند احتساب الاحتمال الرياضي لاي حادثة يجب ان تتوفر الحالة التي تكـــون فيها الاحداث مستبعدة لبعضها الاخر وان كل حدث ياخذ الفرصة التي تاخذها الاحـــداث الاخرى في الوقوع .

ويعرف احتمال الحصول على صفة معينة (حادثة معينة) مثل (A) من تجربة عشـوائية معينة بأنه عدد مرات حدوث الصفة (A) مقسوما على الحالات المتوقعة .

ملاحظة : نجد ان قيمة الاحتمال هي عبارة عن دالة (تطبيق) P مجالها (منطلقـــها) S ومداها $R_{[0,1]}$



$P: S \rightarrow R_{[0,1]}$

تسمى دالة الاحتمال P دالة احتمال منتظم اذا اعطى نفس القيمة الاحتمالية لكل عنصـــر من عناصر فضاء العينة .

<u>ملاحظة:</u> نجد ان قيمه الاحتمال تعتمد بالدرجة الاساس على معرفة كل الحالات الممكنـــة لــ(S) وان هذه الحالات يمكن حصرها بسهوله في الحالات البسيطة ولكن عند زيـــادة

عدد الاحداث يؤدي الى وجود صعوبة في تحديد عدد الحوادث الممكنة ولذلك لابـد مـن اللجوء الى بعض الطرق الرياضية التي تساعد في تحديد مثل هذه الحـالات ومـهما زاد عددها واهم هذه الطرق

1. permutation التباديل

وهي عملية ترتيب n من الاشياء في مجاميع كل منها يتالف من r من الاشياء وحسب القاعدة التالية

$$P_r^n = P_{n,r} = \frac{n!}{(n-r)!}$$

$$P_r^n = P_{n,r} = \frac{n!}{(n-r)!}, n, r \in I^+$$

مثال/ جد عدد الطرق الممكنة لترتيب اربع كرات مرقمة من 1-4

$$P_4^4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4 \times 3 \times 2 \times 1 = 24 \text{ or } 4! = 4 \times 3 \times 2 \times 1 = 24$$

مثال / جد عدد الطرق الممكنة التي يمكن وضع خمس كرات في صندوقين بحيث ان كـل صنـدوق يحتوي على كرة واحدة فقط

$$P_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \times 4 \times 3!}{3!} = 20 \text{ samples}$$

في حالة وجود مكررات في مفردات المجموعة فان عدد الطرق الممكنـة للـترتيب يمكـن حسـابها بالطريقة التالية :

$$P_{n1,n2,\cdots nk}^n = \frac{n!}{n_1! n_2!, \cdots n_k!}$$

where $n = n_1 + n_2 + \cdots + n_k$

مثال / جد عدد الطرق الممكنة لترتيب سبع كرات اربعة منها بيضاء واثنان حمــراء والبـاقي الــوان اخرى .

$$P^{7}_{4,2,1} = \frac{7!}{4!2!1!} = 105 \text{ Methods}$$

2. Combination التوافيق

هي عملية اختيار او انتخاب (Selection) عدد من المفردات بحجم r من مجموعة كبـيرة بحجـم n وبدون ترتيب وتستخدم الصيغة التالية :

$$\binom{n}{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}, n,r \in I^{+}$$

مثال / ما هو عدد العينات التي يمكن تكوينها من مجتمع مؤلف من ست مفردات بحيــث يكون حجم العينة مفردتين اثنين فقط .

$$\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2!4!} = 15 \text{ samples.}$$

مثال / ما هو عدد اللجان التي يمكن تأليفها من اربعة افراد بحيث ان كل لجنــة تحتــوي على أ. فردين اثنين ب. ثلاثة افراد

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \left(\frac{4!}{2!2!}\right) = \frac{24}{2} = 6$$

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4$$

<u>ملاحظة :</u> هناك علاقة بين التباديل والتوافيق وهي :

$$\binom{n}{r} = \frac{P^{n}_{r}}{r!},$$

Binomial theorm نظرية ذات الحدين

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^{r}$$

binomial coefficients معادلات ذات الحدين

$$= a^{n} + n\, a^{n-1} b + \frac{n(n-1)}{1-2} a^{n-2} b^{2} + \cdots + b^{n}$$

ex/ Prove that

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Sol. R.S

Facts about the sets حقائق حول المجموعات

$1. A \cap \Phi = \Phi, A \cup \Phi = A, \left(A^{c}\right)^{c} = A$

$$2. \left.\begin{aligned} A^{c} \cup B^{c} &= (A \cap B)^{c} \\ A^{c} \cap B^{c} &= (A \cup B)^{c} \end{aligned}\right\} \textit{Demo.Law}$$

$3. \Phi^{c} = S, S^{c} = \Phi$

$4. A \cup A^{c} = S, A \cap A^{c} = \Phi$

$5. A_1 \cap A_2^{c} = A_1 - A_2$ A_1 happence but A_2 not happence

$6. A_1 \cup A_2$ A_1 or A_2 happence

$7. (A_1 \cup A_2) - A_1 \cap A_2$ A_1 or A_2 happ , but not both

$8. A_1 \cap A_2$ Both A_1 , and A_2 happence

Axioms of Probability :بديهيات الاحتمالية

1. If $A \subseteq S$, then $0 \leq P(A) \leq 1$

2. $P(S) = 1$

3. If $A_1 A_2, \cdots A_n, \cdots$ are sequence of disjoint events , then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n \cdots) = P(A_1) + P(A_2) + \cdots + P(A_n) \cdots$$

ie. $P\left(\bigcup_{i=1}^{\infty} A\right) = \sum_{i=1}^{\infty} P(Ai)$

Note:- Special case of Ax.3

If A and B are disjoint events , then $P(A \cup B) = P(A) + P(B)$

ex. Toss a coin 3-times

a. Find the Pr. to get 2-H

b. Find the Pr. to get no-H

S has 8 elts

Sol. $S = \{HHH, HHT, \cdots, TTT\}$

a. let A to get 2-H

$A = \{HHT, HTH, THH\}$ A has 3 elts

$$P(A) = \frac{3}{8} \in [0,1]$$

b. let B to get no-H

$B = \{TTT\}$ $P(B) = \frac{1}{8} \in [0,1]$

A^c : to get less (2-H) or (3-H) = {S − A ; A = (2-H)}

$$= P(A^c) = 1 - P(A) \Rightarrow P(A^c) = 1 - \frac{3}{8} = \frac{5}{8} \in [0,1]$$

H.W ex/ toss a die twice (a: Find the pr. That sum. of dots is equal to (8) , D

$P(D) = \frac{5}{36} \in [0,1]$

$D = d_1 + d_2 = 8$

$D = \{(4,4), (2,6), (6,2), (5,3), (3,5)\}$

$d_2 \geq 3$

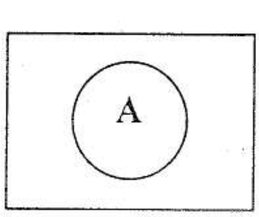
b. To get ~~one (H)~~, P(c) = ? , P (c^c) = ?

c. Find the pr. That $d_2 < 3$) = {(1,2), (2,1), (3,1), (4,1), (5,1), (6,1), (3,2), (4,2), (5,2), (6,2)} ⟹ $P(C) = \frac{10}{36} \in [0,1]$

Theorem 1 :- $P(\Phi) = 0$

Proof :- let A be any event

S



$$A\Phi = \Phi$$

A & Φ are disj.

$$A \cup \Phi = A$$

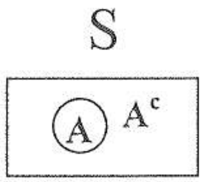
$$P(A \cup \Phi) = P(A)$$

$$P(A) + P(\Phi) = P(A) \text{ by AX.3}$$

$$P(\Phi) = 0$$

Theorem 2 :- $P(A^c) = 1 - P(A)$

Proof :-



$\because AA^c = \Phi$

$\therefore$ A & A^c are disj

$A \cup A^c = S \Rightarrow P(A \cup A^c) = P(S)$

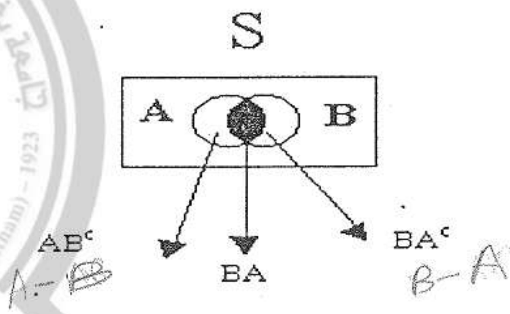
$P(A) + P(A^c) = 1 \qquad by AX,2, AX,3$

$\therefore P(A^c) = 1 - P(A)$

Theorem 3 :- If A and B are joint events , then

$P(A \cup B) = P(A) + P(B) - P(AB)$

Proof :-



AB^c & AB *are* disjoint $\Rightarrow A = AB^c \cup AB$

$P(A) = P(AB^c) + P(AB) \quad by \quad AX.3 \quad \cdots(1)$

BA^c & AB *are* disjoint

$B = BA^c \cup AB$

$P(B) = P(BA^c) + P(AB) \quad by \quad AX.3$

$P(BA^c) = P(B) - P(AB) \cdots\cdots\cdots\cdots\cdots(2)$

$\because AB^c, AB$ & BA^c *are disjo* int

$A \cup B = AB^c \cup AB \cup BA^c$

$P(A \cup B) = P(AB^c) + P(AB) + P(BA^c)$ by Ax-3

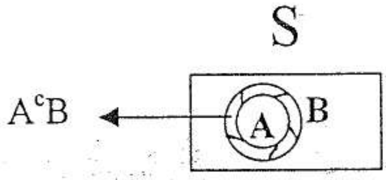
$\therefore P(A \cup B) = [P(A) - \cancel{P(AB)}] + \cancel{P(AB)} + [P(B) - P(AB)]$

$P(A \cup B) = P(A) + P(B) - P(AB)$

Theorem 4 :- If A & B are events such that $A \subseteq B$, then $P(A) \leq P(B)$

Proof :- A and A^cB are disj.

$$B = A \cup A^cB$$
$$P(B) = P(A \cup A^cB)$$
$$P(B) = P(A) + P(A^cB) \quad by \quad AX_3$$
$$P(B) - P(A) = P(A^cB) \geq 0$$
$$P(B) - P(A) \geq 0 \Rightarrow P(A) \leq P(B)$$



H.W For any events A and B , show that

1. $P(AB) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

2. If A and B are joint events , when $P(A) = 0.8; P(B) = 0.5$

Find the conditions and the value of Max $P(AB)$ and Min $P(AB)$

3. If $P(A) = \frac{1}{3}$ & $P(B) = \frac{1}{2}$ Find the value of $P(BA^c)$ when

a. $A \& B$ are disj. events b. $A \subseteq B$ c. $P(AB) = \frac{1}{8}$

4 If A,B and c are dis J. events find

1. $P[(A \cup B) \cap C]$ 2. $P[A^c \cup B^c]$

Theorem 5 :- (H.W.)

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

Theorem 6 :- Convergent of pr. Th. 6 Th. 7

If $A_1, A_2 \cdots, A_n, \ldots$ be asequence of infinite events such that

$A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_n \subset \ldots$

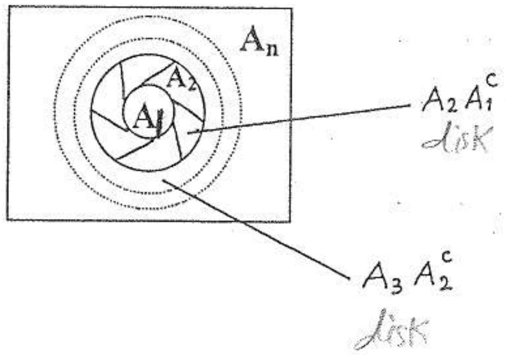
Then $P\left[\bigcup_{i=1}^{\infty} A_i\right] = \lim_{n \to \infty} P(A_n)$ (Conv. from above)

Proof:- $A_1, A_2A_1^c, A_3A_2^c, \ldots A_nA_{n-1}^c$ are disj

$A_2 = A_1 \cup A_2 = A_1 \cup A_2A_1^c$

$A_3 = A_1 \cup A_2 \cup A_3 = A_2 \cup A_3A_2^c$

$A_n = \bigcup_{i=1}^{n} A_i A_{i-1}^c = \bigcup_{i=1}^{n} A_i$



$\therefore P(A_n) = P\left(\bigcup_{i=1}^{n} A_i \quad A_{i-1}^c\right)$

$= \sum_{i=1}^{n} P\left(A_i A_{i-1}^c\right) \quad by AX.3$

$\underset{n\to\infty}{Lim} P(A_n) = \underset{n\to\infty}{Lim} \sum_{i=1}^{n} P\left(A_i \quad A_{i-1}^c\right)$

$= \sum_{i=1}^{\infty} P\left(A_i \quad A_{i-1}^c\right) \ldots (1)$

$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i \quad A_{i-1}^c$

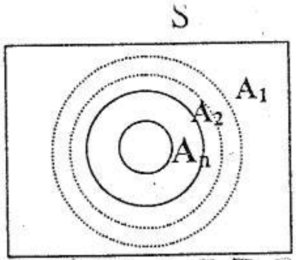
$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i \quad A_{i-1}^c\right)$

$= \sum_{i=1}^{\infty} P\left(A_i \quad A_{i-1}^c\right) \ldots (2)$

$\because (1) = (2) \quad \therefore P\left(\bigcup_{i=1}^{\infty} A_i\right) = \underset{n\to\infty}{Lim} \quad P(A_n)$

(Conv- from below)

Theorem 7 : Let $A_1, A_2, A_3, \ldots, A_n, \ldots$ *be an* infinite sequence of events such that



$$A_1 \supset A_2 \supset A_3 \supset \ldots\ldots \supset A_n \supset \ldots\ldots$$

Then $P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)$

Proof :- $A_1^c \subset A_2^c \subset A_3^c \ldots\ldots \subset A_n^c \subset \ldots\ldots$ $\therefore$ By theor. 6

$A \subset B \Leftrightarrow A^c \supset B^c$

$$P\left(\bigcup_{i=1}^{\infty} A^c i\right) = \lim_{n\to\infty} P(A_n^c)$$

$$P\left[\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right] = \lim_{n\to\infty} [1 - P(A_n)] \quad \textit{by the th2. \& D.Law.}$$

$$1 - P\left[\bigcap_{i=1}^{\infty} A_i\right] = \lim_{n\to\infty} P(A_n)$$

$$P\left[\bigcap_{i=1}^{\infty} A_i\right] = \lim_{n\to\infty} P(A_n)$$

Random Sampling العينات العشوائية

Suppose a population of n-elts .$(a_1, a_2, \ldots\ldots a_n)$ We want to choose a subset of this population has (K) elts($a_1, a_2, \ldots\ldots a_k$)at random $(k \le n)$ These subset is called random samples . there aretow kinds of random sample :-

1. Un Ordered Sample العينات غير المرتبة

Select (k) elts from (n) elts at once (at the same time)

2. Ordered Sample العينات المرتبة

a. one by one without replacement selecte (k) elts. From (n) elts .

one by one without repl .

b. one by one with replacement selecte (k) elts , From (n) elts.

one by one with replac .

Case 1 الحالة الاولى

Choose (k) elts . at the same time from (n) elts .

We use , (combination n , K) to find the number of all samples

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Ex/ Given a set of (4) elts. $\{a,b,c,d\}$

Choose a sample of (2) elts.

a. Find the sample space of all samples .

b. Find the pro. That a sample has elts . (b)

Sol/ a.

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\times3\times2!}{2!\times2!} = \frac{12}{2} = 6 \text{ samples}$$

$$S=\{(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)\}$$

b. Let A be a sample has elts (b)

$$A=\{(a,b),(b,c),(d,d)\}$$

$$\therefore P(A)=\frac{3}{6}=\frac{1}{2}$$

i.e/

طريقة عامة . في السؤال السابق يكون هناك اربعة عناصر اما لو اخذنــا 50 عنصــر

يكون الاستخدام بالشكل التالي :

$(\square , \square)$

$\overset{3}{(a,c,d)}$ b

$P_{3,1} \times b_{1,1}$

$\frac{3!}{2!} \times \frac{1!}{0!}$

$3 \times 1 = 3$

$(\square , \square)$

$(a_2,a_3,\ldots a_{50})$ 49 a_1

$P_{49,1} \times P_{1,1}$

$\frac{49!}{48!} \times \frac{1!}{0!}$

Ex/ Given a set of (3) boys and (4) girls students. Choose a sample of (3) students

a. Find S

b. Find the pr. That a sample has 2 boys

c. Find the pr. That a sample has at least (2) girls .

sol/ a.

$$\binom{7}{3} = \frac{7!}{3!(4!)} = 35$$

$$\therefore S \quad has \quad 35\, samples$$

b. $\left(\frac{3b}{3g}\right) \rightarrow \left(\frac{2b}{1g}\right)$

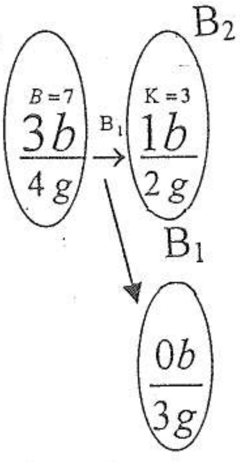
A: be a sample that has (2) boys

$$\binom{3}{2}\binom{4}{1}=\left[\frac{3!}{2!(3-2)!}\right]\times\left[\frac{4!}{1!(4-1)!}\right]=12$$

$3\times 4=12$ [12 *samples has 2boys and girl*]

$$\therefore P(A)=\frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}}=\frac{12}{35}$$

c.



$\binom{3}{1}$ $\binom{4}{2}$ B *be asample has at least (2)girls*

$\binom{3}{0}$ $\binom{4}{3}$ $B=B_1\cup B_2$

$B=B_1\cup B_2$ [B_1 *and* B_2 *are disj.*]

$$P(B)=P(B_1)+P(B_2)$$

$$P(B)=\frac{\binom{3}{1}\binom{4}{2}}{\binom{7}{3}}+\frac{\binom{3}{0}\binom{4}{3}}{\binom{7}{3}}=0.5$$

<u>Case 2 :</u>

a. Choose (k) elts. From (n) . elts. one by one with out replacement .

In this case we use (permutation n, k) to find the number of samples in S.

Where $P_k^n = \dfrac{n!}{(n-K)!}$

Ex/ Given a set of (4) elts. $\{a,b,c,d\}$

Choose a sample of (2) elts. one by one without replacement

a. Finds

b. Find the pr. That a sample has elts. (b) .

Sol/

$$P_2^4 = \frac{4!}{(4-2)!} = 12$$

$$S\ has\quad (12)\quad elts.$$

$$S = \begin{Bmatrix} (a,b), & (b,a), & (c,d), & (d,a) \\ (a,c), & (b,c), & (c,b), & (d,b) \\ (a,d), & (b,d), & (c,d), & (d,c) \end{Bmatrix}$$

$$A = \{(a,b),(b,a),(b,c),(b,d),(c,b),(d,b)\}$$

$$P(A) = \frac{6}{12} = \frac{1}{2}$$

ex/ Given $\{2,3,5,6,8\}$ a set of (5) integers choose a sample of (3) integers one by one without replacement .

a. Find the pr. That the sample con be divided by 5

b. divided by (2) .

Sol/ $P_3^5 = \frac{5!}{2!} = 60$

S has (60) samples

a. let A be a sample which divided by (5)

$\boxed{2}$ $\boxed{}$ $\boxed{}$

2	3	5
3	6	
6	8	
8		

$4 \times 3 \times 1 = 12$

$\therefore$ A has (12) samples

$P(A) = \frac{12}{60}$

b. let B be a sample which divided by (2)

B has (36) samples

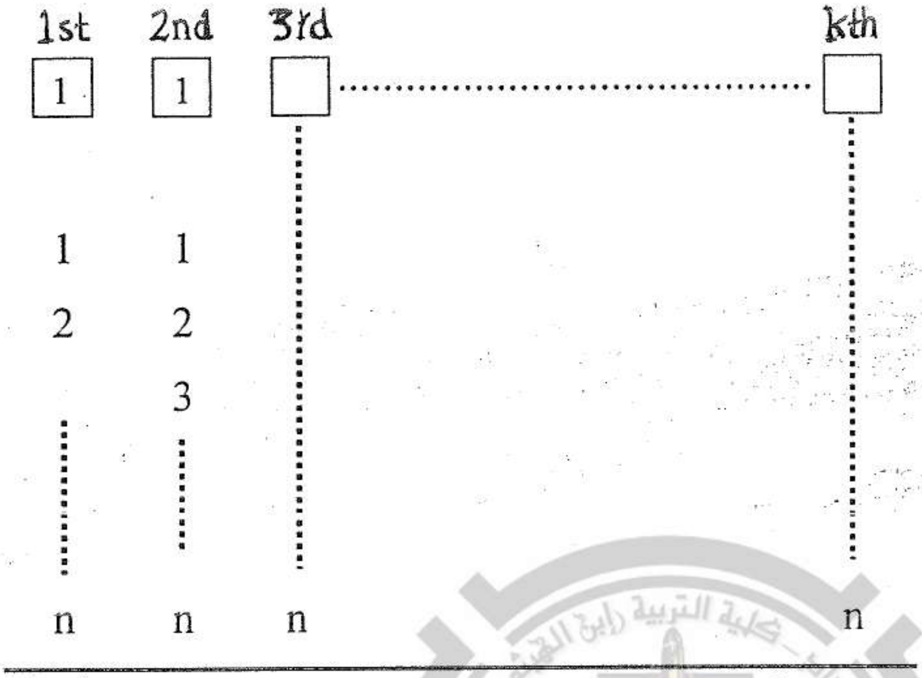
$P(B) = \frac{36}{60} = \in [0,1]$

$\boxed{}$ $\boxed{}$ $\boxed{}$

3	5	2
5	6	6
6	8	8
8		

$4 \times 3 \times 3 = 36$

b. Choose (k) elts. From (n) elts , one by one with replacement.



$$n \times n \times n \ldots \ldots \ldots \times n = n^k$$

$\therefore S \quad has\left(n^k\right) \quad samples$

ex/ Given 4 elts $\{a,b,c,d\}$ choose a sample of 2 elts. One by one with replacement

a. Find S

b. Find the pr. That a sample has elt (b)

Sol/

a. $4^2=16$ S has (16) samples

$$S=\begin{Bmatrix} (a,a), & (b,a), & (c,a), & (d,a) \\ (a,b), & (b,b), & (c,b), & (d,b) \\ (a,c), & (b,c), & (c,c), & (d,c) \\ (a,d), & (b,d), & (c,d), & (d,d) \end{Bmatrix}$$

b. Let A be a sample has (b).

$$A = \{(a,b),(b,a),(b,b),(b,c),(b,d),(c,b),(d,b)\}$$

$$P(A) = \frac{7}{16}$$

Exercises :-

1. A box has (24) bulbs of which (4) are defective .Choose 4 bulbs , find the pr. That they are defective .

2. A set of (11) integers ; (5) of them are negative and the others are positive . Choose a sample of (4) integers and multiply them , then find the pr. That the product is .

a. negative b. positive

3. Given a set of (12) transistors of which (3) are defective،choose a sample of (4) transistors then find the pr. that .

a. Two transistor are defective .

b. at lest one transistors is defective .

4. Find the pr. That two people of (K) people will have the same birthday .

Probability Space

Def :- (δ-Field)

A non-empty collection $\wp$ subsets of a set (S) is called σ -field of subsets of (S) provided the following two properties holds .

1. If $A \in \wp$, *then* $A^c \in \wp$
2. If $A_n \in \wp, \quad n = 1,2,\ldots$

   then $\bigcap_{n-1}^{\infty} A_n \wp$ & $\bigcup_{n-1}^{\infty} A_n \in \wp$

Def:- A probability measure (p) on a (δ)-field of subsets $(\wp)$ is a real valued function having a domain $(\wp)$ and satisfying the following properties

1. $P(S) = 1$
2. $P(A) \geq 0, \forall A \in \wp$
3. If $A_1, A_2, \ldots\ldots A_n$, are disjoint in $\wp$ then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Def:- (Probability Space)

The triple $(S, \wp, P)$ is called a probability space .

Remarks :- the elements of S are called sample points .

Any $A \in \wp$ is know as event clearly A is a collection of sample points.

ex/ Toss a coin once

$$S = \{H, T\}$$

$$\wp = \{\{H\}, \{T\}, S, \Phi\} 2^n$$

ex/ Toss a coin twice

$$S = \begin{Bmatrix} HH, & HT, & TH, & TT \\ a & b & c & d \end{Bmatrix}$$

$$\wp = \begin{Bmatrix} \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\} \\ (b,d), \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\} \\ \{b,c,d\}, S, \Phi \end{Bmatrix}$$

$$\begin{matrix}4\\2\end{matrix} = 16$$

Independent Events : الحوادث المستقلة

Def :- If A and B are events . We say A and B are independent events iff

$$P(A) \times P(B) = P(AB)$$

At the same time A and B are dependent events iff

$$P(A) \times P(B) \neq P(AB)$$

Ex/ Choose (2) integers From $\{1,2,3,4\}$ one by one without (with) replacement.

Joint $\Rightarrow$ Dep.
Ex.
العكس غير صحيح
Joint $\not\Rightarrow$ indep.
while The indep. $\Rightarrow$ Joint

If A: 1^{st} chosen int. is (2)

B: 2^{nd} chosen int. is (1)

Are A and B ind. Events ? why

Sol/

1. Without repl.

$$P^{4}_{2} = \frac{4!}{(4-2)!} = 12$$

S has (12) elts. $= \{(1,2),(2,1),(3,1),(4,1),(1,3),(2,3),(3,2),(4,2),(1,4),(2,4),(3,4),(4,3)\}$

$$A = \{(2,1),(2,3),(2,4)\} \Rightarrow P(A) = \frac{3}{12} = \frac{1}{4}$$

$$B = \{(2,1),(3,1),(4,1)\} \Rightarrow P(B) = \frac{3}{12} = \frac{1}{4}$$

$$AB = \{(2,1)\} \Rightarrow P(AB) = \frac{1}{12}$$

$$P(A) \times P(B) = \frac{1}{16} \neq \frac{1}{12} = P(AB)$$

A and B are dependent

2. With replace.

$$(n^{K}) = 4^{2} = 16$$

$\therefore$ S has (16) samples

$A=\{(2,1),(2,2),(2,3),(2,4)\} \Rightarrow P(A)=\frac{4}{16}=\frac{1}{4}$

$B=\{(1,1),(2,1),(3,1),(4,1)\} \Rightarrow P(B)=\frac{4}{16}=\frac{1}{4}$

$AB=\{(2,1)\} \Rightarrow P(AB)=\frac{1}{16}$

$P(A)\times P(B)=\frac{1}{16}=P(AB)$

$\therefore$ A and B are indep.

Theorem 8 : If A and B are independent event such that $A \neq \Phi, B \neq \Phi$ then A and B are Joint events .

Proof :- $\because$ A *and* B ind $\Rightarrow P(A).P(B)=P(AB)$

T.P/ A and B are Joint

ie/ T.P/ $AB \neq \Phi$

$\because A \neq \Phi \Rightarrow P(A) \neq 0$

$B \neq \Phi \Rightarrow P(B) \neq 0$

$P(A)\times P(B) \neq 0$

$P(AB) \neq 0$ By hyp

$\therefore AB \neq \Phi$

ملاحظة : العكس من النظرية اعلاه غير صحيح بمعنـــى اذا كـانت A و B حـادثتين متصلتيـن (Joint) فانه ليس من الضروري ان تكون الحادثتين A و B مستقلتين (independent) مثـــال على ذلك المثال السابق في حالة بدون ارجاع (Without repel).

Theorem 9 : If A and B are disjoint events , such that $A \neq \Phi, B \neq \Phi$ then A and B are dependent .

Proof :- $\therefore$ A and B are disjoint

$\therefore AB = \Phi \Rightarrow P(AB) = 0 \ldots\ldots(1)$

$\because A \neq \Phi \Rightarrow P(A) \neq 0$

$B \neq \Phi \Rightarrow P(B) \neq 0$

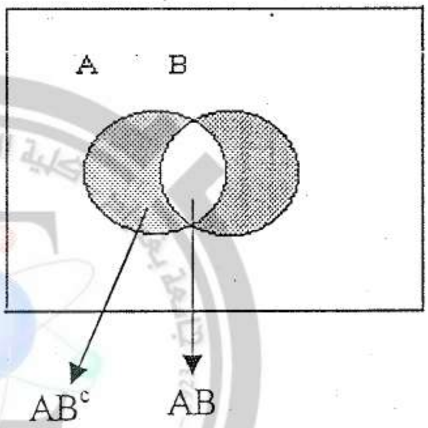
$P(A) \times P(B) \neq 0 \ldots\ldots\ldots\ldots(2)$

$P(A)P(B) \neq P(AB)$

A , B are dependent

Theorem 10 : If A and B are independent events , then .

1. A and B^c are independent .
2. B and A^c are independent .
3. A^c and B^c are independent .



Proof (1) :- Tp. $A \& B^c$ are ind.

ie/ Tp. $P(A)P(B^c) = P(AB^c)$

$A = AB^c \cup AB$

$P(A) = P(AB^c) + P(AB)$ by $AX.3$

$P(AB^c) = P(A) - P(AB)$

$= P(A) - P(A).P(B)$ by hyp.

$= P(A)[1 - P(B)]$

$= P(A)P(B^c)$

Independence of Three Events :

Def :- If A , B and C are events , then A,B and C are independent events iff

1. a. $P(A).P(B) = P(AB)$ (A, B are ind .)

   b. $P(A).P(C) = P(AC)$ (A, C are ind .)

c. $P(B).P(C)=P(BC)$ (B,C are ind.)

2. $P(A).P(B).P(C)=P(ABC)$

Note :- If satisfy only condition (I) , then A,B and C are said to be pairwise independent .

ex/ Given $S=\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$

A: 1^{st} coordinate is (1)

B: 2^{nd} coordinate is (1)

C= 3^{rd} coordinate is (1)

Are A,B and C indep ? why ?

Conditional Probability الاحتمال الشرطي

Def :- Let A,B are events if event A happens first , then event B happens .

Or event A given then event B happens denoted by $(B|A)$.

Were (B\A) is called condition events .

Also P(B\A) is called conditional probability .

Where $P(B\setminus A)=\dfrac{P(AB)}{P(A)}, P(A)\neq 0$

Note :- 1. If A and B are indep. Events

$$P(B\setminus A)=\frac{P(A)P(B)}{P(A)}=P(B)$$

2. From def. of cond. pr.

$P(AB)=P(A).P(B\setminus A)$ multiplication rule

ex/ Toss a dice twice $P(A\cup B)=P(A)+P(B)-P(AB)$

Find the pr. That $d_1+d_2\leq 6$

Given that $d_1 + d_2 = \text{odd}$

لاتعريفين $P(AB)$

A,B indep.
$P(AB) = P(A) \cdot P(B)$

A,B dep.
$P(AB) = P(A) \cdot P(B|A)$

Sol/ $P(B \setminus A) = \dfrac{P(AB)}{P(A)}$

$$A = \begin{Bmatrix} (1,2), (2,1), (3,2), (4,1), (5,2), (6,1) \\ (1,4)\ (2,3)\ (3,4)\ (4,3)\ (5,4)\ (6,3) \\ (1,6)\ (2,5)\ (3,6)\ (4,5)\ (5,6)\ \ 6,5 \end{Bmatrix}$$

$$B = \begin{Bmatrix} (1,1), (2,1), (3,1), (4,1), (5,1) \\ (1,2)\ (2,2)\ (3,2)\ (4,2) \\ (1,3)\ (2,3)\ (3,3) \\ (1,4)\ (2,4) \\ (1,5) \end{Bmatrix}$$

$$P = (A) = \frac{18}{36}$$

$$P(B) = \frac{15}{36}$$

$$AB = \begin{Bmatrix} (1,2), (2,1), (3,2) \\ (1,4), (2,3), (2,4) \end{Bmatrix}$$

$$\therefore P(AB) = \frac{6}{36}$$

$$P(B \setminus A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{6}{36}}{\frac{18}{36}} = \frac{6}{18} = \frac{1}{3} \in [0,1]$$

$$P(A \setminus B) = \frac{P(AB)}{P(B)}$$

$$= \frac{6}{15}$$

Theorem 11 :- let $A_1, A_2, \ldots, A_n$ *be* an events where

$A_i \neq \Phi \qquad i = 1,2,\ldots n$ then

$$P(A_1 A_2 \ldots A_n) = P(A_1).P(A_2 \setminus A_1).P(A_3 \setminus A_1A_2)P(A_4 \setminus A_1A_2A_3) \ldots P(A_n \setminus A_1A_2A_3\ldots A_{n-1})$$

Proof :- right side

$$= P(A_1)\frac{P(A_1A_2)}{P(A_1)}.\frac{P(A_1A_2A_3)}{P(A_1A_2)}.\frac{P(A_1A_2A_3A_4)}{P(A_1A_2A_3)}\cdots\cdots\frac{P(A_1A_2A_3\cdots A_n)}{P(A_1A_2A_3\cdots A_{n-1})}$$

$$= P(A_1A_2A_3\ldots A_n)$$

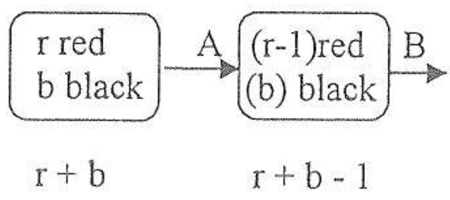
= left side

ex/ A box has (r) red balls and (b) black balls . Choose (2) balls one by one without replac.

1. If the first chosen ball is red find the pr. that both balls has different colour .
2. Find the pr. that the 1^{st} &2^{nd} chosen ball are red .
3. Find the pr. That at most one , ball is red

Sol/ Let A: be the 1^{st} red ball

B: be 1^{st} chosen black ball



$$P(B \setminus A) = \frac{b}{r+b-1}, \quad P(A) = \frac{r}{r+b}$$

2. C: be 2^{nd} chosen ball is red

$$P(AC) = P(A).P(C \setminus A)$$

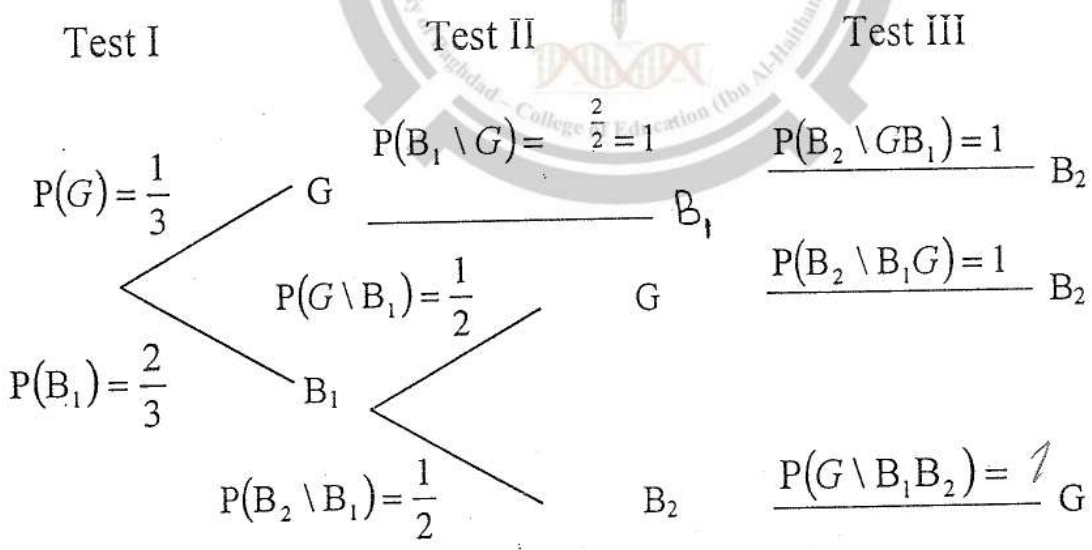
$$= \frac{r}{r+b}.\frac{r-1}{r+b-1}$$

3.

$$P(A \cup B) - P(AB)$$
$$= 1 - P(A)P(B \setminus A)$$
$$= 1 - \frac{r}{r+b} \cdot \frac{b}{r+b-1}$$

ex. 2/ Given (2) bad tubes and (1) good tube . Take the tube one by one until both bad tubes are founds . find the pr. that 2^{nd} bad tube is found on

1. test 1
2. test 2
3. test 3



1. $P(B_2 \text{ on test I}) = P(\Phi) = 0$

2. $P(B_2 \text{ on test II}) = P(B_1B_2)$

$$= P(B_1).P(B_2 \setminus B_1)$$
$$= \frac{2}{3}.\frac{1}{2} = \frac{1}{3}$$

3. $P(B_2 \quad on \quad test \quad III) = P(GB_1B_2 \cup B_1GB_2)$

$$= P(GB_1B_2) + P(B_1GB_2)$$

$$= P(G)P(B_1 \setminus G)P(B_2 \setminus GB_1) + P(B_1)P(G \setminus B_1)P(B_2 \setminus B_1G)$$
$$= \frac{1}{3}.1:1 + \frac{2}{3}.\frac{1}{2}.1$$
$$= \frac{1}{3} + \frac{1}{3}$$
$$= \frac{2}{3}$$

مثلا في تجربة الزهر

roll a dice one

$A_1 = \{1,2\}, A_2 = \{3,4\}, A_3 = \{5,6\}$

S

1, 2	3, 4	5, 6

$A_1 \cup A_2 \cup A_3 = S$ & $\cap A_i = \emptyset$ $i = 1,2,3$

ex/ A box has (4) red and (3) black balls choose a sample of (4) balls one by one without repl . Find the pr. to get a sample r,b,r,b

Sol/

r	4
b	3

$\rightarrow$

1^{st}	2^{nd}	3^{rd}	4^{th}
r ,	b ,	r ,	b

$N = 4+3=7$

Let R_1 : to Choose 1^{st} red ball

$B_1 \backslash R_1$: to choose 1^{st} black ball , given 1^{st} red ball

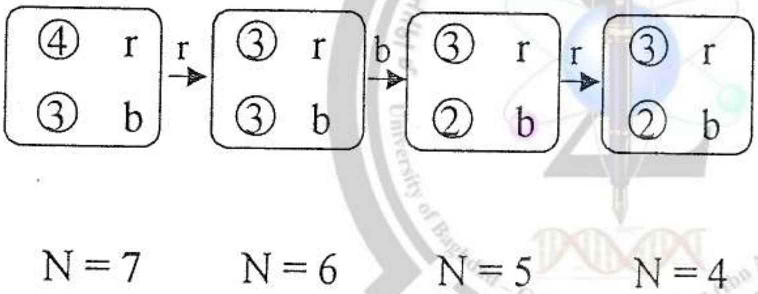
$R_2 \backslash R_1 B_1$: to choose 2^{nd} red ball , given 1^{st} red , 1^{st} black balls

$B_2 \backslash R_1 B_1 R_2$: to choose 2^{nd} black ball , given 1^{st} red , 1^{st} black and 2^{nd} red balls

$$P(R_1 B_1 R_2 B_2) = P(R_1).P(R_1 \backslash B_1)P(R_2 \backslash R_1 B_1).P(B_2 \backslash R_1 B_1 R_2)$$

$$= \frac{4}{7}.\frac{3}{6}.\frac{3}{5}.\frac{2}{4}.$$

$$= \frac{3}{35}$$



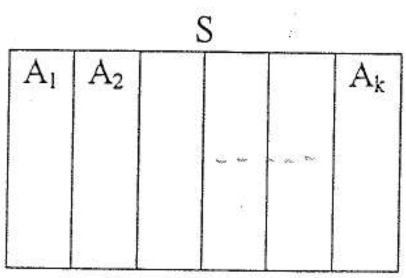
Partition Of a Sample Space (Ex. Rolling a die once)

حوادث متنافية

Def :- A Finite sequence of event $A_1, A_2, \cdots A_K$ Form partition of (S) iff

1. $A_1, A_2, \ldots, A_K$ are disjoint

i.e/ $\bigcap_{i=1}^{K} Ai = \Phi$

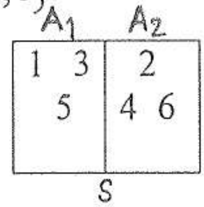


2. $\bigcup_{i=1}^{K} Ai = S$

مثل: مراحل الكلية (١، ٢، ٣، ٤) هي حوادث متنافية لانه طالب الصف ١ لا يداوم في الصف ٢ في نفس الوقت

$\emptyset = \cap 1,2$

ex/ Toss a dice once $S = \{1,2,3,4,5,6\}$



$A_1 = \{1,3,5\}$

$A_2 = \{2,4,6\}$

$$\left.\begin{array}{l} A_1A_2 = \Phi \\ A_1 \cup A_2 = S \end{array}\right\} \Rightarrow A_1, A_2 \; form \quad apartition \quad of \quad S$$

Theorem 12:- If $Ai \; (i = 1,2,\ldots,K), Ai \neq \Phi$ from a partition of S. If

$\Phi \neq B \subset S, \quad then \quad P(B) = \sum_{i=1}^{K} P(Ai)P(B \setminus Ai)$

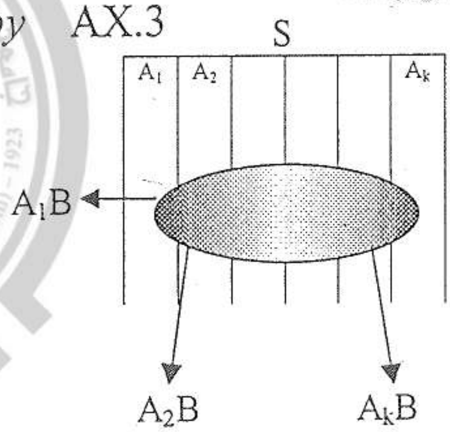
Proof :- $A_1B, A_2B, A_3B, \ldots A_KB$ are disjoint

$B = A_1B \cup A_2B \cup A_3B \cup \ldots\ldots \cup A_KB$

$P(B) = P(A_1B) + P(A_2B) + \ldots\ldots + P(A_KB)$ by AX.3

$= \sum_{i=1}^{K} P(A_iB)$

$= \sum_{i=1}^{K} P(Ai)P(B \setminus Ai)$ by theorem 11



B: choose a student from S

Theorem 13 :- (Bayes Theorem)

If $Aj(j = 1,2,\ldots K)$ From a partitions of S , where $Aj \neq \Phi$, and if

$\Phi \neq B \subset S$ then

$$P(Ai \setminus B) = \frac{P(Ai)P(B \setminus Ai)}{\sum_{j=1}^{K} P(Aj)P(B \setminus Aj)}$$

prior information

Ex:

Bayes

$P(B) = \sum_{i=1}^{4} P(Ai)\, P(B \setminus Ai)$

في EX السابق (قسم الرياضيات)
ما احتمالية ان الطالب من الذكور علماً شرط ان يكون من A4
فانها نظرية Bayes .
اي عندما يعطي شرط نستخدم نظرية Bayes .

where (i) is a one value of (j)

Proof :- by theorem 12

$$P(B)=\sum_{i=1}^{K} P(Ai)P(B \setminus Ai)$$

$$P(Ai \setminus B)=\frac{P(AiB)}{P(B)} \quad \textit{by def of cond.pr.}$$

$$=\frac{P(Ai)P(B \setminus Ai)}{\sum_{j=1}^{K} P(Ai)P(B \setminus Aj)} \quad \textit{by theorem II and theorem } 12$$

Note :

1. $P(Ai \setminus B)$ is called the posterior pr. (it is the pr. Of an event which is the source when a result is given) .

2. $P(Ai)$ is called prior pr.

ex/ Given the following boxes :-

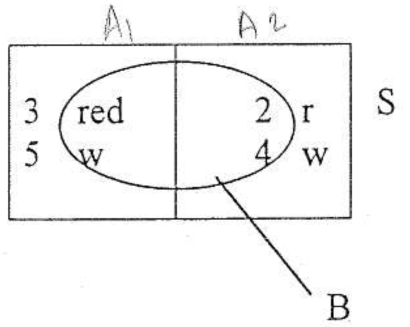
box 1 has (3) red and (5) white balls

box 2 has (2) red and (4) white balls

choose a box , then choose a ball from the chosen box

a. Find the pr. that a white ball is chosen .

b. If a red ball is chosen , Find the pr. that it is from box 2 .

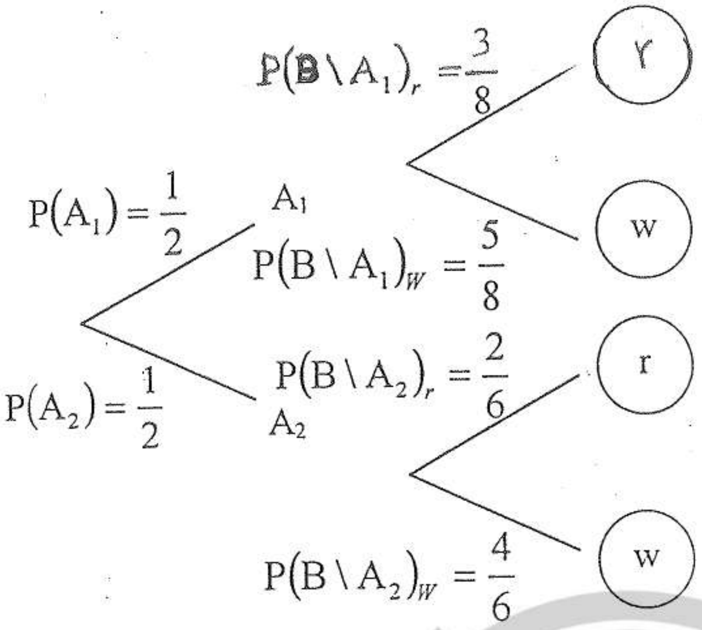


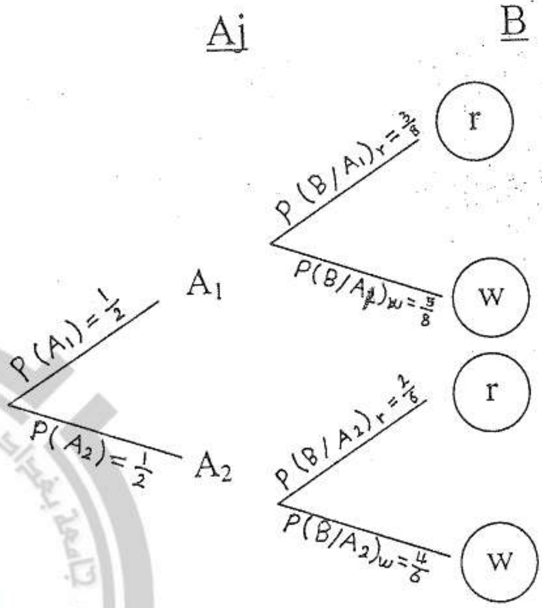
Sol/

Let A_1 : choose box 1

A_2 : choose box 2

B : choose a ball





a. $$P(B)_W = \sum_{j=1}^{2} P(Aj).P(B \backslash Aj)_W$$

$$= P(A1)P(B \backslash A\ )_W + P(A2)P(B \backslash A_2)_W$$

$$= \left(\frac{1}{2}\right)\left(\frac{5}{8}\right) + \left(\frac{1}{2}\right)\left(\frac{4}{6}\right) = ?$$

b. $$P(A_2 \backslash B)_r = \frac{P(A_2)P(B \backslash A_2)_r}{\sum_{j=1}^{2} P(Aj)P(B \backslash Aj)_r}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{2}{6}\right)}{\left(\frac{1}{2}\right)\left(\frac{3}{8}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{6}\right)}$$

<u>H.W</u>. Find p (B) $_r$, $P(A \backslash B)_W$

ex/ Three Machines M_1 , M_2 and M_3 produce glasses

M_1 Produce 20% of glasses

M_2 Produce 30% of glasses

M_3 Produce 50% of glasses

Also

1% of glass produced by M_1 is defective

2% of glass produced by M_2 is defective

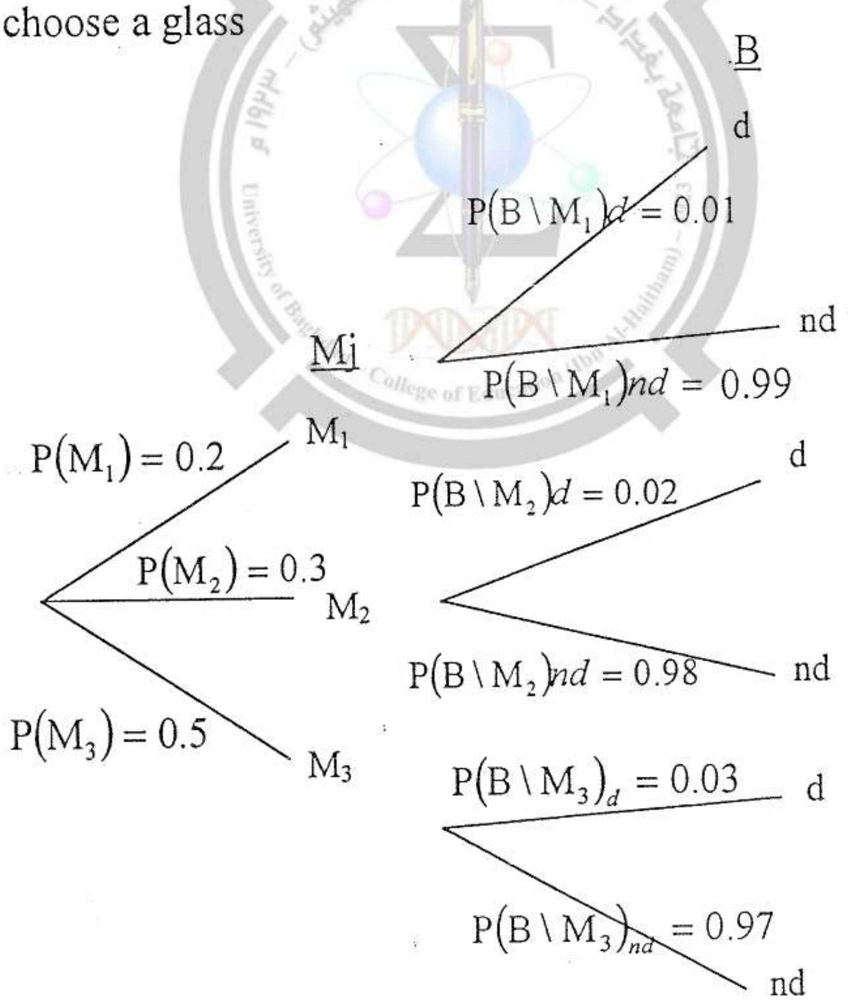
3% of glass produced by M_3 is defective

Choose a glass ,then

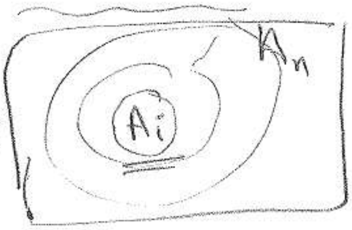
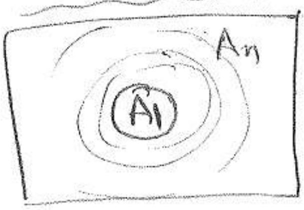
a. Find the pr. That the glass is produced by M_3 if it is defe.

b. Find the pr. That the glass is def.

Let B choose a glass



اخطاء مطبعية في ملزمة ch.2 / مادة الاحتمالية
المرحلة الثالثة / قسم الرياضيات

رقم الصفحة	الخطأ	التصحيح
١٥		
١٧	$A=\{(a,b),(b,c),(d,d)\}$	$A=\{(a,b),(b,c),(b,d)\}$
١٨	b. $\frac{3b}{3g} \longrightarrow \frac{2b}{1g}$	b. $\frac{3b}{4g} \longrightarrow \frac{2b}{1g}$
٢٤	(Defi- σ-Find)	(Defi- σ-field)
٣٠	$P(B,A)=\frac{P(AB)}{P(B)}$	$P(B\|A)=\frac{P(AB)}{P(A)}$
٣٤	$=P(R_1)P(R_1\|R_1)P(R_2\|R_1B_1)$	$=P(R_1)P(B_1\|R_1)P(R_2\|R_1B_1)$
٣٥	Th.13 $P(A_i)=\frac{P(A_i)P(B\|A_i)}{\sum_{j=1}^{k}P(A_j)P(B\|A_j)}$	$P(A_i\|B)=\frac{P(A_i)P(B\|A_i)}{\sum_{i=1}^{k}P(A_j)P(B\|A_j)}$
٣٦	$P(B)=\sum_{j=1}^{k}P(A_i)P(B\|A_i)$ $P(A_i\|B)= \_\_\_ = \frac{P(A_i)P(B\|A_i)}{\sum_{j=1}^{k}P(A_j)P(B\|A_j)}$	$P(B)=\sum_{j=1}^{k}P(A_j)P(B\|A_j)$ $= \_\_\_ = \frac{P(A_i)P(B\|A_i)}{\sum_{j=1}^{k}P(A_j)P(B\|A_j)}$
٤٠	2 Coin 3 Coin 4 Coin	(2) Coins (3) " (4) "

(Some Questions About Chapter two)

سلّة

Q1: Urn has (8) cards which have the number (1, 2, ..., 8); Choose one card then find the pr. that chosen card have a number which divided by 3 or 4.

Sol.
$A = \{3, 6\} \Rightarrow P(A) = \frac{2}{8} = \frac{1}{4}$

$B = \{4, 8\} \Rightarrow P(B) = \frac{2}{8} = \frac{1}{4}$

Since $AB = \emptyset \Rightarrow A, B$ are disjoint events.

$\therefore P(A \underset{or}{\cup} B) = P(A) + P(B)$ (By Ax.3)

$= \frac{1}{4} + \frac{1}{4}$

$= \frac{1}{2} \in [0,1]$

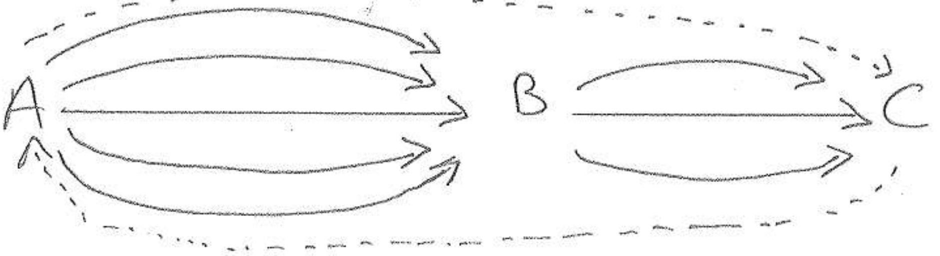
Are A and B independent events?

$\left.\begin{array}{l} P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{2} \\ AB = \emptyset \Rightarrow P(AB) = 0 \end{array}\right\}$ $P(AB) \neq P(A) \cdot P(B)$ $\therefore A, B$ are independent.

OR By Th. (since A, B are disjoint then A, B are depen.

Q2 If there are five roads from A to B and there are three roads from B to C, then how many ways can one make roads from A to C crossing B and return from C to A.

Sol.



$A \longrightarrow C \quad \times \quad A \longleftarrow C$

$15 \qquad\qquad 15$

$P_1^5 \cdot P_1^3 \qquad\qquad P_1^3 \cdot P_1^5$

$5 \times 3 \quad \times \quad 3 \times 5$

$15 \quad \times \quad 15 = (\underline{225}$ ways from A ...

Q3: How many arrangement can be made of the letters of words (MISSISSIPPI) taken all together?

Sol.

الحرف	التكرار
M	1
I	4
S	4
P	2
	$\Sigma = 11$

$$P^{11}_{1,4,4,2} = \frac{11!}{1!\,4!\,4!\,2!} = 34650$$

Q4: Choose a Card from playing Cards, find:

(1) The pr. that a card will be diamond (♦) (A).
(2) The pr. that a card will be a picture (B).
(3) The pr. of getting a Jach (C).
(4) The pr. of getting a queen (D).

Sol.

(1) $P(A) = \dfrac{C^{13}_{1}}{C^{52}_{1}} = \dfrac{13}{52}$

(2) $P(B) = \dfrac{C^{12}_{1}}{C^{52}_{1}} = \dfrac{12}{52}$

(3) $P(C) = \dfrac{C^{4}_{1}}{C^{52}_{1}} = \dfrac{4}{52}$

(4) $P(D) = \dfrac{C^{4}_{1}}{C^{52}_{1}} = \dfrac{4}{52}$

Q5: Three Cards are drawn at random from deck of (52) Cards, let the events:

A: 2 Cards of diamond

B: one number Card; one Jack and the queen of harts.

Sol. $P(A) = C^{13}_{2} \cdot C^{39}_{1} / C^{52}_{3}$

→ follows

Q9:
في مدينة ما ، 40% من المواطنين لهم شعر بني اللون ، 25% منهم لهم عيون بنية اللون و 15% لهم شعر بني وعيون بنية اللون ، اختير مواطن بطريقة عشوائية من المدينة .

(i) اذا كان شعره بني فماهو احتمال ان تكون ايضاً عيناه بنيتان؟

(ii) اذا كانت عينه بنية ، فما هو احتمال ان يكون شعره بنياً ؟

(iii) ماهو احتمال ان لايكون شعره بنياً ولا ان تكون عينه بنية ؟

Sol.
$P(A) = .40$, $P(B) = .25$ و $P(AB) = .15$

(i) $P(B|A) = \dfrac{P(AB)}{P(A)} = \dfrac{.15}{.40} = \dfrac{3}{8} \in [0,1]$

(ii) $P(B|A) = \dfrac{P(AB)}{P(B)} = \dfrac{.15}{.25} = \dfrac{3}{5} \in [0,1]$

(iii) $P(A^cB^c) = P((A \cup B)^c) = 1 - P(A \cup B)$ (By Demo. Law & Th. 2)

$P(A \cup B) = P(A) + P(B) - P(AB)$ (A, B are joint).

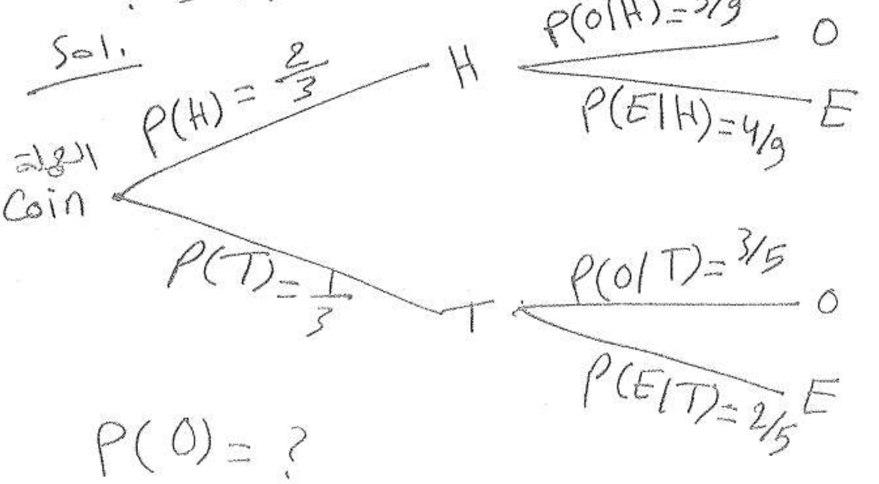
$= .40 + .25 - .15$

$= .50 \Rightarrow 50\%$

$P(A^cB^c) = 1 - P(A \cup B) = 1 - .50 = .50$

Q10: صنعت قطعة نقود بحيث يكون احتمال ظهور الصورة H (2/3) والكتابة T (1/3) . ألقيت هذه القطعة مرة واحدة . نختار بعد ذلك عدداً عشوائياً من 1 الى 9 اذا ظهرت الصورة . اما اذا ظهرت الكتابة نختار بطريقة عشوائية عدداً من 1 الى 5 . ماهو احتمال ان يكون العدد المختار زوجياً ؟

Sol.



$O = \{1,3,5,7,9\}$
$E = \{2,4,6,8\}$

$O = \{1,3,5\}$
$E = \{2,4\}$

$P(O) = ?$

$P(E) = ?$

$$P(B) = \frac{C_1^{40} \cdot C_1^{4} \cdot C_1^{1} \cdot C_0^{7}}{C_3^{52}}$$

Q6: One card are drawn at random from deck of (52) cards let the events:

A: to get 10.
B: to get diamoned (♢).
C: to get no. card.

$52 = 13 \times 4$ = العدد الكلي
$12 = 4 \times 3$ = عدد الصور
$40 = 4 \times 10$ = عدد الارقام

sol.
$$P(A) = P(10) = \frac{C_1^{4}}{C_1^{52}}$$

$$P(B) = P(\diamond) = \frac{C_1^{13}}{C_1^{52}}$$

$$P(C) = P(no.) = \frac{C_1^{40}}{C_1^{52}}$$

Q7: Find pr. that:

1. $P(\text{red card}) = \frac{26}{52}$
2. $P(2) = \frac{4}{52}$
3. $P(K) = \frac{4}{12}$
4. $P(\underbrace{\text{red}}_{26}, 2) = \frac{2}{26}$
5. $P(\text{pic.}) = \frac{12}{52}$
6. $P(\underbrace{\heartsuit}_{13}, 7) = \frac{1}{13}$

Q8 اخذ رجل (5) ورقات من اوراق اللعب واحدة بعد الاخرى. ماهو احتمال ان تكون جميع الاوراق من نوع (♡) ؟

$$P(\heartsuit\heartsuit\heartsuit\heartsuit\heartsuit) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48}$$
(By Th.11)

$$P(O)=\frac{2}{3}\cdot\frac{5}{9}+\frac{1}{3}\cdot\frac{3}{5}=\frac{77}{135}\in[0,1]$$ (باستخدام نظرية بيز التمهيدية)

$$P(E)=\frac{2}{3}\cdot\frac{4}{9}+\frac{1}{3}\cdot\frac{2}{5}=\frac{58}{135}\in[0,1]$$

Q11: في احدى الكليات رسب 25% من الطلبة في امتحان الرياضيات ورسب 15% من الطلبة في امتحان الكيمياء ورسب 10% في امتحان الرياضيات والكيمياء.
اختير أحد الطلبة بطريقة عشوائية:
(i) اذا كان راسباً في الكيمياء، فما هو احتمال ان يكون راسباً في الرياضيات؟
(ii) اذا كان راسباً في الرياضيات، فما هو احتمال ان يكون راسباً في الكيمياء؟
(iii) ماهو احتمال ان يكون راسباً في الرياضيات أو الكيمياء؟

Sol.

$M=$ {الطلبة الراسبون في الرياضيات} ، $C=$ {الطلبة الراسبون في الكيمياء}

$MC=$ {الطلبة الراسبون في الرياضيات والكيمياء}

$$P(M)=.25,\ P(C)=.15,\ P(MC)=.10$$

$$\text{(i) } P(M|C)=\frac{P(MC)}{P(C)}=\frac{.10}{.15}=\frac{2}{3}\in[0,1]$$

$$\text{(ii) } P(C|M)=\frac{P(MC)}{P(M)}=\frac{.10}{.25}=\frac{2}{5}\in[0,1]$$

$$\text{(iii) } P(M\cup C)=P(M)+P(C)-P(MC)=\frac{3}{10}=.30$$

Q12: حادثتان A,B بحيث ان $(P(A|B)=\frac{1}{4})$ وان $(P(B|A)=\frac{1}{2})$ و $P(A)=\frac{1}{4}$ ، اذكر هل العبارات التالية صحيحة؟
① A,B are independent ?
② Is $A\subset B$?

Sol. $$P(B|A)=\frac{P(AB)}{P(A)}=\frac{P(AB)}{1/4}=\frac{1}{2}$$

$$\Rightarrow \frac{1}{4}\cdot\frac{1}{2}=P(AB)\Rightarrow \boxed{P(AB)=\frac{1}{8}}$$

1

$\Longrightarrow$ fallows

$$P(A/B) = \frac{P(AB)}{P(B)} = \frac{1/8}{P(B)} = \frac{1}{4}$$

$$\Rightarrow \frac{\frac{1}{8}}{1/4} = P(B) \Rightarrow P(B) = \frac{1}{\cancel{8}} \cdot \cancel{4} = \frac{1}{2}$$

$$\therefore \boxed{P(B) = \frac{1}{2}}$$

لا يمكن (باستخدام المغالطة)

(1) $P(AB) = P(A)P(B)$
$\frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2}$ $\Big\} \Rightarrow A, B$ are independent events.

(2) By Th.4 If $A \subseteq B \Rightarrow P(A) \leq P(B)$

$$P(A) = \frac{1}{4}, \quad P(B) = \frac{1}{2}$$

$$P(A) \leq P(B)$$

$$\frac{1}{4} \leq \frac{1}{2}$$

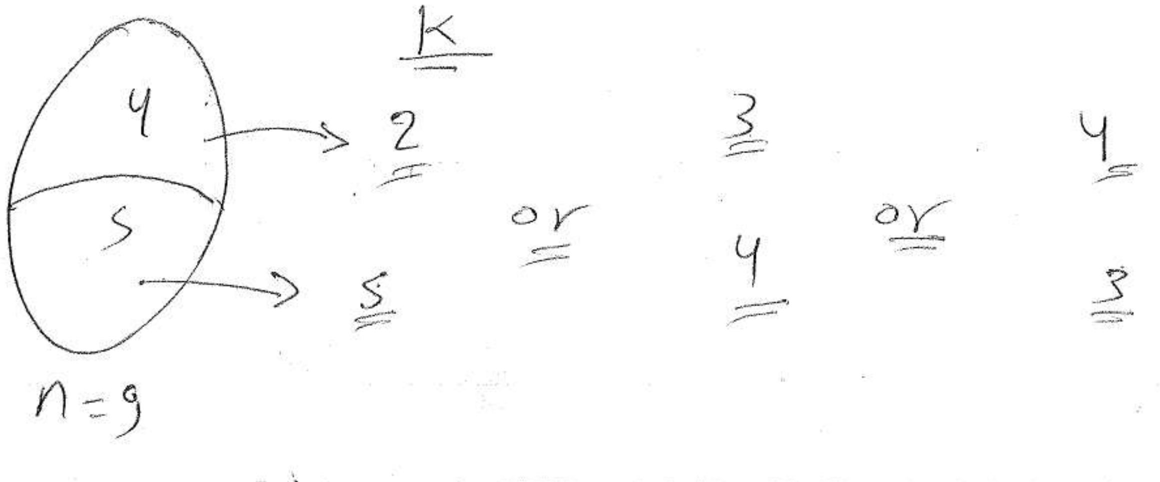
$$\Rightarrow A \subseteq B$$

Q13: A student is to answer (7) out of (9) questions on an exam. Find the pr. that he must answer at least (2) of the first (4) questions.

Sol.

Let W: be a student answer at least (2) of the first (4) questions.

$$P(W) = \frac{C^4_2 C^5_5 + C^4_3 C^5_4 + C^4_4 C^5_3}{C^9_7} \in [0,1]$$



Chapter Two
Introduction to Probability

- Exercises -

(PP.14)

For any events A and B, Show that :

1. $P(AB) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

Proof:

$\because AB \subset A$ (By Fact)

$\therefore P(AB) \leq P(A)$ (By Th.4) ... ①

$\because A \subset A \cup B$ (By Fact)

$\therefore P(A) \leq P(A \cup B)$ (By Th.4) ... ②

$\because$ A, B are joint events (since $P(AB) \neq \emptyset$)

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(AB)$ (By Th.3)

$\Rightarrow P(AB) = P(A) + P(B) - P(A \cup B)$

But $0 \leq P(AB) \leq 1$ (By Axiom 1)

$\Rightarrow 0 \leq P(A) + P(B) - P(A \cup B) \leq 1$

$\Rightarrow P(A) + P(B) - P(A \cup B) \geq 0$

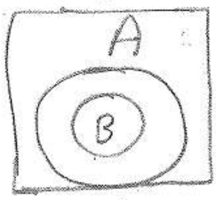
$\Rightarrow P(A) + P(B) \geq P(A \cup B)$

i.e. $P(A \cup B) \leq P(A) + P(B)$ ... ③

By ① + ② + ③ we get :

$P(AB) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

. If A and B are joint events, when $P(A)=0.8$, $P(B)=0.5$. Find the Conditions and the Value of Max $P(AB)$ and Min $P(AB)$.



Sol. $\because$ A and B are joint.

$\Longrightarrow \quad AB \neq \emptyset$

Case ① If $B \subseteq A$ (Since $P(B) \leq P(A)$ / By Th. 4)

$B \subseteq A \Longrightarrow AB = B$

$\Longrightarrow P(AB) = P(B) = 0.5$

Case ② If $B \not\subseteq A$ & A, B are joint events:

$P(A \cup B) = P(A) + P(B) - P(AB)$ (By Th. 3)

$= 0.8 + 0.5 - P(AB)$

$P(AB) = 1.3 - P(A \cup B) \geq 0$ (By Ax. 1)

$0 \leq P(A \cup B) \leq 1$ (By Axiom 1)

$0 \geq -P(A \cup B) \geq -1$

$1.3 \geq 1.3 - P(A \cup B) \geq -1 + 1.3$

$1.3 \geq 1.3 - P(A \cup B) \geq 0.3$

$1.3 \geq P(AB) \geq 0.3 \Longrightarrow P(AB) \geq 0.3$

$\therefore$ If $A \subset B \Rightarrow$ Max $P(AB) = 0.5$

If $A \not\subset B \Rightarrow$ Min $P(AB) \geq 0.3$

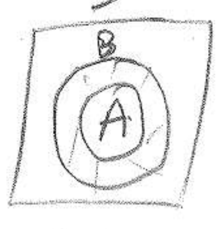
---

3. If $P(A) = \frac{1}{3}$ & $P(B) = \frac{1}{2}$. Find the Value of $P(BA^c)$ when:

a. A & B are disjoint events

b. $A \subseteq B$

c. $P(AB) = \frac{1}{8}$

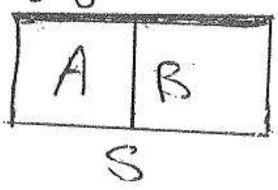


$A \subseteq B$

Sol.

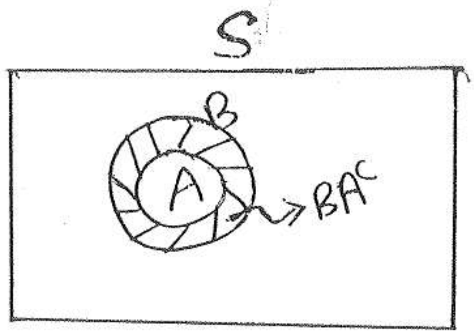
a. If A and B are disjoint.

$\because$ A, B are disjoint



$P(BA^c) = P(B) = \frac{1}{2}$

b. If $A \subseteq B$



$B = A \cup BA^c$

where A & BA^c are disjoint events.

$P(B) = P(A) + P(BA^c)$ (By Ax. 3)

$\frac{1}{2} = \frac{1}{3} + P(BA^c)$

$P(BA^c) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

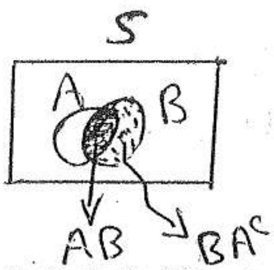
c. $P(AB) = \frac{1}{8} \Rightarrow P(AB) \neq 0 \Rightarrow AB \neq \emptyset$ (By Th. 1).

$B = AB \cup BA^c$ (when AB, BA^c are disjoint events)

$P(B) = P(AB) + P(BA^c)$ (By Ax. 3)

$\frac{1}{2} = \frac{1}{8} + P(BA^c)$

$\Rightarrow P(BA^c) = \frac{3}{8} \in [0,1]$



4. If A, B and C are disjoint events, find:

1. $P[(A \cup B) \cap C] = P[AC \cup BC] = P(\emptyset \cup \emptyset)$ (since $AB = \emptyset$, $AC = \emptyset$, $BC = \emptyset$, A, B, C are disj.)
$= P(\emptyset)$
$= 0$ (By Th. 1)

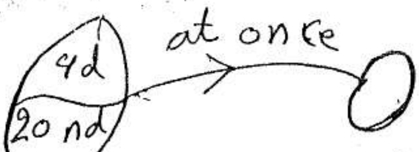
2. $P(A^c \cup B^c) = P[(AB)^c] = P(\emptyset)^c = P(S) = \underline{1}$

(By Demo. Laws) By $AB = \emptyset$ (A, B are disj. events) (By Ax. 2)

- Exercises -

1. A box has (24) bulbs of which (4) are defective. Choose (4) bulbs, find the pr. that they are defective.

Sol.



$r=4$

$n=24$

$$n(s) = C_r^n = C_4^{24} \quad \left(\begin{array}{c}\text{Choose}\\ \text{at once}\end{array}\right)$$

Let E: be defective bulbs.

$$n(E) = \binom{4}{4}_{d}\binom{20}{0}_{nd} = 1 \cdot 1 = 1 \quad \left(\text{Since } C_n^n = 1,\ C_0^n = 1\right)$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{C_4^4 \cdot C_0^{20}}{C_4^{24}} = \frac{1}{C_4^{24}} \in [0,1].$$

---

2. A set of (11) integers; (5) of them are negative and the others are positive. Choose a sample of (4) integers and multiply them, then find the pr. that the product is:
   a. negative b. positive.

Sol.

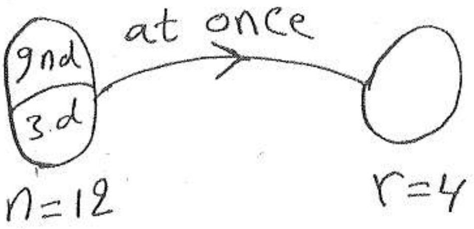
$r=4$

$n=11$

$$n(S) = C_r^n = C_4^{11} = \frac{11!}{4!\,7!} = 330$$

$\therefore$ S has (330) samples.

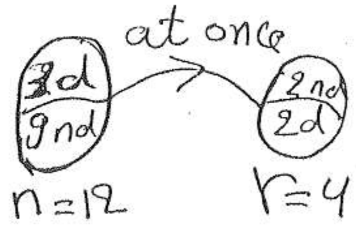
a. Let A be the sample that the product multible of (4) chosen integers is positive.



$$n(S) = C^{12}_{4} = \frac{12!}{4!\,(12-4)!}$$
$= 495$ samples

ⓐ Let A : be a sample has 2d (2-defective) and 2nd (2-not defective).

$n(A) = C^{3}_{2}\, C^{9}_{2}$



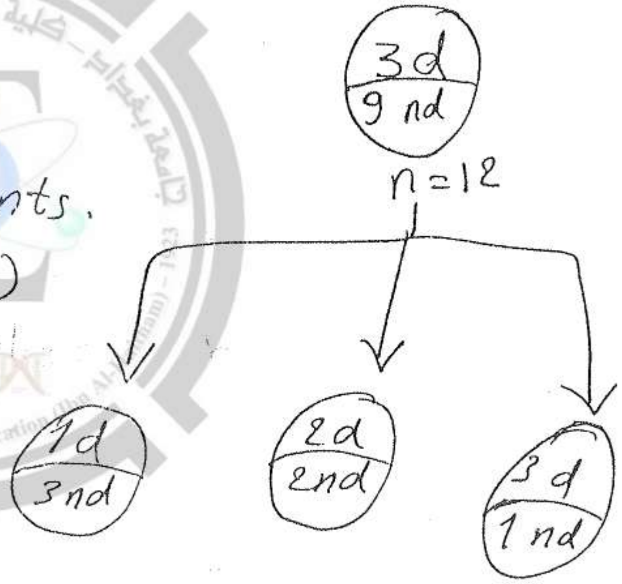
$$\therefore P(A) = \frac{C^{3}_{2}\, C^{9}_{2}}{C^{12}_{4}}$$

ⓑ Let B: be the sample that has at least one defective than.

$B = B_1 \cup B_2 \cup B_3$

B_1, B_2, B_3 are disjoint events.

$\therefore P(B) = P(B_1) + P(B_2) + P(B_3)$



$P(B_1) = C^{3}_{1}\, C^{9}_{3} / C^{12}_{4}$

$P(B_2) = C^{3}_{2}\, C^{9}_{2} / C^{12}_{4}$

$P(B_3) = C^{3}_{3}\, C^{9}_{1} / C^{12}_{4}$

$$P(B) = \frac{C^{3}_{1} C^{9}_{3} + C^{3}_{2} C^{9}_{2} + C^{3}_{3} C^{9}_{1}}{C^{12}_{4}} \in [0,1]$$

PP. (42)

— Exercises —

① Given (9) coins. (2) coins have (H) on one side and (T) on the other, (3) coins have (H) on both side and (4) coins have (T) on both side. Choose a coin and find the pr. that H happears.

i.e. $+\cdot+\cdot+\cdot+ = + \rightarrow A_1$
$+\cdot-\cdot- = + \rightarrow A_2$
$-\cdot-\cdot- = + \rightarrow A_3$

$A = A_1 \cup A_2 \cup A_3$ (3- Cases)
& A_1, A_2 and A_3 are disjoint events.

$\therefore P(A) = P(A_1) + P(A_2) + P(A_3)$

$$P(A_1) = \frac{C^6_4 \; C^5_0}{C^{11}_4} = \frac{15}{330} \in [0,1]$$

$$P(A_2) = \frac{C^6_2 \; C^5_2}{C^{11}_4} = \frac{150}{330} \in [0,1]$$

$$P(A_3) = \frac{C^6_0 \; C^5_4}{C^{11}_4} = \frac{5}{330} \in [0,1]$$

$$\therefore P(A) = \frac{15+150+5}{330} = \frac{170}{330} = \frac{17}{33} \in [0,1]$$

b. Let B: be the sample that the Product multiple of (4) chosen integer is negative.

i.e. $+\cdot+\cdot+\cdot- = - \rightarrow B_1$
$+\cdot-\cdot-\cdot- = - \rightarrow B_2$

$B = B_1 \cup B_2$
B_1, B_2 are disjoint events

$\therefore P(B) = P(B_1) + P(B_2)$

OR since the product is positive or negative, then:

$$P(B) = P(A^c) = 1 - P(A) \quad \text{(By Th. 2)}$$
$$= 1 - \frac{17}{33}$$
$$= \frac{16}{33}$$

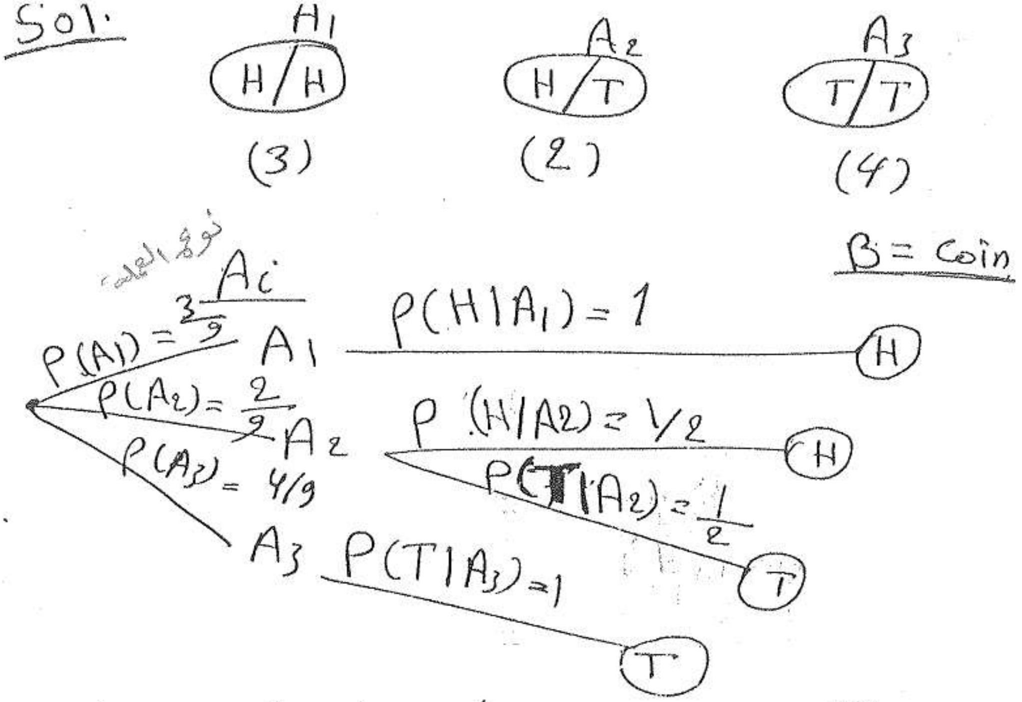
S
$\boxed{+ \mid -}$
$\cdot$ = product

3. Given a set of (12) transistors of which (3) are defective. Choose a sample of (4) transistors, then find the pr. that:

a. Two transistors are defective
b. At least one transistors is defective.

Sol.

Sol.



Let B : be a coin to get H

$$P(\text{to get } H) = P[(A_1H)\cup(A_2H)]$$

$$= P(A_1H)+P(A_2H)$$

$$= P(A_1)\cdot P(H|A_1) + P(A_2)\cdot P(H|A_2)$$

$$= \left(\frac{3}{9}\right)\cdot(1) + \left(\frac{2}{9}\right)\left(\frac{1}{2}\right)$$

$$= \frac{4}{9} \in [0,1]$$

---

② Choose a sample of (5) balls from (6) white and (10) black balls one by one with out replace. Find the Pr. to get a sample (w, w, b, w, b).

Sol. $\boxed{\begin{matrix} 6\ w \\ 10\ b \end{matrix}}$ $n=16$ $\xrightarrow[\text{with rep.}]{(1-1)}$ $\boxed{\quad}$ $r=5$

$$P(w_1 w_2 b_1 w_3 b_2) = ? \quad (\text{By multip. rule})$$

$$P(w_1 w_2 b_1 w_3 b_2) = P(w_1)\cdot P(w_2|w_1)\cdot P(b_1|w_1w_2)\cdot P(w_3|w_1w_2b_1)\cdot P(b_2|w_1w_2b_1w_3)$$

$$= \frac{6}{16}\cdot\frac{5}{15}\cdot\frac{10}{14}\cdot\frac{4}{13}\cdot\frac{9}{12} \in [0,1]$$

Given (one) blue card and (4) red cards which are named A, B, C and D, choose (2) cards one by one without replace. Find the pr. that:

a- both cards are red, given that card A is chosen.

b. both cards are red, given one red card is chosen.

Sol.

$$\underbrace{\boxed{\text{blue}}}_{n_1} + \underbrace{\boxed{A \mid B \mid C \mid D}}_{n_2} \longrightarrow \underbrace{\boxed{n_1+n_2}}_{n}$$

$$n(S) = P_2^5 = \frac{5!}{3!} = \frac{5\times4\times\cancel{3!}}{\cancel{3!}} = 5\times4 = 20 = n(S)$$

a- $\underbrace{\text{Both cards are red}}_{E}$, given that $\underbrace{\text{card A is chosen}}_{F}$.

$$E = \left\{ \begin{matrix} (A,B), & (B,A), & (C,A), & (D,A) \\ (A,C), & (B,C), & (C,B), & (D,B) \\ (A,D), & (B,D), & (C,D), & (D,C) \end{matrix} \right\} \longrightarrow n(E) = 12$$

$$P(E) = \frac{12}{20} \in [0,1]$$

$$F = \left\{ \begin{matrix} (A,b), & (b,A) \\ (A,B), & (B,A) \\ (A,C), & (C,A) \\ (A,D), & (D,A) \end{matrix} \right\} \longrightarrow n(F) = 8$$

$$P(F) = \frac{8}{20} \in [0,1]$$

$$EF = \left\{ \begin{matrix} (A,B), & (B,A) \\ (A,C), & (C,A) \\ (A,D), & (D,A) \end{matrix} \right\} \longrightarrow n(EF) = 6$$

$$P(EF) = \frac{6}{20} \in [0,1]$$

$\longrightarrow$

L. Side $\binom{n+1}{r} = \dfrac{(n+1)!}{r!\,((n+1)-r)!}$

*

$$= \frac{(n+1)\,n!}{r!\,((n+1)-r)!}$$

$$= \frac{n!\,(r+n-r+1)}{r!\,(n-r+1)!}$$

$$= \frac{r\,n! + n!\,(n-r+1)}{r!\,(n+1-r)!}$$

$$= \frac{r\,n!}{r!\,(n+1-r)!} + \frac{n!\,(n-r+1)}{r!\,(n+1-r)!}$$

$$= \frac{\cancel{r}\,n!}{\cancel{r}\,(r-1)!\,(n+1-r)!} + \frac{\cancel{(n-r+1)}\,n!}{r!\,\cancel{((n+1)-r)}\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-(r-1))!} + \frac{n!}{r!\,(n-r)!}$$

$$= \binom{n}{r-1} + \binom{n}{r}$$

$$\therefore \quad \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{6}{20}}{\frac{8}{20}} = \frac{6}{8} = \frac{3}{4} \in [0,1]$$

b- $\underbrace{\text{Both Cards are red}}_{E}$, given $\underbrace{\text{one red card is chosen}}_{G}$.

$$G = \begin{Bmatrix} (A,b), & (b,A) \\ (B,b), & (b,B) \\ (C,b), & (b,C) \\ (D,b), & (b,D) \end{Bmatrix} \rightarrow n(G) = 8$$

$$P(G) = \frac{8}{20} \in [0,1]$$

$$EG = \emptyset \rightarrow P(EG) = P(\emptyset) = 0$$

$$P(E/G) = \frac{P(EG)}{P(G)} = \frac{0}{8/20} = 0 \in [0,1]$$

Ex. prove that: pp. 10

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

sol.

R. Side $\binom{n}{r-1} + \binom{n}{r} =$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= n!\left[\frac{(n-\cancel{r}+1)+\cancel{r}}{r(r-1)!(n-r+1)(n-r)!}\right]$$

$$= \frac{(n+1)n!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!} = \frac{(n+1)!}{r!((n+1)-r)!}$$

$$= \binom{n+1}{r} = \text{L.S}$$

" In Playing Cards "

Example (1): How many 5 - Cards will have 3 aces and 2 Kin

sol. $n = C_3^4 \cdot C_2^4$

aces(A) King

How many 5-cards will have 3 hearts and 2 spaces ?

$n = C_3^{13} \cdot C_2^{13}$

hearts(♡) spaces (♠)

Note : Ace (A), Heart (♡), Club (♣), Diamond(◊), Spade

Jack (J), Queen (Q) , King (K) = the face Cards (Pic.) صور

These are standard deck of (52) Cards has four suits (

♡, ♣, ◊, ♠ with (13) cards in each suit.

(◊, ♡) → red Cards (26) & (♣, ♠) → black Cards (2

Example (2):

Let E = the drawn card is a spade (♠).

F = the drawn card is a face card. (صورة)

Let G = the drawn card is a heart (♡).

H = the drawn card is a club (♣).

① Are E & F independent events?

② Are G & H independent events?

sol. ① $P(E) = \left(\frac{13}{52}\right)$, $P(F) = \left(\frac{12}{52}\right)$

$P(E \cap F) = P(J♠, Q♠, K♠) = \frac{3}{52}$

$P(E \cap F) = \frac{3}{52} = P(E)\,P(F) = \left(\frac{13}{52}\right) \cdot \left(\frac{12}{52}\right) = \frac{3}{52}$

Then E & F are indep. events.

② $P(G) = \frac{13}{52}$, $P(H) = \frac{13}{52}$, $G \cap H = \phi$

$\therefore P(G \cap H) = P(\phi) = 0$ (by Th.1)

$$P(G)P(H) = \left(\frac{13}{52}\right)\left(\frac{13}{52}\right) = \frac{1}{16} \neq P(GH) = 0$$

Example (3): A single card is drawn from a standard 52-card deck. Test the following events for independence:

(A) E = the drawn card is a red card.
F = the drawn card's number is divisible by 5.

(B) G = the drawn card is a king.
H = the drawn card is a queen.

① Are E & F indep. events?
② Are G & H indep. events?

H.W.

Example (4): In a single card is drawn from a standard 52-card deck. Find the pr. of each of the following events.

① $P(\text{ace}) = \frac{4}{52}$
$\{\heartsuit A, \diamondsuit A, \clubsuit A, \spadesuit A\}$

② $P(\text{face card}) = \frac{12}{52}$

③ $P(\text{spade}) = \frac{13}{52}$

④ $P(\text{spade or heart}) = \frac{13}{52} + \frac{13}{52} = \frac{26}{52}$

⑤ $P(\text{red card}) = \frac{26}{52}$,
$(\heartsuit \,\&\, \diamondsuit)$
13 13

⑥ $P(\text{red or face}) = \frac{26}{52} + \frac{12}{52} - \frac{6}{52}$

(face & Red) تقاطع
(J, K, Q (2 red) = 6)

Example (5): Find the pr. of randomly drawing two aces from an ordinary deck of 52 playing cards; if we sample:
(a) without replacement. (b) with replacement.

Sol. (a) $P = \frac{4}{52} \cdot \frac{3}{51} = P(AA)$ (by multiplication rule)

اول كارد ↗ ثاني كارد ↗

(b) $P = \frac{4}{52} \cdot \frac{4}{52} = P(AA)$ (" " ").

(c) at once $\longrightarrow P = \frac{C^4_2}{C^{52}_2} = P(2A)$
2

Example (1): If a coin is tossing twice.

$S = \{HH, HT, TH, TT\}$

and Let:

A = a head on the first toss $= \{HH, HT\}$

B = a head on the second toss $= \{HH, TH\}$

Are A & B independent events?

Sol. $P(A) = \frac{2}{4} = \frac{1}{2}$, $P(B) = \frac{2}{4} = \frac{1}{2}$

$AB = \{TH\} \rightarrow P(AB) = \frac{1}{4}$

$P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cup B)$

$\therefore$ A & B are indep. events.

Example (2): Roll a die once.

$S = \{1, 2, 3, 4, 5, 6\}$

Find the pr. that:

H.W.

E_1: The die shows an event number.

E_2: The die shows a (1).

E_3: The die shows a multiple of (3) $= \{3, 6\}$ (مضاعفات)

E_4: The die shows a number less than (5) $= \{1, 2, 3, 4\}$

E_5: The die shows a number (7) $= \phi$

E_6: The die shows a number less than (10).

$= \{1, 2, 3, 4, 5, 6\} = S$

Example (3): Rolling a die twice. Find the pr. that the sum (مجموع) of the numbers rolled is greater than (3).

Sol. $P(\text{sum} > 3) = ?$

$$P(\text{sum} \leq 3) = P(\text{sum is } 2) + P(\text{sum is } 3)$$

$$= \Big(\underset{\{(1,1)\}}{\frac{1}{36}} + \underset{\{(2,1),(1,2)\}}{\frac{2}{26}}\Big) = \frac{3}{36} = \frac{1}{12}$$

$$\therefore P(\text{sum} > 3) = 1 - P(\text{sum} \leq 3) \quad (\text{by Th. } P(A) = 1 - P(A'))$$

$$= 1 - \frac{1}{12} = \frac{11}{12}.$$

Example (4): Suppose that two dice are rolled.

(A) what is the pr. that a sum of (7) or (11) turns up? (تظهر صدفة)

(B) what is the pr. that both dice turn up the same or that a sum less than (5) turns up?.

Sol.

(A) $A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \rightarrow n(A) = 6$

$B = \{(5,6), (6,5)\} \rightarrow n(B) = 2$

$n(S) = 6^2 = 36$, $AB = \phi$ (A & B are disjoint).

$$\therefore P(A \cup B) = P(A) + P(B)$$

$$= \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$$

(B) $A = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

$AB = \{(1,1)(2,2)\} \longrightarrow$ A & B are joint events.

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$= \frac{6}{36} + \frac{6}{36} - \frac{2}{36} = \frac{10}{36} = \frac{5}{18}$$

Now; (C) what is the pr. that a sum of (2) or (3) turns up? .

(D) what is the pr. that both dice turn up the same or that a sum greater than (8) turns up? (H.W.)

Example (5): What is the pr. that a number selected at random from the first (500) positive integers is (exactly) divided by (3) or (4)?

$A: \frac{500}{3} = 166$, $B: \frac{500}{4} = 125$

AB: the largest integer less than or equal to $\frac{500}{12} = 41$

$$\therefore P(A \cup B) = P(A) + P(B) - P(AB)$$

$$= \frac{166}{500} + \frac{125}{500} - \frac{41}{500} = .5$$

Example (6): what is the pr. that a number selected at random from the first (140) positive integer is (exactly) divedid by (4) or (6)?

(H.W.)

كلية معينة
راسبون في الرياضيات

Example (7): In a Certain College, 25% of the students failed mathematics, 15% of the students failed Chemistry, and 10% of the students failed both mathematics and Chemistry. A student is Selected at random.

(i) If the student failed chemistry, What is the pr. that he failed mathematics?

(ii) If he failed mathematics, What is the pr. that he failed chem.?

(iii) What is the pr. that he failed math. or Chemistry?

Sol. Let $M = \{$students who failed math.$\}$

$C = \{$students who failed chem.$\}$

$P(M) = .25$, $P(C) = .15$, $P(MC) = .10$

(i) $P(M|C) = \dfrac{P(MC)}{P(C)} = \dfrac{.10}{.15} = \dfrac{2}{3}$

(ii) $P(C/M) = \dfrac{P(MC)}{P(M)} = \dfrac{.10}{.25} = \dfrac{2}{5}$

(iii) $P(M \cup C) = P(M) + P(C) - P(MC)$

$= .25 + .15 - .10 = .30 = \dfrac{3}{10} = 30\%$

Example (8): Rolling a die twice. Find the pr. that:

(a) the first die shows a (2) or the sum of the results is (6) or (7).

Sol. $P(A) = \dfrac{6}{36}$, $P(B) = \dfrac{11}{36}$, $P(AB) = \dfrac{2}{36}$ s.t.

$A = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\}$

$B = \{(5,1),(5,2),(6,1),(4,2),(4,3),(3,3),(3,4),(2,4),(2,5),(1,5),(1,6)\}$.

$\therefore P(A \cup B) = P(A) + P(B) - P(AB)$ (by Th.3)

$= \dfrac{6}{36} + \dfrac{11}{36} - \dfrac{2}{36}$ (since A & B are joint events.)

$= \dfrac{15}{36} = \dfrac{5}{12} \in [0,1]$

(b) The sum of the results is 11, or the second die shows a (5).

Sol. $P(\text{sum is } 11) = \dfrac{2}{36}$, $P(2^{nd}\text{ die shows a}(5)) = \dfrac{6}{36}$.

تابع

$P(\text{sum is } 11 \text{ and } 2^{nd} \text{ die shows a } (5)) = \frac{1}{36}$

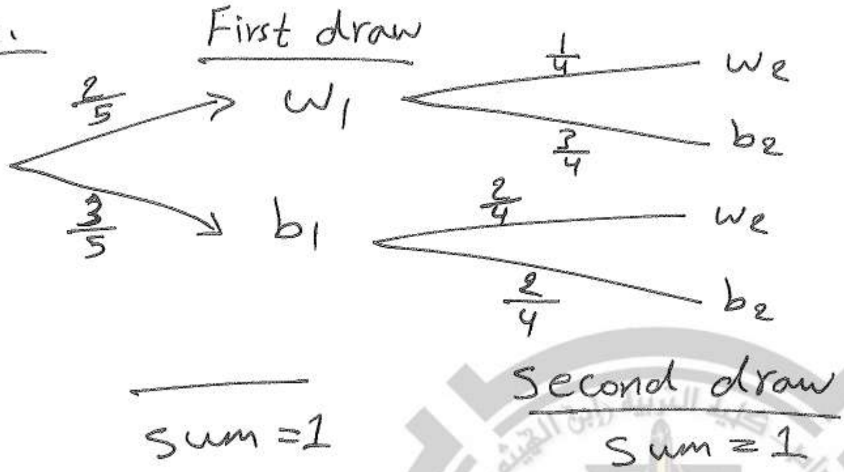
$P(\text{sum is } 11 \text{ or } 2^{nd} \text{ die shows a } (5)) = \frac{2}{36} + \frac{6}{36} - \frac{1}{36}$

$= \frac{7}{36}$

Example (9): Two balls are drawn without replacement, from a box containing (3) blue and (2) white balls. What is the pr. of drawing a white ball on the 2^{nd} draw?

Sol.

(i)

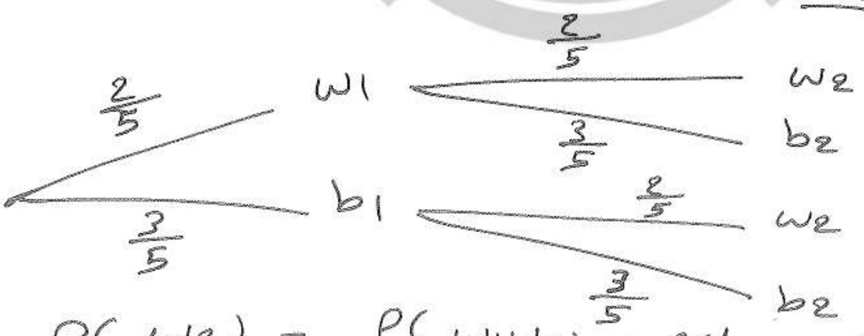


$P(w_2) = P(w_1 w_2) + P(b_1 w_2)$ by (نظرية بيز التمهيدية) (Th. 12)

$= \left(\frac{2}{5}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)$

$= \frac{1}{10} + \frac{3}{10} = \frac{4}{10} = \frac{2}{5} \in [0,1]$

(ii) If two balls are drawn with replacement?



$P(w_2) = P(w_1 w_2) + P(b_1, w_2)$ (by Th. 12)

$= \left(\frac{2}{5}\right)\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)$ (نظرية بيز التمهيدية)

$= (.16) + (.24)$

$= .40$

المطلوب: ان تكون الكرة بيضاء (w) وهنا السحبة (2) : w_2

#

س١: اذا كان احتمال ان يعيش رجل (١٠) سنوات اخرى هو $(\frac{1}{4})$.
واحتمال ان تعيش زوجته (١٠) سنوات اخرى هو $(\frac{1}{3})$. جد احتمال:

① ان يعيش الاثنان (١٠) سنوات اخرى .
② ان يعيش احدهما على الاقل (١٠) سنوات اخرى .
③ ان يموت الاثنان خلال السنوات العشر
④ ان تعيش الزوجة (١٠) سنوات (ويموت الرجل) .
⑤ ان يعيش احدهما على الاكثر (١٠) سنوات اخرى .

Sol. A: ان يعيش الرجل (١٠) سنوات اخرى .
B: ان تعيش زوجته (١٠) سنوات اخرى . (الحوادث مستقلة indep.)

$$P(A) = \frac{1}{4}, \quad P(B) = \frac{1}{3}$$

① $P(AB) = P(A)P(B)$ (by def. of independent events)
$= (\frac{1}{4})(\frac{1}{3}) = \frac{1}{12}$

② $P(A \cup B) = P(A) + P(B) - \underline{P(AB)}$ يعيشان كلاهما
$= (\frac{1}{4} + \frac{1}{3} - \frac{1}{12})$

③ $P(A^c B^c) = 1 - P(A \cup B)$ by ($A^c B^c = (A \cup B)^c$ Demorgan law)
$= 1 -$

④ $P(A^c B) = P(A^c) \cdot P(B)$ by Th. (Th. 10)
$= \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$ ($P(A^c) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$)

⑤ $P = P(A^c B) + P(AB^c) + P(A^c B^c)$
تعيش المرأة ويموت الرجل — أو — يعيش الرجل وتموت المرأة — أو — يموت الاثنان

$P(B^c) = 1 - P(B)$
$= 1 - \frac{1}{3}$
$= \frac{2}{3}$

Complete ---

مثال :اختيرت ورقة من اوراق العب البالغة (٥٢) ورقة جد

أ) احتمال الورقة التي تحمل الرقم ١٠

ب)احتمال ان تكون الورقة نوع ◊

ج)اذا اختيرت (٤) اوراق , ما احتمال ان تكون احدهم J .

د) اذا اختيرت (٣) ورقات ما احتمال ان تكون احدهم 3 والاخرى صورة والثالثة أي ورقة اخرى من اوراق اللعب .

الحل :— أ)

$$\boxed{P(10) = \frac{\binom{4}{1}}{\binom{52}{1}} = \frac{4}{52} = \frac{1}{13}}$$

ب)

$$\boxed{P(\lozenge) = \frac{\binom{13}{1}}{\binom{52}{1}} = \frac{13}{52} = \frac{1}{4}}$$

ج)

$$\boxed{P(J,\underbrace{C,C,C}_{\text{كارتات اخرى}}) = \frac{\binom{4}{1}\binom{48}{3}}{\binom{52}{4}}}$$

د)

$$\boxed{P(3,\underset{\text{صورة}}{P},c) = \frac{\binom{4}{1}\binom{12}{1}\binom{36}{1}}{\binom{52}{3}} = \frac{4*12*36}{\binom{52}{3}}}$$

مثال / البيانات التالية تمثل مجموعة من الطلبة مصنفين حسب القسم والجنس

	قسم الفيزياء B_1	قسم علوم الحياة B_2
A1 ذكور (٣٩٠)	٢١٠	١٨٠
A2 اناث (٢٤٠)	١١٥	١٢٥
العدد الكلي (٦٣٠)	٣٢٥	٣٠٥

اذا اختير طالب واحد من العينة ما احتمال أ)ان يكون الطالب من قسم الفيزياء ب) ان يكون ذكر ج) ان تكون انثى ومن علوم الحياة د) ان يكون ذكر او من الفيزياء

الحل :- أ)

$$P(B_1)=\frac{\binom{325}{1}}{\binom{630}{1}}=\frac{325}{630}=\frac{65}{120}$$

ب)

$$P(A_1)=\frac{\binom{390}{1}}{\binom{630}{1}}=\frac{390}{630}=\frac{13}{21}$$

ج)

$$P(A_2B_2)=\frac{\binom{125}{1}}{\binom{360}{1}}=\frac{125}{630}=\frac{25}{126}$$

د)

$$P(A_1\cup B_1)=P(A_1)+p(B_1)-P(A_1B_1)$$

$$=\frac{\binom{390}{1}}{\binom{630}{1}}+\frac{\binom{325}{1}}{\binom{630}{1}}-\frac{\binom{210}{1}}{\binom{630}{1}}$$

$$=\frac{390+325-210}{630}$$

$$=\frac{505}{630}=\frac{101}{126}$$

مثال : مجتمع يضم (١٠٠) شخص تم اجراء فحص لمعرفة صنف الدم لكل منهم وكانت النتائج كالتالي

صنف الدم	A	B	AB	O	
العدد	٢٢	١١	٦٠	٧	المجموع (١٠٠) شخص

ما احتمال أ) شخص يحمل صنف الدم AB

ت) شخص لا يحمل صنف الدم O

ث) ثلاث اشخاص احدهم يحمل صنف الدم o

ج) شخصين احدهم يحمل صنف الدم A والاخر B

أ)

$$P(AB)=\frac{\binom{60}{1}}{\binom{100}{1}}=\frac{60}{100}=0.6$$

ب)

$$P(O)=\frac{\binom{7}{1}}{\binom{100}{1}}=\frac{7}{100}=0.07$$

يحمل صنف O

$P(O^c)=1-P(O) \longrightarrow P(O^c)=1-0.07=0.93$

لا يحمل صنف O

ح) F : تمثل ثلاث اشخاص احدهم يحمل الصنف O

$$P(F)=\frac{\binom{7}{1}\binom{93}{2}}{\binom{100}{3}}$$

د) E: تمثل القيمة المكونة من شخصين احدهم يحمل الصنف A والاخر الصنف B

$$P(A,B) = P(E) = \frac{\binom{22}{1}\binom{11}{1}}{\binom{100}{2}} = \frac{22*11}{\frac{100!}{2!(98)!}}$$

مثال / البيانات التالية تمثل عدد الاشخاص الذين هم بحاجة الى طبيب عام او اسنان من نوع الدعم المالي

الحل :-

مصدر الدعم / الاختصاص		طبيب عام B1	طبيب اسنان B2	
A1	حكومي	٢٨٠	٤٧٠	٧٥٠
A2	شخصي	١٤٠	١١٠	٢٥٠
		٤٢٠	٥٨٠	١٠٠٠

اذا اختير شخص وبشكل عشوائي فما احتمال ان

أ) ان يكون الدعم حكومي ب) يراجع طبيب اسنان

ج)ذو دعم شخصي او يراجع طبيب عام

د) ذو دعم حكومي ويراجع طبيب اسنان

الحل :- أ)

$$P(A1) = \frac{\binom{750}{1}}{\binom{1000}{1}}$$

ب)

$$P(B2) = \frac{\binom{580}{1}}{\binom{1000}{1}}$$

ج)

$$P(A_2 \cup B1) = P(A2) + P(B1) - P(A2B1)$$

$$P(A2 \cup B_1) = \frac{\binom{250}{1}}{\binom{1000}{1}} + \frac{\binom{420}{1}}{\binom{1000}{1}} - \frac{\binom{140}{1}}{\binom{1000}{1}}$$

$$= \frac{53}{100}$$

د)

$$P(A1B2) = \frac{\binom{470}{1}}{\binom{1000}{1}}$$

$$= \frac{470}{1000} = 0.47$$

اسئلـــــة عـــامـــــة

١) اختيرت ورقة واحدة من أوراق اللعب البالغة ٥٢ فما احتمال

١) ان يكون نوع ◊ ٣) ان يكون K او Q

٢) ان تكون صورة ٤) ليست اس A

٢) اذا اختيرت ثلاث ورقات فما احتمال توضيح

١) ان تكون جميعها حمراء حمراء 26

٢) احتوائها على الاقل A (اس) (على الاكثر) سوداء 26

٣) احتوائها على الاكثر رقم ١٠

٤) اثنان صور الكلي 52

الحل :- ١)

$$P(RRR) = \frac{\binom{26}{3}}{\binom{52}{3}}$$

$$= \frac{\frac{26!}{3!23!}}{\frac{52!}{3!99!}} = \frac{24}{204}$$

٢) على الاقل A ⇐ AXX ، AAX ، AAA المكملة هي XXX (المكملة لايوجد فيها A)

$$P(A^C)=\frac{\binom{48}{3}}{\binom{52}{3}}= \longrightarrow P(A)=1-P(A^C)=1-\frac{\binom{48}{3}}{\binom{52}{3}}$$

٣)

$$P(10)=\frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}}+\frac{\binom{48}{3}}{\binom{52}{3}}3$$

٤)

P (C, صورة وصورة) $= \dfrac{\binom{12}{2}\binom{40}{1}}{\binom{52}{3}}$

الحوادث المستقلة "Independent Events"

يقال للحادثين B,A في الفضاء العيني لتجربة عشوائية معينة انها مستقلان اذا لم يؤثر احدهما على وقوع الاخر .

وهذا يعني ان B,A حادثين مستقلين اذا وفقط اذا تحققت العلاقة التالية

$$\boxed{P(A)\times P(B)=P(AB)}$$

ملاحظة : كل حادثين مستقلين في Ω فانهما يجب ان يكون متصلين والعكس غير صحيح

مثال :- اذا كان B,A حادثين مستقلين في Ω بحيث ان

$P(A)=0.4\ ,\ P(B)=0.2$

جد :- $P(A\cup B)$

الحل :-

$$\begin{aligned}P(A\cup B)&=P(A)+P(B)-P(AB)\\&=P(A)+P(B)-P(A)\cdot P(B)\\&=0.4+0.2-(0.4)(0.2)\\&=0.6-0.08\\&=0.52\end{aligned}$$

نظرية ١

نظرية (١) اذا كان كل من A و B حادثين مستقلين في تجربة عشوائية معينة فان :-

١) B^C , A حادثين مستقلين ايضا [indep. Events] $\xrightarrow{\text{معناه}}$ $P(A).P(B^c) = P(AB^c)$

٢) B, A^C حادثين مستقلين ايضا [indep. Events] ← $P(A^c).P(B) = P(A^cB)$

٣) B^C,A^C حادثين مستقلين ايضا [indep. Events] ← $P(A^c)P(B^c) = P(A^cB^c)$

مثال :- اذا كان احتمال نجاح احمد في امتحان معين هو (٠,٨) واحتمال نجاح سعيد في نفس الامتحان هو (٠,٧) جد :-

١) احتمال نجاح احمد وعدم نجاح سعيد

٢) احتمال نجاح احدهما على الأكثر

الحل :-

نفرض ان نجاح احمد هو A ← P(A) = 0.8

نفرض ان نجاح سعيد هو B ← P(B) = 0.7

نلاحظ استقلال الحادثين

١)

$$\boxed{P(AB^C) = P(A)P(B^C) = (0.8)[1-0.7] = 0.24}$$

٢)

$$\boxed{P(AB^C) + P(A^C B) + P(A^C B^C)}$$

$$\boxed{\begin{aligned} &= P(A)P(B^C) + P(A^c).P(B) + P(A^C)P(B^C) \\ &= (0.8)(0.3) + (0.2)(0.7) + (0.2)(0.3) \\ &= 0.44 \end{aligned}}$$

نظرية (٢) اذا كان كل من A,B حادثين منفصلين (disjoint events) بحيث ان A ≠Ø, B ≠Ø

فان A,B حادثين معتمدين (غير مستقلين) (dependent)

الحوادث المعتمدة "Dependent Events"

يقال ان A , B حادثين معتمدين (غير مستقلين) اذا وفقط اذا $P(A).P(B) \neq P(AB)$

مثال :-سحب عنصرين من مجموعة مكونة من اربعة عناصر {1,2,3,4} عنصر عنصر بدون ارجاع (ومع الارجاع) فاذا كانت الحادثين B,A كما يلي :-

A : العنصر الاول فيها هو (٢).

B : العنصر الثاني بها هو (١) .

هل ان A,B حادثين مستقلين ؟

الحل :- ١) طريقة السحب الاولي (بدون ارجاع)

$$P_2^4 = \frac{4!}{2!} = 12$$

$$\Omega = \begin{Bmatrix} (1,2) & (2,1) & (3,1) & (4,1) \\ (1,3) & (2,3) & (3,2) & (4,2) \\ (1,4) & (2,4) & (3,4) & (4,3) \end{Bmatrix}$$

$$A = \{(2,1),(2,3),(2,4)] \subset S \Rightarrow P(A)\frac{3}{12} = \frac{1}{4}$$

$$B = \{(2,1),(3,1),(4,1)\} \subset S \Rightarrow P(B) = \frac{3}{12} = \frac{1}{4}$$

$$AB = \{(2,1)\} \subset S \Rightarrow P(AB) = \frac{1}{12}$$

$$P(A)xP(B) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

نلاحظ الان

$$P(AB) = \frac{1}{12}$$

$$\therefore P(A)xP(B) \neq P(AB)$$

∴ A و B غير مستقلين

الحادثين A,B غير مستقلين

٢)طريقة السحب الثانية (مع الارجاع)

عدد عناصر هو $(4^2)=16$

$$\Omega = \left\{ \begin{matrix} (1,1) & (2,1) & (1,11) & (4,11) \\ (1,2) & (2,2) & (3,2) & (4,2) \\ (1,3) & (2,3) & (3,3) & (3,3) \\ (1,4) & (2,4) & (3,4) & (4,4) \end{matrix} \right\}$$

$$A = \{(2,1),(2,2),(2,3),(2,4)\} \Rightarrow P(A) = \frac{4}{16} = \frac{1}{4}$$

$$B = \{(1,1),(2,1),(3,1),(4,1)\} \Rightarrow P(B) = \frac{4}{16} = \frac{1}{4}$$

$$AB = \{(2,1)\} \Rightarrow P(AB) = \frac{1}{16}$$

$$P(A)xP(B) = \frac{1}{4}x\frac{1}{4} = \frac{1}{16}$$

$$\therefore P(AB) = P(A)xP(B)$$

$\therefore$ A,B حادثين مستقلين

اسئلة وتوضيح حول ورق اللعب :

اسود	احمر	اسود	احمر
♣	♦	♠	♥
A	A	A	A
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9
10	10	10	10
J	J	J	J
Q	Q	Q	Q
K	K	K	K

العدد الكلي $4 \times 13 = 52$

عدد الصور $3 \times 4 = 12$

اسئلة : اختيرت ورقة من الاوراق البالغة (52) جد :

أ. احتمال ان تكون الورقة تحمل رقم (10) .

ب. احتمال ان تكون الورقة من نوع ♢ .

جـ. اذا اختيرت (4) ما هو احتمال ان تكون احدهم A .

د. اذا اختيرت (5) ورقات فما هو احتمال ان تكون احدهم A والثانية صورة والثالثة اي ورقة إخرى من اوراق اللعب .

الحل :-

Let B = to get 10

$$P(B)=\frac{\binom{4}{1}}{\binom{52}{1}}=\frac{4}{52}=\frac{1}{13}$$

ب.

Let C = to get ◊ (◊ diomoned)

$$\therefore P(C)=\frac{\binom{13}{1}}{\binom{52}{1}}=\frac{13}{52}=\frac{1}{4}$$

جـ.

D = To get 4 cards one of them is A

52 – 4 = 48

$$\therefore P(D)=\frac{\binom{4}{1}\binom{48}{3}}{\binom{52}{1}}$$

Let E = to get (3) cards one of then A and the second is a picture and the third any other card

$$\therefore P(E)=\frac{\binom{4}{1}\binom{12}{1}\binom{36}{1}}{\binom{52}{3}}$$