وزارة التعليم العالي والبحث والعلمي
جامعة ديالى
كلية تربية المقداد / قسم الرياضيات

# محاضرات مادة

# الإحصاء والاحتمالية

## للعام الدراسي (2023-2024)

## المرحلة الثالثة

# Chapter Three

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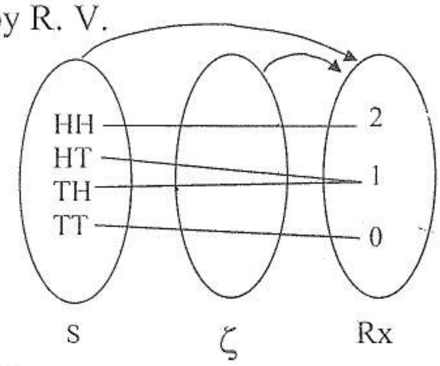
# *Chapter Three*

## ***"Random variables and Probability Distribution"***

Def.: A random variable X is a function that mapps all elements $s \in S$ (all event in $\zeta$ ) to a real numbers (Rx) denoted by R. V.




Ex 1.: Toss a coin twice.

Let X = number of H show that X is a R. V.

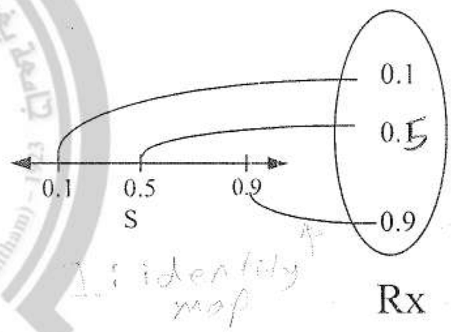
S = {HH, HT, TH, TT}

x(HH) = 2, x(HT) = 1, x(TH) = 1, x(TT) = 0

Rx = {x; x = 0 , 1, 2} countable

Note: We shall use X to denoted of R. V.X and x to denote of value of R. V. X, $x \in X$: 0, 1, 2 (in ex. "1").

Ex2.: Choose a point from interval (0, 1).




Let X be the chosen point, to show X is a R. V.

S consist all point in (0, 1)

∴ S has infinite number of points

The points in S are mapped to a real numbers.

then X is a R. V.

Rx = {x; 0 < x < 1} uncountable.

Def.: A random variable X is a say to be discret r. v. if Rx is countable.

See ex. "1" above, denoted by d. r. v.

Def.: A random variable x is say to be continuous r. v. if Rx is uncountable denoted by c.r.v.

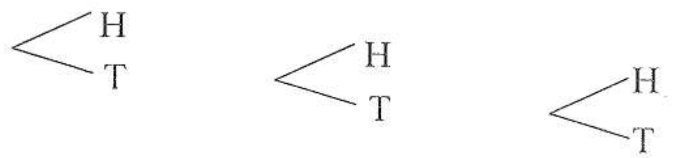
See ex, "2" above.

Ex.3: Toss a coin until first H appears.

Let x: number of tosses. show that x is d.r.v.

Sol.:

x = 1 x = 2 x = 3




$x \in X$: 1, 2, 3, 4, ….

$\therefore R_x = \{x;\ x \in N\}$ countable

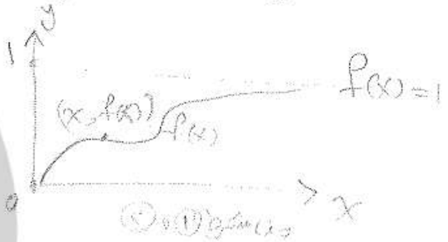
$\therefore$ x is d.r.v.

***Def.: Probability Mass function (P. M. f).***

Let X be a d.r.v.

A function f is a p.m.f. of X if $f(x) = p(X = x)$, and satisfy the following conditions:-

1. $f(x) \geq 0,\ \forall x \in X$ , 2. $\sum_{\forall x \in X} f(x) = 1$




Note: 1. condition (1) shows the graph of f(x) above of the x – axis.

2. Also, if $A \subset S$ then $P(x \in A) = \sum_{x \in A} f(x) = 1$

Ex.: Given $f(x) = \begin{cases} \dfrac{x}{10}, & \text{for } x = 0, 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$

Show that f(x) is a p.M.f.

p.d.f. = {(0, 6/36), (1, 10/36), (2, 8/36), …}

ex. Two dices are thrown. Find the p.M.f. f(x) and sketch f(x).

let $X = |d_1 - d_2|$

Sol. $S = \{(d_1, d_2);\ 1 \leq d_i \leq 6\}$, $n(S) = 36 = 6^2$

$R_X = \{0, 1, 2, 3, 4, 5\}$ = discrete

$f(0) = P(X = 0) = \frac{6}{36} > 0$

$f(1) = P(X = 1) = \frac{10}{36} > 0$

$f(2) = P(X = 2) = \frac{8}{36}$

$f(3) = P(X = 3) = \frac{6}{36} > 0$

$f(4) = P(X = 4) = \frac{4}{36}$

$f(5) = P(X = 5) = \frac{2}{36}$

$\sum_{i=0}^{5} f(x_i) = 1$

cond. "1": T. p. $f(x) \geq 0 \ \forall x \in X$

$$f(0)=0,\ f(1)=\frac{1}{10},\ f(2)=\frac{2}{10},\ f(3)=\frac{3}{10},\ f(4)=\frac{4}{10}$$

$\therefore\ f(x) \geq 0 \ \forall x \in X$

$\therefore$ cond (1) satisfied.

cond "2": T.p $\sum_{x=0}^{4} f(x) = 1$

$$\sum_{x=0}^{4} f(x) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = \frac{10}{10} = 1.$$

$\therefore$ f(x) is a p. M.f

$$* \ p(x=1) = f(1) = \frac{1}{10}$$

$* \ p(x = 8) = f(8) = 0$

$* \ p(x \geq 3) = p[(x = 3) \cup (x = 4)] = p(x = 3) + p(x = 4)$

$$= f(3) + f(4) = \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$$

or by note 2 $p(x \geq 3) = \sum_{x=3}^{4} f(x) = f(3) + f(4) = \frac{7}{10}$

$* \ p(x \leq 2) = p[(x = 2) \cup (x = 1)] = p(x = 2) + p(x = 1)$

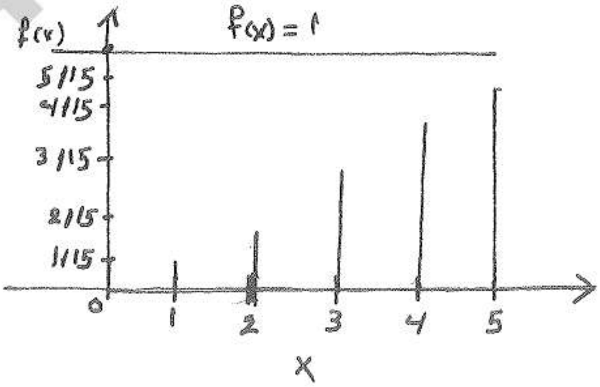
$$= f(2) + f(1) = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$$

كيف نثبت ان دالة صحيحة (p.m.f) ؟

Ex.: Given a p. m . f. $f(x) = \begin{cases} \frac{x}{k}, & \text{for } x = 1,2,3,4,5 \\ 0, & \text{otherwise} \end{cases}$

find the value of k and sketch f(x).

Sol.: $\because$ f(x) is a p. m .f.




$\therefore$ by cond "2" we get $\sum_{x=1}^{5} f(x) = 1$

$f(1) + f(2) + f(3) + f(4) + f(5) = 1$

$$\frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} = 1 \ \Rightarrow \ \therefore k = 15$$

$$\therefore f(x) = \begin{cases} \frac{x}{15}, & \text{for } x = 1,2,3,4,5 \\ 0, & \text{otherwise} \end{cases}$$

Ex.: Toss a coin 3-times. Let x = number of H. find the p. M.f. of x and sketch it's graph.

Sol.: S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$\therefore$ S has (8) elements

x = no. of H, $x \in X$; x= 0, 1, 2, 3 $\to x_1, x_2, x_3, x_4$ فاهي مجموعة الاحتمالية ؟ $P(X = x_i)$

Rx = {x; x = 0, 1, 2, 3}, Rx is a countable

x is d. r. v.

Event (X = x): to get xH, x = 0, 1, 2, 3 when toss a coin 3- times

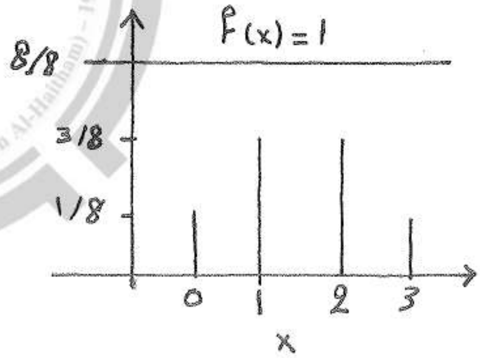
$\binom{3}{x}$ : number of samples in event (X = x) when toss a coin 3-times

$$f(x) = p(X = x) = \begin{cases} \dfrac{\binom{3}{x}}{8} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

pr. d. f. = $\{(x_i, f(x_i)); \forall x \in X\}$

| x | $f(x) = \dfrac{\binom{3}{x}}{8}$ |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |

$$\sum_{x=0}^{3} f(x) = 1$$




Def.: "probability distribution"

A probability dist. of a r.v.X is a set of all ordered pair of x and f(x), $\forall x \in X$.

i. e.: pr. dist.of x = $\{(xi, f(xi)); \forall xi \in X\}$ of X.

In ex. above: pr. dist. of $x = \left\{(0,\frac{1}{8}), (1,\frac{3}{8}),(2,\frac{3}{8}),(3,\frac{1}{8})\right\}$

Ex.: Given (3) red and (5) white balls choose a sample of any (4) balls, let x = number of white ball in a sample.

a. Find the P. m. f of x

b. Find the pr. that a sample has (2) white balls.

c. Find the pr. that a sample has at least (3) white balls.

Sol.:

a.


n=8 k=4

$\binom{8}{4}$: no. of sample in S

x = no. of w ball in a sample, x = 1, 2, 3, 4

Event (X = x): to get x w ball and ( 4-x) r ball in a sample.

$\therefore$ Number of samples $\in$ (X = x) is $\binom{5}{x}\binom{3}{4-x}$

$$f(x) = P(X = x) = \begin{cases} \dfrac{\binom{5}{x}\binom{3}{4-x}}{\binom{8}{4}}, & for\ x = 1,2,3,4 \\ 0 & otherwise \end{cases}$$

b. $p(x = 2) = f(2) = \dfrac{\binom{5}{2}\binom{3}{2}}{\binom{8}{4}}$

c. $p(x \ge 3) = \sum_{x=3}^{4} f(x) = f(3) + f(4)$

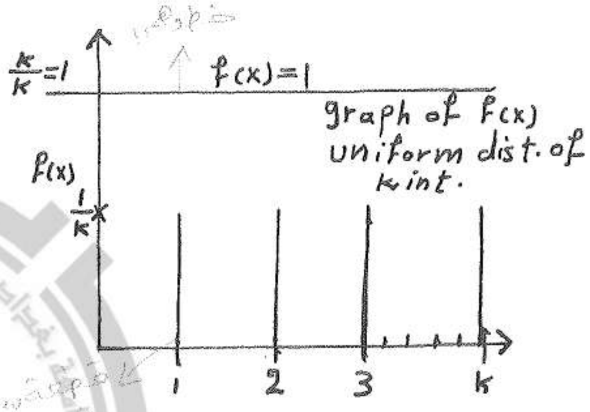
## Uniform Distribution of discreet random variable:

Def.: Given k integers: 1, 2, 3, …., k choose one integer

Let x = the choosen int., S = {1, 2, 3, …., k}

$$\therefore f(x) = p(X = x) = \begin{cases} \dfrac{1}{k}, & for\ x = 1, 2, 3, .......k \\ 0 & otherwise \end{cases}$$

is called uniform dist. of k integers.




To show f(x) is a p. m. f. ?

cond. "1": T. p. $f(x) \geq 0$

$$\because k > 0 \Rightarrow \frac{1}{k} > 0$$

$$\therefore f(x) = \frac{1}{k} > 0 \qquad for\ x = 1,2,3,....,k$$
$$= 0 \qquad otherwise$$
$$\therefore f(x) \geq 0$$

cond "2": T. p $\sum_{x=1}^{k} f(x) = 1$

$$\sum_{x=1}^{k} f(x) = \underbrace{\frac{1}{k} + \frac{1}{k} + .... + \frac{1}{k}}_{k-times} = \frac{k}{k} = 1$$

$\therefore f(x)$ *is a p.m.f.*

Exercise: 1

1. A r.v.x. has a discret dist. with p.m.f., $f(x) = \begin{cases} cx, & x = 1, 2, 3, 4, 5 \\ 0, & o.w. \end{cases}$
   Find the pr. dist. of X.

2. x has a uniform dist. on six integers: 2, 3, 4. 5, 6, 7 find the p.m .f. of x.

3. Given a set of integers 2, 3,…., 15 choose one integer which divisable by 3.

Let x = the chosen int. find the p.m.f. of x.

4. Given a set of integers $\{1, 2, \ldots, 10\}$ choose an integer and determine it's divisors.

Let x = number of divisors. find the p.M.f. of x.

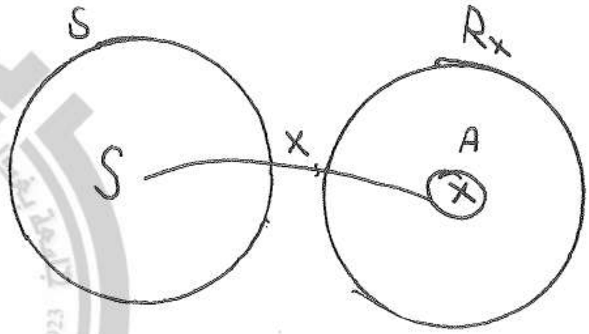
Probability density function (P.d.f):-

Let x be a c.r.v.

A function f(x) is a p.d.f of x if for any interval $A \subset Rx$ $p(x \in A) = \int_A f(x)dx$. and satisfy the following conditions:

1. $f(x) \geq 0 \ \forall x \in R$, 2. $\int_{-\infty}^{\infty} f(x)dx = 1$

$X(s) = x \in Rx$, Rx is uncountable $A \subset Rx$




A is a set of real no.

Suppose that $A = \{x; a < x < b\}$.

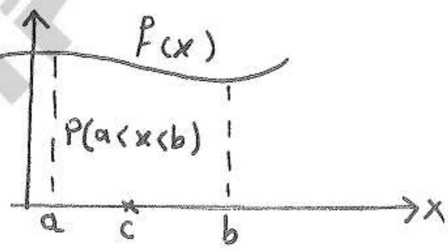
$p(x \in A) = p(a < x < b) = \int_a^b f(x)dx$

= Area under curve from a to b.

f(x) is cont. over (a, b)

let $c \in (a, b)$

f(x) is cont. at (c)




$p(x = c) =$ no. Area $= 0 \Rightarrow \int_c^c f(x)dx = 0$

$p(a \leq x \leq b) = p(x = a) + p(a < x < b) + p(x = b)$

$= p(a < x < b)$

Note: When x is a d. r. v.

1. $p(a \leq x \leq b)$ not necessary equal to $p(a < x < b)$.

2. $p(x = c) = f(c)$ not necessary equal to zero.

Ex "1": Given a p.d.f, $f(x)=\begin{cases} kx & \text{for } 0<x<2 \\ k & \text{for } 2\le x<4 \\ 0 & \text{otherwise} \end{cases}$

a. find the value of k and sketch f(x).

b. find $p(x>1)$, $p(x<3)$, $p(\frac{3}{2}<x<\frac{5}{2})$, $p(x<1 \mid \frac{1}{2}<x<\frac{3}{2})$

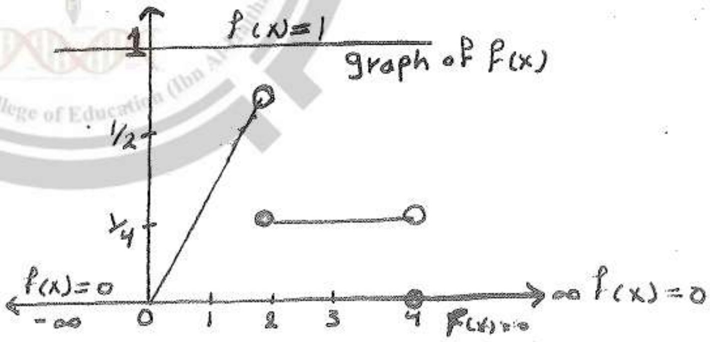
Sol.: a. by cond. "2" of p.d.f. $\Rightarrow \int_{-\infty}^{\infty} f(x)dx=1$

$$1=\int_0^2 kxdx+\int_2^4 kdx \Rightarrow 1=\frac{k}{2}x^2 \Big|_0^2 + kx \Big|_2^4$$

$$\Rightarrow 1=\frac{k}{2}[4-0]+k[4-2] \Rightarrow 1=2k+2k$$

$$\Rightarrow 1=4k \Rightarrow k=\frac{1}{4}$$

$$f(x)=\begin{cases} \frac{x}{4} & \text{for } 0<x<2 \\ \frac{1}{4} & \text{for } 2\le x<4 \\ 0 & \text{o.w.} \end{cases}$$




MATHEMATICS

b. $p(x>1)=\int_1^2 \frac{x}{4}dx+\int_2^4 \frac{1}{4}dx$

$$=\frac{1}{8}x^2 \Big|_1^2+\frac{1}{4}x \Big|_2^4=\frac{1}{8}(4-1)+\frac{1}{4}(4-2)=\frac{3}{8}+\frac{1}{2}=\frac{7}{8}$$

$p(x<3)=\int_0^2 \frac{x}{4}dx+\int_2^3 \frac{1}{4}dx$

$$=\frac{1}{8}x^2 \Big|_0^2+\frac{1}{4}x \Big|_2^3=\frac{1}{8}(4-0)+\frac{1}{4}(3-2)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$p(\frac{3}{2}<x<\frac{5}{2})=\int_{\frac{3}{2}}^{2} \frac{x}{4}dx+\int_{2}^{\frac{5}{2}} \frac{1}{4}dx$

$$=\frac{1}{8}x^2 \Big|_{\frac{3}{2}}^{2}+\frac{1}{4}x \Big|_{2}^{\frac{5}{2}}=\frac{1}{8}(4-\frac{9}{4})+\frac{1}{4}(\frac{5}{2}-2)$$

$$=\frac{1}{8}(\frac{7}{4})+\frac{1}{4}(\frac{1}{2})=\frac{7}{32}+\frac{1}{8}=\frac{11}{32}$$

$P(x<1|\frac{1}{2}<x<\frac{3}{2})=P(A|B)=\frac{P(AB)}{P(B)}$

$A=\{x;\ x<1\}=\{x;\ 0<x<1\}$, $B=\{x;\ \frac{1}{2}<x<\frac{3}{2}\ \}$

$AB=\{x,\ \frac{1}{2}<x<1\}$

$P(B)=\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{x}{4}dx=\frac{1}{8}x^2 \Big|_{\frac{1}{2}}^{\frac{3}{2}}=\frac{1}{8}(\frac{9}{4}-\frac{1}{4})=\frac{1}{4}$

$P(AB)=\int_{\frac{1}{2}}^{1} \frac{x}{4}dx=\frac{1}{8}x^2 \Big|_{\frac{1}{2}}^{1}=\frac{1}{8}(1-\frac{1}{4})=\frac{1}{8}(\frac{3}{4})=\frac{3}{32}$

$P(x<1|\frac{1}{2}<x<\frac{3}{2})=\frac{3/32}{1/4}=\frac{3}{8}$

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Ex "2": Given a p. d. f $f(x)=\begin{cases} ke^{-x} & \text{for } x>0 \\ 0 & \text{for o.w} \end{cases}$

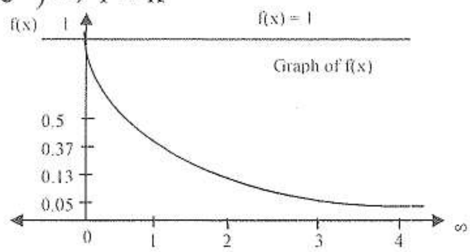
a. find the value of k and sketch f(x)

b. find p(x < 2).

Sol.: a. by cond. "2" of p.d.f $\int_{-\infty}^{\infty} f(x)dx = 1$

$$1 = k\int_{-\infty}^{\infty} e^{-x}dx \Rightarrow 1 = -ke^{-x}\Big|_0^{\infty} \Rightarrow 1 = -k(e^{-\infty} - e^{0}) \Rightarrow 1 = k$$

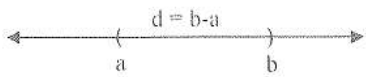
$$f(x)=\begin{cases} e^{-x} & \text{for } x>0 \\ 0 & \text{for o.w} \end{cases}$$




| x | $f(x) = e^{-x}$ |
|---|---|
| 0 | $e^{0} = 1$ |
| 1 | $e^{-1} = \frac{1}{e} = \frac{1}{2.7} = 0.37$ |
| 2 | $e^{-2} = 0.13$ |
| 3 | $e^{-3} = 0.05$ |
| . | |
| ∞ | $e^{-\infty} = 0$ |

$$b.\ \ p(x<2) = \int_{-\infty}^{2} f(x)dx = \int_{-\infty}^{0} o\,dx + \int_{0}^{2} e^{-x}dx = -e^{-x}\Big|_0^2$$
$$= -\left[e^{-2} - e^{0}\right] = 1 - e^{-2} = 1 - 0.13 = 0.87$$

Uniform distribution on interval (a, b):




Given an interval (a, b), b > a

Choose a point x from (a, b) then a < x < b

A function $f(x)=\begin{cases} \dfrac{1}{b-a} & \text{for } a<x<b \\ 0 & \text{o.w} \end{cases}$

Is called uniform dist. on (a, b)

To show that f(x) a p.d.f

Cond. "1": T.p $f(x) \geq 0$

$$\because b - a > 0 \ \text{since}\ a > b$$

$$\therefore \frac{1}{b-a} > 0 \Rightarrow f(x) = \frac{1}{b-a} > 0 \quad \text{for}\ a < x < b$$

$f(x) \geq 0$ $\qquad = 0 \qquad$ o.w

Cond "2": T.p $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} 0dx + \int_{a}^{b} \frac{1}{b-a}dx + \int_{b}^{\infty} 0dx$$

$$= \int_{a}^{b} \frac{1}{b-a}dx = \frac{1}{b-a} x \Big|_{a}^{b}$$

$$= \frac{1}{b-a}(b-a) = 1$$

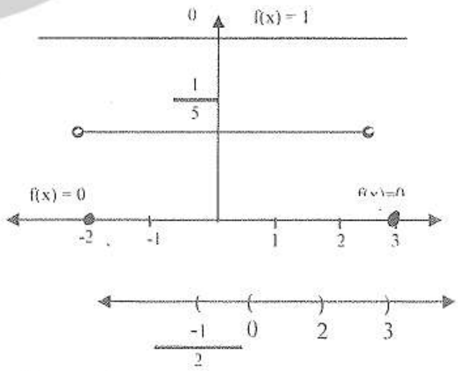
$\therefore f(x)$ is a p.d.f

Ex "1": If x has a uniform dist. on (-2,3) [x ~ unif. (-2,3)]

a. find a p.d.f of x and sketch f(x).

b. Find $f(x > 0 \mid -\frac{1}{2} < x < 2)$

Sol.: a. $$f(x) = \begin{cases} \frac{1}{3-(-2)} = \frac{1}{5} & \text{for} -2 < x < 3 \\ 0 & \text{o.w} \end{cases}$$




b. $$p(x > 0 \mid -\frac{1}{2} < x < 2) = \frac{p(AB)}{P(B)}$$

$$A = \{x;\ 0 < x < 3\}$$

$$B = \left\{x;\ -\frac{1}{2} < x < 2\right\}$$

$$AB = \{x;\ 0 < x < 2\}$$

$$P(B) = \int_{-\frac{1}{2}}^{2} \frac{1}{5}dx = \frac{1}{5} x \Big|_{-\frac{1}{2}}^{2} = \frac{1}{5}(2 + \frac{1}{2}) = \frac{1}{5}(\frac{5}{2}) = \frac{1}{2}$$

$$P(AB) = \int_{0}^{2} \frac{1}{5}dx = \frac{1}{5} x \Big|_{0}^{2} = \frac{1}{5}(2-0) = \frac{2}{5}$$

$$\therefore p(x < 0 \mid -\frac{1}{2} < x < 2) = \frac{2/5}{1/2} = \frac{4}{5}$$

H.W.: 2

1. $x \sim$ unif. $(-1, 3)$ find p.d.f of x and find $p(0 \le x < 2)$, $p(x < 1 | -1 < x < 2)$.

2. Given $f(x) = \dfrac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$. Show that f(x) is a p.d.f.

Note: f(x) above is called cauchy dist.

Comulative Distribution Function (c. d. f)

***(Distribution Function (d. f))*** or (Comulative Distribution fun.) c.d.F

Def.: Let x be a r. v. either d. r.v. or c.r.v

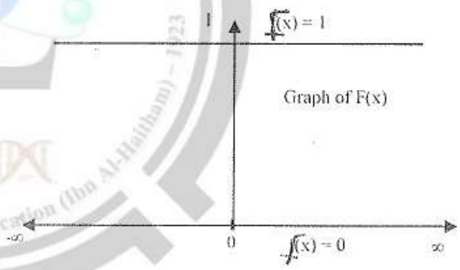
A function F(x) is a c.d.f of x iff

$F(x) = p(X \le x); -\infty < x < \infty$

Event $(X \le x)$ = all point in $(-\infty, x]$, $0 \le p(X \le x) \le 1 \Rightarrow 0 \le F(x) \le 1$

That is mean F(x) is bounded by zero and one.

i.e.: The graph of F(x) lies between the line F(x) = 0 and the line F(x) = 1




Note: We shall use F(x) to denote of a c.d.f of x and f(x) to denote of p.m.f or p.d.f .

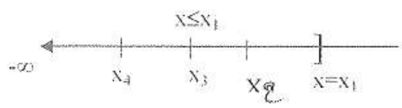
To find F(x) when x is d.r.v.

Let x be a d.r.v. with p.m.f f(x)

$Rx = \{xj \in I, j = 1, 2, 3, \ldots\}$ countable

$F(x) = p(X \le x)$ by def.

Choose x and let $x = x_1$ ($x_1 \in I$)




Event $(X \le x) = \{xj \le xi, i,j = 1,2,\ldots\}$

$(X \le x) = \{(X=x_1) \cup (X=x_2) \cup (X=x_3) \cup \ldots \cup (X=xj) \cup \ldots\}$

$$P(X \le x) = P(X=x_1)+P(X=x_2)+P(X=x_3)+\ldots+P(X=xj)+\ldots$$
$$= f(x1) + f(x2) + f(x3) + \ldots.$$

$$p(X \le x) = \sum_{xj \le x} f(xj)$$

$$F(x) = \sum_{xj \le x} f(xj)$$ c.d.f of x when it's d.r.v.

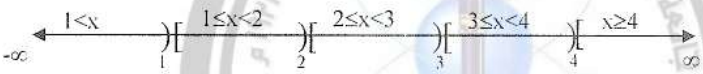
Ex. "1": Given a p.m.f $f(x) = \begin{cases} \dfrac{x}{10} & \text{for } x = 1,2,3,4 \\ 0 & \text{o.w} \end{cases}$

a. Find the c.d.f F(x) and sketch it's graph.

b. Find $p(x \le 2)$, $p(x > 3)$, $p(x < 2)$, $p(1 < x \le \frac{3}{2})$, $p(1 \le x < \frac{3}{2})$

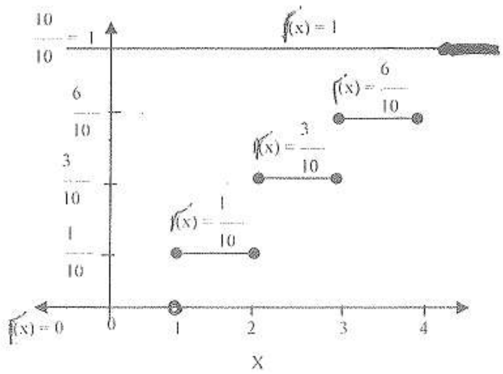
Sol.: a. Rx = {x, x = 1, 2, 3, 4} count.

$$F(x) = \sum_{xj \le x} f(xj)$$




| Interval of x | $xj \in I$ | $f(x) = \frac{x}{10}$ | $F(x) = \sum_{xj \le x} f(xj)$ |
|---|---|---|---|
| $x < 1$ | 0 | 0 | $F(x) = 0$ |
| $1 \le x < 2$ | 1 | $\frac{1}{10}$ | $F(x) = 0 + \frac{1}{10} = \frac{1}{10}$ |
| $2 \le x < 3$ | 2 | $\frac{2}{10}$ | $F(x) = 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$ |
| $3 \le x < 4$ | 3 | $\frac{3}{10}$ | $F(x) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = \frac{6}{10}$ |
| $x \ge 4$ | 4 | $\frac{4}{10}$ | $F(x) = 1$ |

$$F(x) = \begin{cases} 0 & for\, x < 1 \\ \frac{1}{10} & for\, 1 \le x < 2 \\ \frac{3}{10} & for\, 2 \le x < 3 \\ \frac{6}{10} & for\, 3 \le x < 4 \\ 1 & for\, x \ge 4 \end{cases}$$



NOTE: 1- F(X) discont. at x=1,2,3,4.

2- F(X) is cont. to the right for each interval of continiouity.

b. $p(x \leq 2) = F(2) = \frac{3}{10}$

($F(a) = p(x \leq a)$) or $p(x \leq 2) = \sum_{X=1}^{2} f(x) = f(1) + f(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$

$$P(X>3) = 1 - P(X \leq 3) = 1 - F(3) = 1 - \frac{6}{10} = \frac{4}{10}$$

Or $p(x>3) = p(x=4) = f(4) = \frac{4}{10}$

$P(x<2) \neq F(2)$ $[p(x \leq x) = F(x)]$

$P(x<2) = p(x=1) = F(1) = \frac{1}{10}$

$P(1 < x \leq \frac{3}{2}) = p(\phi) = 0$

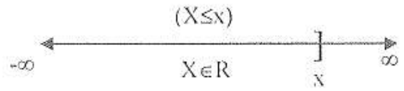
Since $\nexists$ integer $\in (1, \frac{3}{2}]$

$P(1 \leq x < \frac{3}{2}) = p(x=1) = f(1) = \frac{1}{10}$

H.W: Given an integers ,5,6,7,…..,15 Choose one even integer. Let x be the chosen integer. Find F(x) and sketch it's $\rightarrow$ graph.

***To find F(x) when x is c.r.v.***

Let x be a c.r.v. with p.d.f f(x)




$F(x) = p(X \leq x)$ by def.

To find $p(X \leq x)$

$\because$ x has a p.d.f $f(x) \Rightarrow p(X \le x) = \int_{-\infty}^{x} f(t)dt$

$\therefore F(x) = \int_{-\infty}^{x} f(t)dt$ [c.d.f of x when x is c.r.v]

f(x) is cont. $\forall x \in (-\infty, x)$

Ex. "1": Given a p.d.f $f(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

a. Find F(x) and sketch it's graph.

b. Find $p(x \le \frac{1}{3}),\ p(x < \frac{1}{3}),\ p(x > 1),\ p(x \le -\frac{1}{2}),\ p(-\frac{1}{3} < x < \frac{1}{4})$

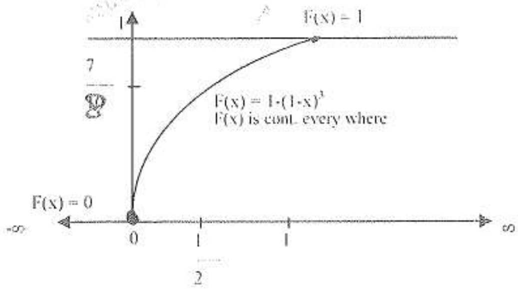
Sol.: a. $F(x) = \int_{-\infty}^{x} f(t)dt$

$= \int_{-\infty}^{0} 0dt + \int_{0}^{x} 3(1-t)^2 dt = 3\int_{0}^{x}(1-t)^2 dt$

$= -3\frac{(1-t)^3}{3} |_0^x = -[(1-x)^3 - 1] = 1-(1-x)^3$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-(1-x)^3 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

| x | $F(x) = 1-(1-x)^3$ |
|---|---|
| 0 | 0 |
| $\frac{1}{2}$ | $\frac{7}{8}$ |
| 1 | 1 |



b. $p(x \le \frac{1}{3}) = F(\frac{1}{3}) = 1-(1-\frac{1}{3})^3 = 1-\frac{8}{27} = \frac{19}{27}$

$p(x < \frac{1}{3}) \ne F(\frac{1}{3})$

$$p(x < \frac{1}{3}) = 3\int_0^{\frac{1}{3}}(1-x)^2\,dx = -3\frac{(1-x)^3}{3}\Big|_0^{\frac{1}{3}} = -(1-x)^3\Big|_0^{\frac{1}{3}}$$

$$= -[(1-\frac{1}{3})^3 - 1] = -[(\frac{2}{3})^3 - 1] = 1 - \frac{8}{27} = \frac{19}{27}$$

$p(x > 1) = 1 - p(x \le 1) = 1 - F(1) = 1 - 1 = 0$

$p(x \le -\frac{1}{2}) = F(-\frac{1}{2}) = 0$

$$p(-\frac{1}{3} < x < \frac{1}{4}) = \int_{-\frac{1}{3}}^{\frac{1}{4}} f(x)\,dx = \int_{-\frac{1}{3}}^{0} o\,dx + \int_0^{\frac{1}{4}} 3(1-x)^2\,dx$$

$$= -(1-x)^3\Big|_0^{\frac{1}{4}} = -[(1-\frac{1}{4})^3 - 1] = 1 - (\frac{3}{4})^3$$

$$= 1 - \frac{27}{64} = \frac{37}{64}$$

Note: If $F$ is conts at $x = a$, then $P(X \le a) = P(X < a) = F(a)$
only when $X$ is contin

H.W4

1. Given a p.d.f $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w} \end{cases}$

a. Find c.d.f of x

b. Find $p(x > 4.2)$ and $p(x < 3.5)$ and $p(x \le -1)$

2. Given a p.d.f $f(x) = \begin{cases} Ce^{-x} & \text{for } x > 0 \\ 0 & \text{o.w} \end{cases}$

a. Find the value of c and sketch F(x).

b. Find F(x) and sketch it's graph.

c. $p(x \le 3)$, $p(x < 3)$.

# Chapter Three

## « Random Variables and Probability Distribution »

### Exercise/P.6

Q₁: A r.v. X has a discret distribution with p.m.f.
P.6

$$f(x) = \begin{cases} cx & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Find the pr. dist. of X.

Sol. $\because f(x)$ is a p.m.f.

$\therefore$ Condition ② is satisfied $\sum_{x=1}^{5} f(x) = 1$

$$\sum_{x=1}^{5} f(x) = f(1) + f(2) + f(3) + f(4) + f(5) = 1$$
$$= c[1+2+3+4+5] = 1$$
$$= c\left[\frac{5(5+1)}{2}\right] = 1$$
$$= 15c = 1 \Rightarrow c = \frac{1}{15}$$

$$\therefore f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{pr. dist. of } X = \{(x_i, f(x_i)),\ \forall x_i \in X\}$$
$$= \left\{\left(1, \tfrac{1}{15}\right), \left(2, \tfrac{2}{15}\right), \left(3, \tfrac{3}{15}\right), \left(4, \tfrac{4}{15}\right), \left(5, \tfrac{5}{15}\right)\right\}$$

---

Q₂: X has a uniform dist. on six integers: 2, 3, 4, 5, 6, 7. Find the p.m.f. of X.

Sol. $X = 2, 3, 4, 5, 6, 7 \Rightarrow$ 6-integers

$X \sim$ uniform (6)

$$\therefore f(x) = \begin{cases} \frac{1}{K} & \text{for } x = 1, 2, \dots, K \\ 0 & \text{o.w} \end{cases} = \begin{cases} \frac{1}{6} & \text{for } x = 2, 3, 4, 5, 6, 7 \\ 0 & \text{o.w.} \end{cases}$$

---

$Q_3$: Given a set of integers $2, 3, \ldots, 15$ Choose one integer which divisable by 3.

Let $X$ = the chosen int. find the p.m.f. of $X$.

Sol. $X = \{3, 6, 9, 12, 15\}$ = d.r.v.

$R_x = \{3, 6, 9, 12, 15\}$ is Countable.

$X \sim$ uniform $(5)$

$$\underset{\text{p.m.f.}}{f(x)} = \begin{cases} \frac{1}{5} & \text{for } x = 3, 6, 9, 12, 15 \\ 0 & \text{o.w.} \end{cases}$$

---

$Q_4$: Given a set of integers $\{1, 2, \ldots, 10\}$, choose an integer and determine it's divisors.

Let $X$ = no. of divisors

Find the p.m.f. of $X$.

Sol. $X$ = no. of divisors. $X$ = r.v.

| integer | divisor | no. of divisor |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1, 2 | 2 |
| 3 | 1, 3 | 2 |
| 4 | 1, 2, 4 | 3 |
| 5 | 1, 5 | 2 |
| 6 | 1, 2, 3, 6 | 4 |
| 7 | 1, 7 | 2 |
| 8 | 1, 2, 4, 8 | 4 |
| 9 | 1, 3, 9 | 3 |
| 10 | 1, 2, 5, 10 | 4 |

$f(x)$ = p.m.f. = $\frac{x \text{ تكرار}}{10}$

$X$

$x = 1 \xrightarrow{\text{تكرار}} 1 \Rightarrow \frac{1}{10}$

$x = 2 \xrightarrow{\text{تكرار}} 4 \Rightarrow \frac{4}{10}$

$x = 3 \xrightarrow{\text{تكرار}} 2 \Rightarrow \frac{2}{10}$

$x = 4 \xrightarrow{\text{تكرار}} 3 \Rightarrow \frac{3}{10}$

$$\therefore \underset{\text{p.m.f.}}{f(x)} = \begin{cases} \frac{1}{10} & \text{for } x = 1 \\ \frac{4}{10} & \text{for } x = 2 \\ \frac{2}{10} & \text{for } x = 3 \\ \frac{3}{10} & \text{for } x = 4 \\ 0 & \text{o.w.} \end{cases}$$

Q1: $X \sim unif.(-1,3)$. Find p.d.f. of X and find $P(0 \leq X < 2)$, $P(X<1 / -1<X<2)$

Sol. given $X \sim unif.(-1,3)$

$$\therefore f(x) = \begin{cases} \frac{1}{3-(-1)} & \text{for } -1<x<3 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \frac{1}{4} & \text{for } -1<x<3 \\ 0 & \text{o.w.} \end{cases}$$

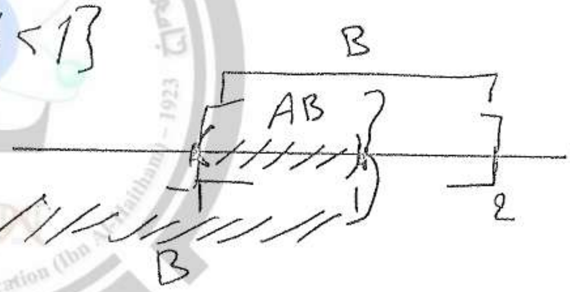
$$P(0 \leq X<2) = \int_0^2 f(x)\,dx = \int_0^2 \frac{1}{4}\,dx = \frac{1}{4}x \Big|_0^2 = \frac{1}{4}[2-0] = \frac{1}{2}$$

$$P(\underbrace{X<1}_{A} / \underbrace{-1<X<2}_{B}) = P(A|B) = \frac{P(AB)}{P(B)}$$

$A = \{x : x<1\} = \{x : -1<x<1\}$

$B = \{x : -1<x<2\}$

$AB = \{x : -1<x<1\}$




$$P(B) = \int_{-1}^{2} \frac{1}{4}\,dx = \frac{1}{4}x \Big|_{-1}^{2} = \frac{1}{4}[2+1] = \frac{3}{4} \in [0,1]$$

$$P(AB) = \int_{-1}^{1} \frac{1}{4}\,dx = \frac{1}{4}x \Big|_{-1}^{1} = \frac{1}{4}(1+1) = \frac{1}{2} \in [0,1]$$

$$P(X<1 / -1<X<2) = \frac{1/2}{3/4} = \frac{2}{3} \in [0,1]$$

---

Q2: Given $f(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty<x<\infty$. Show that $f(x)$ is a p.d.f.

Note: $f(x)$ is called cauchy dist.

Sol. T.P. cond.① $f(x) \geq 0 \quad \forall x \in R_x = (-\infty,\infty)$

$x^2 \geq 0 \Rightarrow \pi(1+x^2) \geq 0 = \dfrac{1}{\pi(1+x^2)} \geq 0$

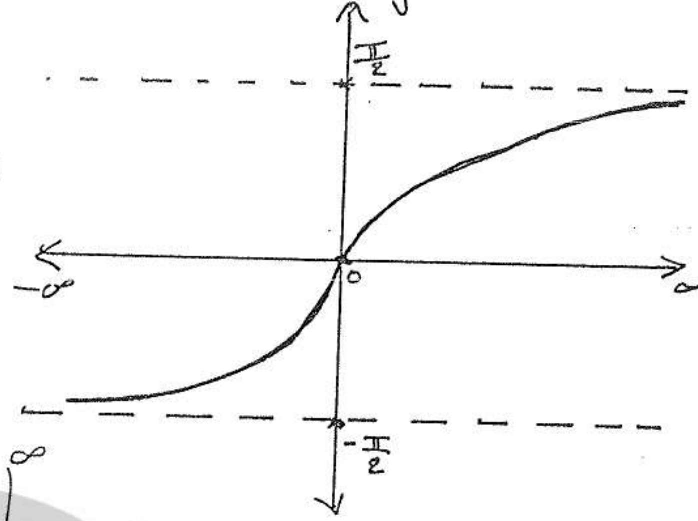
$= f(x) \geq 0$

$x \in (-\infty, \infty)$

if $x < \infty \rightarrow f(x) \geq 0$

if $x > -\infty \rightarrow f(x) \geq 0$

$\therefore f(x) \geq 0 \quad \forall x \in (-\infty, \infty)$




T.P. Cond. ② is satisfied

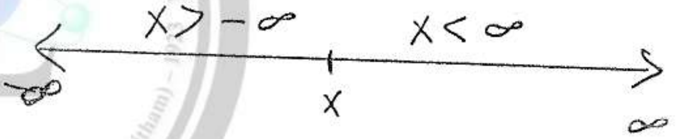
<u>i.e.</u> <u>T.P.</u> $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{1}{1+x^2}\,dx = \dfrac{1}{\pi} \tan^{-1} x \Big]_{-\infty}^{\infty}$

$= \dfrac{1}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right] = \dfrac{1}{\pi}\left[\dfrac{\pi}{2} - \left[-\dfrac{\pi}{2}\right]\right] = \dfrac{1}{\cancel{\pi}}(\cancel{\pi}) = 1$

$\therefore$ Cond. ② is satisfied.

$\therefore f(x)$ is a p.d.f.

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<u>H.w. P.14</u> Given an integer 5, 6, 7, ..., 15. Choose one even integer. Let X be the chosen integer. Find F(x) and sketch it's graph.

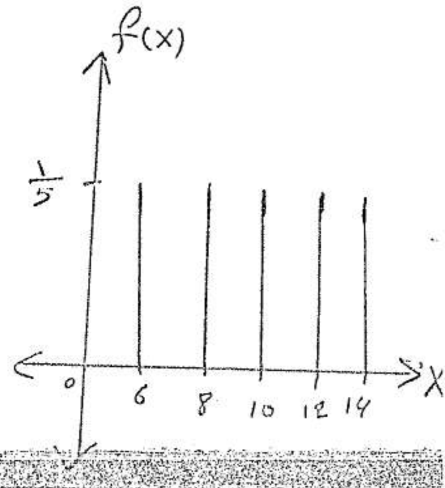
<u>Sol.</u> $X$ = The even integer

$\therefore X = 6, 8, 10, 12, 14$

$R_X = \{x : x = 6, 8, 10, 12, 14\}$ = d.r.v.

$X \sim$ unif. (5)

$\therefore f(x) = \begin{cases} \dfrac{1}{5} & \text{for } x = 6, 8, 10, 12, 14 \\ 0 & \text{o.w.} \end{cases}$

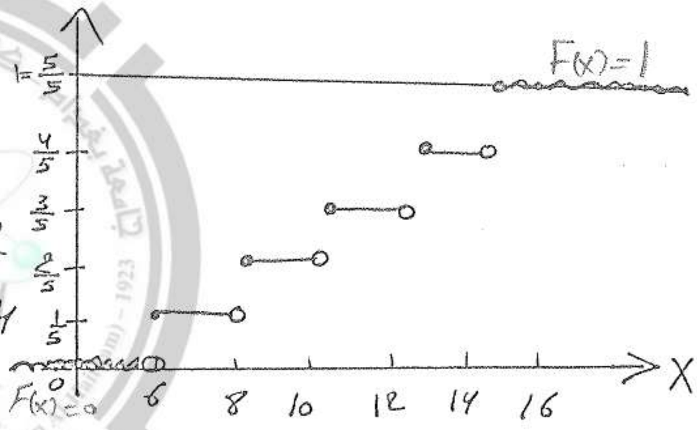


| interval of X | X integer | p.m.f $f(x)=\frac{1}{5}$ | $F(x)=$ c.d.f. |
|---|---|---|---|
| $X<6$ | no. int. | $0$ | $0 \leftarrow$ تبدأ بالصفر |
| $6\le X<8$ | $6$ | $\frac{1}{5}$ | $0+\frac{1}{5}=\frac{1}{5}$ |
| $8\le X<10$ | $8$ | $\frac{1}{5}$ | $0+\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$ |
| $10\le X<12$ | $10$ | $\frac{1}{5}$ | $0+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}$ |
| $12\le X<14$ | $12$ | $\frac{1}{5}$ | $0+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{4}{5}$ |
| $X\ge 14$ | $14$ | $\frac{1}{5}$ | $0+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{5}{5}=1$ |

تنتهي بالواحد

$$F(x)=\begin{cases} 0 & \text{for } X<6 \\ 1/5 & \text{for } 6\le X<8 \\ 2/5 & \text{for } 8\le X<10 \\ 3/5 & \text{for } 10\le X<12 \\ 4/5 & \text{for } 12\le X<14 \\ 1 & \text{for } X\ge 14 \end{cases}$$

(c.d.f.)




$0\le F(x)\le 1$

## Exercise / P.16

Q$_1$: Given a p.d.f.

$$f(x)=\begin{cases} \frac{1}{x^2} & \text{for } x>1 \\ 0 & \text{o.w.} \end{cases}$$

a- Find c.d.f. of X

b. Find $P(X>4.2)$ and $P(X<3.5)$ and $P(X\le -1)$

Sol. $F(x)=\int_{-\infty}^{x} f(t)\,dt = \int_{1}^{x}\frac{1}{t^2}\,dt = \frac{-1}{t}\Big|_1^x = -\left[\frac{1}{x}-1\right]=1-\frac{1}{x}$

$$\therefore F(x) = \begin{cases} 1 - \frac{1}{x} & \text{for } x \geq 1 \\ 0 & \text{o.w.} \end{cases}$$

Note: $F(x)$ is conts. at $x = 1$

since $\lim_{x \to 1^+} F(x) = 1 - \frac{1}{1} = 0 = F(1^+)$

$\lim_{x \to 1^-} F(x) = 0 = F(1^-)$

Now, $P(X > 4.2) = 1 - P(X \leq 4.2)$

$= 1 - F(4.2)$

$= 1 - \left(1 - \frac{1}{4.2}\right) = 0.238$

$$P(X < 3.5) = \int_1^{3.5} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{3.5} = -\left[\frac{1}{3.5} - 1\right] = 0.714$$

$P(X \leq -1) = F(-1) = 0$

---

Q2: Given a p.d.f.

$$f(x) = \begin{cases} ce^{-x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

a. Find the value of $c$ and sketch $F(x)$.

b. Find $F(x)$ and sketch it's graph.

c. $P(X \leq 3)$, $P(X < 3)$.

Sol. $\because f(x)$ is p.d.f.

a. $\therefore$ cond. ② is satisfied : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} f(x)\,dx = c\int_0^{\infty} e^{-x}\,dx = -ce^{-x}\Big|_0^{\infty} = -c\left[e^{-\infty} - e^{0}\right] = -c(-1) = c = 1$$

$\therefore \boxed{c = 1}$

$F(1^-) = \lim_{x \to 1^-} F(x) = 0$

$\therefore F(1^+) \neq F(1^-) \Rightarrow F(x)$ is not conts. at $x=1$

---

• H.W / P. 20 Given a c.d.f.

$$F(x) = \begin{cases} 1 - \frac{4}{x^2} & \text{for} \quad x > 2 \\ 0 & \text{for} \quad x \leq 2 \end{cases}$$

1. Find $f(x)$ and sketch $F(x)$ & $f(x)$ (H.W.)
2. Find $P(X \leq 4)$, $P(X > 8)$, $P(2 < X \leq 8)$, $P(X = 2)$.

Sol: $F(X)$ is conts at $x = 2$

Since $F(2^+) = F(2^-)$

$F(2^+) = \lim_{x \to 2^+} F(x) = 1 - \frac{4}{4} = 0$

$F(2^-) = \lim_{x \to 2^-} F(x) = 0$

$$f(x) = F'(x) = \begin{cases} \frac{8}{x^3} & \text{for} \quad x > 2 \\ 0 & \text{for} \quad x \leq 2 \end{cases}$$

$P(X \leq 4) = F(4) = 1 - \frac{4}{16} = 1 - \frac{1}{4} = \frac{3}{4}$ (By def. of $F(x)$)

$P(X > 8) = 1 - P(X \leq 8)$

$= 1 - F(8)$

$= 1 - (1 - \frac{1}{16}) = 1 - \frac{15}{16} = \frac{1}{16}$

$P(2 < X \leq 8) = F(8) - F(2)$ By Th. $P(a < X \leq b) = F(b) - F(a)$

$= (1 - \frac{4}{64}) - 0$

$= 1 - \frac{1}{16} = \frac{15}{16}$

$P(X = 2) = F(2^+) - F(2^-)$ By Th. $P(X = x) = F(x^+) - F(x^-)$

$= 0$

Since $F(x)$ is conts. at $x=2$

$\therefore f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$

b. $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$= \int_{0}^{x} e^{-t}\,dt = -e^{-t}\Big|_{0}^{x} = -[e^{-x} - \underbrace{e^{0}}_{=1}] = 1 - e^{-x}$

$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{o.w.} \end{cases} \Leftarrow$ $F(x)$ is conts at $x = 0$ since $\left\{ \begin{array}{l} F(0^+) = F(0^-) \\ F(0^+) = 1 - e^{0} = 1 - 1 = 0 \\ F(0^-) = 0 \end{array} \right\}$

c. $P(X \leq 3) = F(3) = 1 - e^{-3}$

$P(X < 3) = \int_{0}^{3} e^{-x}\,dx = -e^{-x}\Big|_{0}^{3} = -[e^{-3} - e^{0}]$

$= 0.9502$

or $P(X < 3) = P(X \leq 3) = F(3)$

since (C.R.V.) & $F$ is conts. at $x = 3$

---

H.W / P.17 Given a c.d.f. $F(x)$:

$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 1 \\ 0 & \text{o.w.} \end{cases}$

Find the p.d.f. $f(x)$.

Sol. $F(x)$ is conts. function. $\Rightarrow F(x)$ is diff.

$f(x) = \dfrac{dF(x)}{dx} = F'(x)$

$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$

But is $F(x)$ conts. at $x = 1$?

$F(1^+) = \lim_{x \to 1^+} F(x) = 1 - e^{-1} \neq 0$

$Q_1$: Toss a Coin 3-times. If $X$ be the number of head (H). Find the p.f. of $X$.

Sol. $X \equiv$ number of H.

$x = 0, 1, 2, 3$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

TTT HTT HHT HHH

$R_X = \{ x : x = 0, 1, 2, 3 \}$ = Countable

$\therefore X$ is d.r.v.

$\binom{3}{x} \equiv$ number of sample in event $(X = x)$ when toss a Coin 3-times.

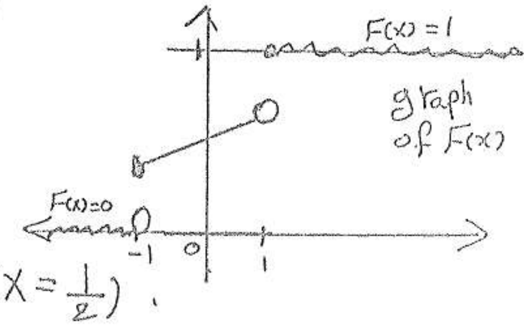
$$P(X=x) = f(x) = \begin{cases} \dfrac{\binom{3}{x}}{8} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

p.f.

---

$Q_2$: Given a d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \dfrac{x+2}{4} & \text{for } -1 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$




① Sketch $F(x)$ and find $P(X=-1)$ and $P(X=\frac{1}{2})$.

② If $F(x)$ conts. at $x = \frac{1}{2}$? why?

Sol. $P(X=-1) = F(-1^{+}) - F(-1^{-})$ By Th. $P(X=x) = F(x^{+}) - F(x^{-})$

$= \dfrac{-1+2}{4} - 0 = \dfrac{1}{4}$

$P(X=\frac{1}{2}) = \dfrac{\frac{1}{2}+2}{4} - \dfrac{\frac{1}{2}+2}{4} = 0$

$F(x)$ is conts. at $x = \frac{1}{2}$ since $P(X=\frac{1}{2}) = 0$

---

$Q_3$: Given a p.d.f. $f(x)$:

$$f(x) = \begin{cases} 1-|x| & \text{for } |x|<1 \\ 0 & \text{o.w.} \end{cases}$$

Find the d.f. $F(x)$ and find $P(X>0)$ & $P(|X|>\frac{1}{2})$.

sol:

$|X|<1 \Rightarrow -1<X<1$

$|x| = -x \Rightarrow -1<X<0$

$|X| = X \Rightarrow 0 \le X < 1$

$$f(x) = \begin{cases} 1+X & \text{for } -1<x<0 \\ 1-X & \text{for } 0 \le X < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$F(X) = \begin{cases} \int_{-1}^{x} (1+t)\,dt & \text{for } -1 \le X < 0 \\ \int_{0}^{x} (1-t)\,dt & \text{for } 0 \le X < 1 \\ 0 & \text{for } X<-1 \\ 1 & \text{for } X \ge 1 \end{cases}$$

Find $P(X>0) = 1-P(X \le 0)$

$= 1-F(0) = 1-\int_{0}^{x}(1-t)\,dt = \ldots$

$P(|X|>\frac{1}{2}) = 1-P(|X| \le \frac{1}{2})$

$= 1-P(-\frac{1}{2} \le X \le \frac{1}{2})$

$= 1-\left[\int_{-\frac{1}{2}}^{0} f(x)\,dx + \int_{0}^{\frac{1}{2}} f(x)\,dx\right]$

$= 1-\left[\int_{-\frac{1}{2}}^{0}(1+x)\,dx + \int_{0}^{\frac{1}{2}}(1-x)\,dx\right]$

$= \ldots$

$Q_4$: Two dice are thrown once, let $X$ be the absolute difference between the two numbers shown by the dice. Find the p.m.f. and c.d.f. of $X$.

Sol. $S = \{ (d_1, d_2) ; 1 \leq d_i \leq 6 ; i = 1, 2 \} = 6^2 = 36$

$$\underset{R.V.}{X} = \left\{ \begin{array}{lll} |1-1| = 0 , & |2-1| = 1 , & \ldots , |6-1| = 5 \\ |1-2| = 1 & |2-2| = 0 & |6-2| = 4 \\ |1-3| = 2 & |2-3| = 1 & |6-3| = 3 \\ \vdots & \vdots & \vdots \\ |1-6| = 5 & |2-6| = 4 & |6-6| = 0 \end{array} \right\}$$

$\therefore X = 0, 1, 2, 3, 4, 5$

Event $(X=0) = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6) \} = (6)$ elements

$f(x) = P(X = x)$

$f(0) = P(X = 0) = \frac{6}{36}$

Event $(X=1) = \{ (2,1), (1,2), (2,3), (3,2), (4,3), (3,4), (4,5), (5,4), (5,6), (6,5) \} = (10)$ elements

$f(1) = P(X=1) = \frac{10}{36}$

Event $(X=2) = \{ (1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4) \} = (8)$ elements.

$f(2) = P(X=2) = \frac{8}{36}$

Event $(X=3) = \{ (1,4), (4,1), (2,5), (5,2), (3,6), (6,3) \} = (6)$ elements.

$f(3) = P(X=3) = \frac{6}{36}$

Event $(X=4) = \{ (1,5), (5,1), (2,6), (6,2) \} = (4)$ elements.

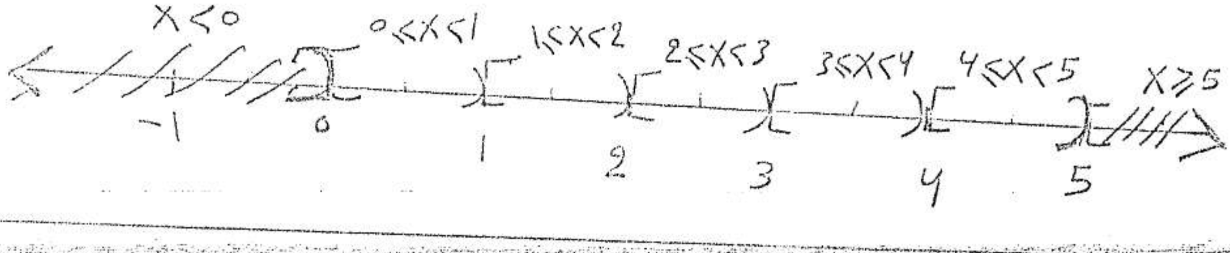
$f(4) = P(X=4) = \frac{4}{36}$

Event $(X=5) = \{ (1,6), (6,1) \} = (2)$ elements

$f(5) = P(X=5) = \frac{2}{36}$

$$\therefore f(x) = \begin{cases} \frac{6}{36} & \text{for } x = 0 \\ \frac{10}{36} & \text{for } x = 1 \\ \frac{8}{36} & \text{for } x = 2 \\ \frac{6}{36} & \text{for } x = 3 \\ \frac{4}{36} & \text{for } x = 4 \\ \frac{2}{36} & \text{for } x = 5 \\ 0 & \text{o.w.} \end{cases}$$

$$\underset{(d.f.)}{F(x)} = \begin{cases} 0 & \text{for } x < 0 \\ \frac{6}{36} & \text{for } 0 \le x < 1 \\ \frac{16}{36} & \text{for } 1 \le x < 2 \\ \frac{24}{36} & \text{for } 2 \le x < 3 \\ \frac{30}{36} & \text{for } 3 \le x < 4 \\ \frac{34}{36} & \text{for } 4 \le x < 5 \\ 1 & \text{for } x \ge 5 \end{cases}$$



Q5: Given a p.f.

$$f(x) = \begin{cases} K\left(\frac{2}{3}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Find the value of K.

Sol: Since $f(x)$ is p.m.f. then Cond. ② $\sum_{\forall x \in R_x} f(x) = 1$ is satisfied.

By Cond. ② we get: $\sum_{x=1}^{\infty} K\left(\frac{2}{3}\right)^x = 1$

$$1 = K \sum_{x=1}^{\infty} \left(\frac{2}{3}\right)^x$$

$$1 = K\left[\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^n + \dots\right]$$

$$1 = \left(\frac{2}{3}\right)K\Big[\underbrace{1 + \left(\frac{2}{3}\right) + \dots + \left(\frac{2}{3}\right)^{n-1} + \dots}_{\text{Geometric Series}}\Big]$$

لأن $r = \left(\frac{2}{3}\right) < 1 \Rightarrow$ the series is convergent.

$$1 = \left(\frac{2}{3}\right)K\left[\frac{1}{1-\left(\frac{2}{3}\right)}\right] = \left(\frac{2}{3}\right)K\left[\frac{1}{\frac{1}{3}}\right] = \left(\frac{2}{3}\right)K(3) = 2K$$

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$$\therefore 2K = 1 \Rightarrow \boxed{K = \frac{1}{2}}$$

$$\therefore f(x) = \begin{cases} \frac{1}{2}\cdot\left(\frac{2}{3}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

---

Q6: Given a p.f.:

$$f(x) = \begin{cases} \frac{e^{-1}}{x!} & \text{for } x = 0, 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Show that $f(x)$ is a p.m.f.

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Sol. T.P. Cond.① : $f(x) \geq 0 \quad \forall x \in R_x$

$\frac{1}{e} > 0 \quad , \frac{1}{x!} > 0 \quad \forall x = 0, 1, 2, \ldots$

since if $x = 0 \Rightarrow \frac{1}{e} > 0 \, , \frac{1}{\underset{=1}{0!}} = 1 > 0$

& $x = 1 \Rightarrow \frac{1}{e} > 0 \, , \frac{1}{1!} = 1 > 0$

$\vdots$

$\Rightarrow \frac{1}{e\,x!} > 0 \Rightarrow$ Cond.① is satisfied.

T.P. Cond.② : $\sum_{x=0}^{\infty} \frac{1}{e\,x!} = 1$

$$\sum_{x=0}^{\infty} \frac{1}{e\,x!} = \frac{1}{e} \sum_{x=0}^{\infty} \frac{1}{x!} = \frac{1}{\cancel{e}} \cdot \cancel{e} = 1$$

$\therefore$ Cond.② is satisfied. $\Rightarrow \therefore f(x)$ is p.m.f.

---

Q7: Given a p.d.f.

$$f(x) = \begin{cases} \frac{1}{8} x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the Value $(t)$ s.t. $P(X \leq t) = \frac{1}{4}$

(b) Find the Value $(t)$ s.t. $P(X \leq t) = \frac{1}{2}$

Sol.

(a) $P(X \leq t) = \frac{1}{4} \quad \ldots ①$

$\frac{1}{8} \int_0^t x\,dx = \frac{1}{16} x^2 \Big|_0^t = \frac{1}{16} t^2 \Rightarrow P(X \leq t) = \frac{1}{16} t^2 = \frac{1}{4}$ (by①)

$\frac{1}{4} t^2 = 1 \Rightarrow t^2 = 4 \Rightarrow t = \sqrt{4} = \pm 2 \Rightarrow \boxed{t = 2}$ only

since $\{0 \leq x \leq 4\} = R_x$ is positive.

(b) $P(X \leq t) = \frac{1}{2}$

$P(X \leq t) = \frac{1}{8} \int_0^t x\,dx = \frac{1}{16} x^2 \Big|_0^t = \frac{1}{16} t^2 \Rightarrow \frac{1}{16} t^2 = \frac{1}{2} \Rightarrow$ نتج

$\frac{1}{8}t^2 = 1 \Rightarrow t^2 = 8 \Rightarrow t = \pm 2\sqrt{2} \Rightarrow t = +2\sqrt{2}$ (only)

since $R_X = \{x : 0 \leq x \leq 4\}$ is positive.

Now; Find $P(X \leq 2\sqrt{2})$ ?

$$P(X \leq 2\sqrt{2}) = \frac{1}{8}\int_0^{2\sqrt{2}} x\,dx = \frac{1}{16}x^2\Big|_0^{2\sqrt{2}} = \frac{1}{16}(2\sqrt{2})^2 = \frac{1}{16}(8) = \frac{1}{2}$$

(c) Find the c.d.f. of $X$.

$$F(x) = \int_0^x f(t)\,dt = \int_0^x \frac{1}{8}t\,dt = \frac{x^2}{16}$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{16} & \text{for } 0 \leq x < 4 \\ 1 & \text{for } x \geq 4 \end{cases}$$

(الدالة f غير مستمرة عند $x=4$)

Note $F(x)$ is conts. at $x=0$ since $F(0^+) = \frac{0^2}{16} = 0 = F(0^-)$.

$F(x)$ is conts. at $x=4$

---

Q8: Suppose that the c.d.f. of a r.v. $X$ is as follows:

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{x^2}{9} & \text{for } 0 < x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}$$

Find and sketch (H.W.) the p.d.f. of $X$.

Sol. $f(x) = \frac{dF(x)}{dx} = F'(x)$ and $F(x)$ is conts. and diff.

$$f(x) = F'(x) = \frac{2}{9}x$$

$$\therefore f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 < x \leq 3 \\ 0 & \text{o.w.} \end{cases}$$

Note $F(3^+) = 1$

$F(3^-) = \frac{3^2}{9} = 1$

$\therefore F(3^+) = F(3^-)$

$\therefore F(x)$ is conts. at $x=3$

Q9: Answer by (true) or (False) and given the reason

① If $X$ is c.r.v., then $P(a \le X \le b) \ne P(a < X < b)$.

(False) since $P(a \le X \le b) = P(a < X < b)$ if $X$ is c.r.v, because $P(X=a)=0$ & $P(X=b)=0$ when $X$ is c.r.v.

② If $X$ is d.r.v. with c.d.f. $F(x)$, then $f(x) = \frac{d}{dx}F(x)$.

(False), only when $X$ is c.r.v., $f(x) = \frac{d}{dx}F(x)$.

③ If $X$ is a r.v. with c.d.f. $F(x)$, then $P(a < X < b) = F(b) - F(a)$.

(False), <u>since</u> by Th. $P(a < X \le b) = F(b) - F(a)$

④ If a c.d.f. $F(x)$ conts. at $x=a$, then $P(X=a)=0$.

(True), since if $F(x)$ conts. at $x=a$, then $F(a^+) = F(a^-)$ and $P(X=a) = F(a^+) - F(a^-)$ (by Theorem).

⑤ $F(\infty) = 0$ & $F(-\infty) = 1$

(False), $F(\infty) = 1$ and $F(-\infty) = 0$ (By Theorem).

⑥ state (Cauchy dist.):

$$f(x) = \begin{cases} \frac{1}{\pi(1+x^2)} & \text{for } -\infty < x < \infty \\ 0 & \text{o.w.} \end{cases}$$ and p.m.f.

(False), Cauchy dist. is p.d.f. since $R_X = \{x; -\infty < x < \infty\}$ and $f(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$ <u>(only)</u>.

---

<u>H.W.</u>: Given a c.d.f. $F(x) = \begin{cases} e^{x-3} & \text{for } x \le 3 \\ 1 & \text{for } x > 3 \end{cases}$

① Find and sketch the p.d.f. $f(x)$.

② Is $F(x)$ conts. at $x=3$? why?

$Q_{10}$: Show that there does not exist any number (C) s.t. the following function $f(x)$ would be a p.d.f.

$$f(x) = \begin{cases} \frac{C}{1+x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

Sol. If cond. ② is satisfied then $f(x)$ is p.d.f.

I.e. T.P. $\int_{-\infty}^{\infty} f(x)\,dx \overset{?}{=} 1$ (if we can that) (لايوجد عدد (C) يجعل هذه f(x) هي دالة احتمالية)

$$\int_0^{\infty} \frac{C}{1+x}\,dx = C\,Ln(1+x)\Big|_0^{\infty} = C[Ln(\infty) - Ln(1)]$$
$$= C[\infty - 0] = \infty$$

$\therefore$ C is not exist.

Then does not exist any no. (C) s.t. $f(x)$ be a p.f.

---

H.W. Given a p.d.f. $f(x)$:

$$f(x) = \begin{cases} Ce^{-2x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value C and sketch it's graph.
② Find the c.d.f. of X ($F(x)$) and sketch it's graph.

---

$Q_{11}$: Show that the following function be a pr.f.

Sol. T.P. cond. ①

$$f(x) = \begin{cases} 2\sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

Note: $2\sin x\cos x = \sin 2x$

Sol. T.P. cond. ① $f(x) \geq 0 \quad \forall x \in (0, \frac{\pi}{2})$

$2\sin x \cos x \geq 0 \quad \forall x \in (0, \frac{\pi}{2})$

T.P. cond. ②

$$\int_0^{\frac{\pi}{2}} 2\sin x\cos x\,dx = \int_0^{\frac{\pi}{2}} \sin 2x\,dx = \frac{-1}{2}\cos 2x\Big|_0^{\frac{\pi}{2}} = \frac{-1}{2}[\cos\pi - \cos 0]$$
$$= \frac{-1}{2}[-1-1] = 1$$

$\therefore$ By cond. ① and ② we get: $f(x)$ is p.d.f.

---

$Q_{12}$: Show that if the following function be pr. f.:

$$f(x) = \begin{cases} \frac{2}{\pi} \sin^2(\frac{x}{2}) & \text{for } 0 < x < \pi \\ 0 & \text{o.w.} \end{cases}$$

Sol. T.P. Cond. (1)

i.e. $f(x) \geq 0 \quad \forall x \in (0, \pi)$

$\sin^2 \frac{x}{2} \geq 0 \Rightarrow \frac{2}{\pi} \sin^2 \frac{x}{2} \geq 0 \quad \forall x \in (0, \pi)$

T.P. Cond. (2)

Note: $\sin^2 x = 1 - \cos x$

$$\frac{1}{\pi} \int_0^{\pi} 2 \sin^2 \frac{x}{2} \, dx = \frac{1}{\pi} \int_0^{\pi} (1 - \cos x) \, dx$$

$$= \frac{1}{\pi} (x - \sin x) \Big|_0^{\pi} = \frac{1}{\pi} [(\pi - 0) - (0 - 0)]$$

$$= \frac{\pi}{\pi} = 1$$

$\therefore f(x)$ is a pr. d. f.

---

$Q_{13}$: A box has (4) red and (3) black balls. Choose a sample of (3) balls one by one without replacement. Let X be a number of black in a sample. Find the p.m.f. of X and sketch it's graph. Also find the pr. that a sample has at least (2) red balls.

Sol.

$$\begin{array}{|cc|} 4 & r \\ 3 & b \end{array} \xrightarrow[\text{without rep.}]{(1-1)} \underset{K=3}{\boxed{\phantom{xx}}}$$

$X \equiv$ no. of black balls in a sample.

$\therefore X = 0, 1, 2, 3 =$ d.r.v.

$f(x) = P(X = x) =$ p.m.f.

Event $(X = x)$: a sample have $(x)$ black and $(3 - x)$ red balls.

$\rightleftarrows$

Event $(X=x)$ has $P^3_x \, P^4_{3-x}$ and $n(S) = P^7_3$

no. of black balls — no. of red balls

$$f(x) = P(X=x) = \begin{cases} \dfrac{P^3_x \, P^4_{3-x}}{P^7_3} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

<u>H.w.</u> Find $F(x)$ of this problem.

---

Q14: Consider the following c.d.f

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2}(x^3+1) & \text{for } -1 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

Find ① $P(-1 < X < 1)$, $P(-1 < X \le 1)$

② p.d.f. $f(x)$

③ $P(-2 < X < 2)$

<u>Sol.</u>

① $P(-1 < X < 1) = P(-1 < X \le 1) = F(1) - F(-1)$ (by Th.)

$= 1 - \frac{1}{2}((-1)^3 + 1)$

$= 1$

X is c.r.v. & F(x) is conts. $dx = 1$

② $f(x) = F'(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } -1 \le x < 1 \\ 0 & \text{o.w.} \end{cases}$ <u>Note:</u> F(x) is conts. at $x = -1$

p.d.f. of X

③ $P(-2 < X < 2) = P(-2 < X \le 2) = F(2) - F(-2)$ (by Th.)

$= 1 - 0$

$= 1$

<u>Note</u> $F(-2) = F(-2^+) - F(-2^-)$

$= 0 - 0 = 0 \Rightarrow F$ is conts. at $x = -2$

Q15: Given a p.d.f.

$$f(x) = \begin{cases} b \sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

Find the value of b.

Sol: $\because f(x)$ is a p.d.f. $\Rightarrow$ Cond. (2) is satisfied.

$$\therefore b\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx = 1$$

$$1 = b\left[\int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2x\right] dx \qquad (\text{since } \sin 2x = 2\sin x \cos x)$$

$$\frac{1}{2}\sin 2x = \sin x \cos x$$

$$1 = -\frac{b}{2} \cos 2x \Big|_0^{\frac{\pi}{2}}$$

$$1 = \frac{-b}{2}\left[\cos \pi - \cos 0\right]$$

$$1 = \frac{-b}{2}\left[-1 - (1)\right] \Rightarrow b = 1$$

$$\therefore f(x) = \begin{cases} \sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

---

Q16: Given a d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < -2 \\ \frac{x+3}{5} & \text{for } -2 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

(1) Find $P(X=-2)$, $P(-3 < X < -2)$

(2) Find $f(x)$

(3) Is $F(x)$ conts. at $x=2$? why?

Sol.

(1) $P(X=-2) = F(-2^{+}) - F(-2^{-})$     by Th1. $P(X=x) = F(x^{+}) - F(x^{-})$

$$= \left(\frac{-2+3}{5}\right) - 0 = \frac{1}{5}$$

$P(-3 < X < -2) \neq P(-3 < X \leq -2)$

since $\begin{pmatrix} F(x) \text{ is discont.} \\ \text{at } x = -2 \end{pmatrix}$

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(2) $f(x) = \dfrac{dF(x)}{dx} = F'(x) = \begin{cases} \frac{1}{5} & \text{for } -2 < X < 2 \\ 0 & \text{o.w.} \end{cases}$

$$P(-3 < x < -2) = \int_{-3}^{-2} f(x)\,dx = 0$$

(3) $F(x)$ is conts. at $x = 2$ since $F(2^-) = F(2^+)$

$F(2^+) = 1$ , $F(2^-) = \frac{2+3}{5} = \frac{5}{5} = 1$

---

Q17: Given the following p.m.f.

$$f(x) = \begin{cases} K \left(\frac{2}{3}\right)^{x-2} & \text{for } x = 2, 3, 4, \dots \\ 0 & \text{o.w.} \end{cases}$$

Find the value of K.

sol. $\because$ $f(x)$ is p.m.f. then cond. ② is satisfied.

$$\therefore K \sum_{x=2}^{\infty} \left(\frac{2}{3}\right)^{x-2} = 1$$

$\underline{\text{Note}}$ $K \sum_{x=2}^{\infty} \left(\frac{2}{3}\right)^{x-2} = K \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^{x}$

$$\therefore 1 = K \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^{x} = K\left[\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right) + \dots + \left(\frac{2}{3}\right)^n + \dots\right]$$

$$1 = K\Big[\underbrace{1 + \left(\frac{2}{3}\right) + \dots + \left(\frac{2}{3}\right)^n + \dots}_{\text{Geometric series}}\Big]$$

لأن $\Rightarrow |r| = \left|\frac{2}{3}\right| < 1 \Leftrightarrow$ the series is convergent

$$1 = K\left[\frac{1}{1 - \left(\frac{2}{3}\right)}\right] = K \cdot \frac{1}{(1/3)} = 3K$$

$$\Longrightarrow 3K = 1 \Longrightarrow \boxed{K = \tfrac{1}{3}}$$

$$\therefore f(x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w} \end{cases}$$

Q18: Given a set of 3 boys and 4 girls. Choose a sample of four students. If $y$ be a random variable represents the number of boys in a sample:

1- Find the pr. distribution of $y$.

2- Find the c.d.f. of $y$. and sketch it's graph.

3- Find the pr. of the following by two methods:

(i) If a sample has at least (2) girls.

(ii) If a sample has at most (1) girl.

Sol.

$$\underbrace{\boxed{\begin{matrix} 3\ b \\ 4\ g \end{matrix}}}_{n=7} \xrightarrow{\text{at once}} \underbrace{\boxed{\phantom{xx}}}_{K=4}$$

$Y \equiv$ no. of boys in a sample.

$y = \{0, 1, 2, 3\} =$ d.r.v.

$n(s) = C^7_4$

$$(i)\ f(y) = P(Y=y) = \begin{cases} \dfrac{C^3_y\, C^4_{4-y}}{C^7_4} & \text{for } y = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

$y = 0 \rightarrow f(0) = P(y=0) = \dfrac{C^3_0\, C^4_4}{C^7_4} = \dfrac{1}{35}$

$y = 1 \rightarrow f(1) = P(y=1) = \dfrac{C^3_1\, C^4_3}{C^7_4} = \dfrac{12}{35}$

$y = 2 \rightarrow f(2) = P(y=2) = \dfrac{C^3_2\, C^4_2}{C^7_4} = \dfrac{18}{35}$

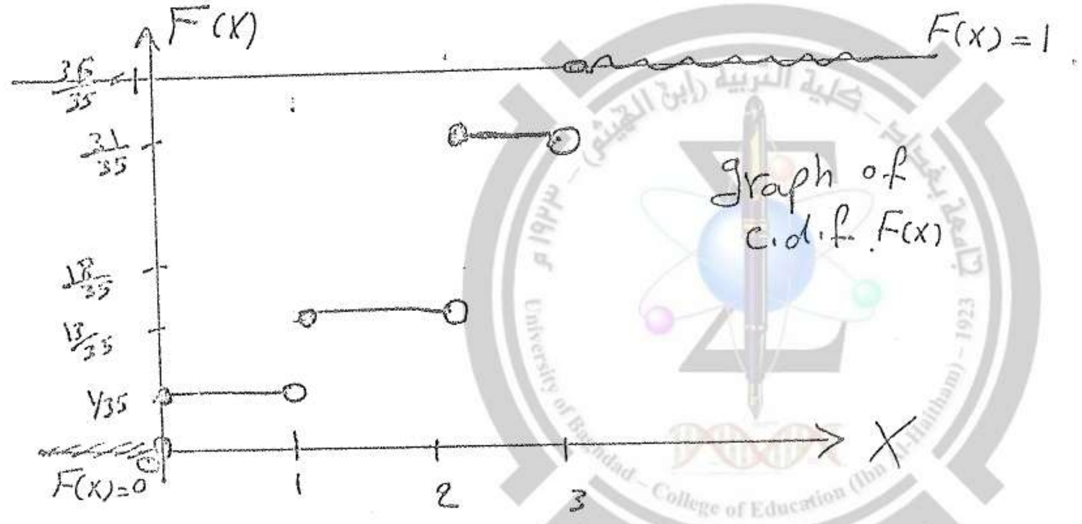
$y = 3 \rightarrow f(3) = P(y=3) = \dfrac{C^3_3\, C^4_1}{35} = \dfrac{4}{35}$

pr. dist. of $X = \{(x_i, f(x_i))\ ;\ \forall x_i \in R_X\}$

$\therefore$ pr. dist. of $X = \{(0, \frac{1}{35}), (1, \frac{12}{35}), (2, \frac{18}{35}), (3, \frac{4}{35})\}$

(ii) To find c.d.f. $F(y)$

| interval | integer | $f(y)$ | $F(y)$ |
|---|---|---|---|
| $y<0$ | no integer | $0$ | $0$ |
| $0\le y<1$ | $0$ | $1/35$ | $1/35$ |
| $1\le y<2$ | $1$ | $12/35$ | $13/35$ |
| $2\le y<3$ | $2$ | $18/35$ | $31/35$ |
| $y\ge 3$ | $3$ | $4/35$ | $1$ |




(iii) $P(y\le 2)=\sum_{y=0}^{2} f(y)=f(0)+f(1)+f(2)$
$$=\frac{1}{35}+\frac{12}{35}+\frac{18}{35}=\frac{31}{35}$$

or $P(y\le 2)=F(2)$
$$=\frac{31}{35}$$

| $g$ | $b$ |
|---|---|
| 2 | 2 |
| 3 | 1 |
| 4 | 0 |

$\Big\} \to y\le 2$

$K=4$

$P(y=3)=f(3)=\frac{4}{35}$

or $P(y=3)=F(3^+)-F(3^-)$
$$=1-\frac{31}{35}=\frac{4}{35}$$

| | $g$ | $b$ |
|---|---|---|
| لا يجوز | ~~0~~ | ~~4~~ |
| | 1 | 3 $\Rightarrow y=3$ |

$K=4$

Q19 : A box has (3) bad and (4) good tubes. Choose (3) tubes at once from this box.
Let X be a r.v. represents the number of bad tubes in a sample.
1- Find the pr. function of X and sketch it's graph.
2- Find the c.d.f. of X and sketch it's graph.
3- (i) Find $P(X<1 \mid 1<X<3)$.
(ii) Find the pr. that a sample has at least (1) good tube.

Sol:

$$\underset{n=7}{\boxed{\begin{matrix}3\,b\\4\,g\end{matrix}}} \xrightarrow{\text{at once}} \underset{K=3}{\boxed{\quad}} \quad , \; n(S) = C^{7}_{3}$$

$X \equiv$ no. of bad tube in a sample.

$\therefore R_X = \{0, 1, 2, 3\}$

$X = 0, 1, 2, 3 \equiv$ d.r.v.

Event $(X = x)$ : a sample has $x$-bad and $(K-x)$ good tubes.

(1) $f(x) = P(X = x) =$ p.m.f.

$$\therefore f(x) = \begin{cases} \dfrac{C^{3}_{x}\, C^{4}_{3-x}}{C^{7}_{3}} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

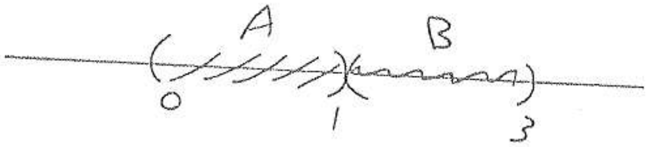
(2) c.d.f. $F(x) = \sum\limits_{x_j \le x} f(x_j)$

| interval | integer | $f(x)$ = p.m.f. | c.d.f. $F(x)$ |
|---|---|---|---|
| $x<0$ | no integer | 0 | 0 |
| $0 \le x < 1$ | 0 | 4/35 | 4/35 |
| $1 \le x < 2$ | 1 | 18/35 | 22/35 |
| $2 \le x < 3$ | 2 | 12/35 | 34/35 |
| $x \ge 3$ | 3 | 1/35 | 1 |

* Sketch it's graph.

$$(3)\ (i)\ P(\underbrace{X<1}_{A} \,/\, \underbrace{1<X<3}_{B}) = P(A|B) = \frac{P(AB)}{P(B)}$$

$$A = \{ X : 0<X<1 \}$$
$$B = \{ X : 1<X<3 \}$$




$$AB = \emptyset$$

$$\therefore P(X<1 \,/\, 1<X<3) = \frac{P(\emptyset)}{P(X=2)} = \frac{0}{f(2)} = \frac{0}{18/35} = 0$$

$$(ii)\ P(X\leq 2) = F(2)$$
$$= \frac{34}{35}$$

$\underline{\underline{\text{or}}}$ $$P(X\leq 2) = f(0) + f(1) + f(2)$$
$$= \frac{4}{35} + \frac{18}{35} + \frac{12}{35}$$
$$= \frac{34}{35}$$

| g | b |
|---|---|
| 1 | 2 |
| 2 | 1 |
| 3 | 0 |

$\rightarrow X\leq 2$

$K = 3$

Probability (Ch.3) — Group (1) — المرحلة الثالثة / اجب على جميع الاسئلة

**Q1** Given a p.d.f. of X : الفصل الثالث

$$f(x) = \begin{cases} 2x & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Find the c.d.f. of X ($F(x)$).

**Q2** If a r.v. $X \sim unif(0,10)$, Calculate (احسب) the pr. that:
① $P(X<3)$ & ② $P(X>6)$

**Q3** Given a c.d.f. of X :

$$F(x) = \begin{cases} 0 & \text{for } x<0 \\ \frac{x^2}{16} & \text{for } 0\le x<4 \\ 1 & \text{for } x\ge 4 \end{cases}$$

Find the pr. of: ① $P(X=4)$, ② $P(X>0)$

**Q4** Given a c.d.f. of X $F(x)$, if $F$ is continuous at $x=2$, IS
$P(0<X<2) = F(2)-F(0)$ ? why?

---

Probability / Ch.3 — Group (8) — اجب على جميع الاسئلة

**Q1** Show that the following function is pr. function :

$$f(x) = \begin{cases} \frac{x+2}{25} & \text{for } x=1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

**Q2** Given a c.d.f. of X :

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x>0 \\ 0 & \text{for } x\le 0 \end{cases}$$

Find: ① $P(X\le 2)$, ② $P(1<X<3)$, ③ $P(X>4)$, ④ $P(X=0)$.

**Q3** Answer (اجب) by (true) or (False) and given the reason (السبب) :
If a c.d.f. $F(x)$ continuous at $x=2$, then $P(X=2)\ne 0$.

**Q4** Given a set of integers $\{2,4,6,8,10\}$. Find the p.m.f. $f(x)$ of X and sketch its graph.

Probability / Ch.3 Group (2) اجب على جميع الاسئلة

**Q1** A box has (3) yellow and (2) red cards. Choose a sample of (3) cards. Let X be a number of yellow cards in a sample, then find:

① the pr.f. of X ($f(x)$).
② the c.d.f. of X ($F(x)$).
③ the pr. that a sample has (2) yellow cards.

**Q2** Show that there does not exist any number (c) such that the following function f(x) would be a p.f. of X:

$$f(x) = \begin{cases} c e^{x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

**Q3** Given a p.d.f. of X:

$$f(x) = \begin{cases} \frac{1}{8} x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{o.w.} \end{cases}$$

Find the value (t) such that $P(X \leq t) = \frac{1}{16}$.

---

المرحلة الثالثة

Probability / Ch.3 Group (7) اجب على جميع الاسئلة

**Q1** Consider the c.d.f. of X:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

① Find the p.f. of X ($f(x)$).
② Is F continuous at $x = 1$? Why?.
③ Find $P(X = 0)$.

**Q2** Show that the following function is pr.f. of X:

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^{x} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{o.w.} \end{cases}$$

**Q3** رمي العملة
Toss a coin fourth times ($4^{th}$ times). Find the c.d.f. of a r.v. X ($F(x)$), when X represents the numbers of tail (T) in a sample.

Probability / Ch.3 Group (3) اجب على جميع الاسئلة

Q1 Given the d.f. $F(x)$ of $X$:

$$F(x)=\begin{cases}0 & \text{for } x<-1\\ \frac{x+1}{2} & \text{for } -1\leq x<1\\ 1 & \text{for } x\geq 1\end{cases}$$

Find: ① $P(X=1)$, ② $P(X<-1)$, ③ $P(-\frac{1}{2}<X<\frac{1}{2})$

Q2 Find the value of (c) from the following Pr. function:

$$f(x)=\begin{cases}c\left(\frac{1}{4}\right)^x & \text{for } x=1,2,3,\dots\\ 0 & \text{o.w.}\end{cases}$$

Q3 Is the following function is Pr. function? Why?

$$f(x)=\begin{cases}\frac{1}{5} & \text{for } x=0,1,2,3,4,5\\ 0 & \text{o.w.}\end{cases}$$

Q4 Find the c.d.f. $F(x)$ from the following Pr.f. of $X$:

$$f(x)=\begin{cases}x & \text{for } 0<x<1\\ 2-x & 1\leq x<2\\ 0 & \text{o.w.}\end{cases}$$

---

Probability/Ch.3 Group (6) اجب على جميع الاسئلة

Q1 Is the following function is Pr. function? Why?

$$f(x)=\begin{cases}\frac{x-2}{5} & \text{for } x=1,2,3,4,5\\ 0 & \text{o.w.}\end{cases}$$

Q2 Given a C.d.f. $F$ of $X$ as follows:

$$F(x)=\begin{cases}1-(1+x)e^{-x} & \text{for } x>0\\ 0 & \text{for } x\leq 0\end{cases}$$

① Find $P(X\leq -2)$, $P(X>4)$

② Is $F$ continuous at $x=0$? why?

Q3 State Cauchy distribution. (Note: only state) اكتب التوزيع فقط

Q4 Given a c.d.f. of $X$:

$$F(x)=\begin{cases}1-e^{-x} & \text{for } x\geq 0\\ 0 & \text{o.w.}\end{cases}$$

Find the p.f. of $X$ ($f(x)$).

Probability/Ch.3 Group (5) المرحلة الثالثة
جب على جميع الاسئلة

Q1 Given a p.d.f. of X:

$$f(x) = \begin{cases} ce^{x} & \text{for } x<0 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value of c.
② Find F(x)

Q2 Show that the folloing function is p.f.:

$$f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^{x} & \text{for } x=1,2,3,\cdots \\ 0 & \text{o.w.} \end{cases}$$

Q3 If $X \sim$ uniform (0,1), find p.f. f(x) of X.

Q4 Given a c.d.f. of X:

$$F(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{1}{2}(x^3+1) & \text{for } -1<x<1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find the p.f. of X (f(x)).

---

Probability/Ch.3 Group (4) المرحلة الثالثة
اجب على جميع الاسئلة

Q1 Find the Pr. distribution of X from the following Pr.f. of X:

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } x=1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

Q2 Show that there does not exist any number (c) such that the following function f(x) would be a p.f.:

$$f(x) = \begin{cases} ce^{-x} & \text{for } x<0 \\ 0 & \text{o.w.} \end{cases}$$

Q3 Given a set of 3-boys and 3-girls. A sample of (2) students are choosing. If Z be a r.v. represents (تمثل) the number of boys in a sample, then find:
① the pr. distribution of Z.
② the c.d.f. of Z (F(z)).
③ the Pr. that a sample has 3-boys.

حلول + أسئلة الامتحان الثاني (Ch.3) لمادة الاحتمالية / المرحلة (٣) / ٩/١/ ٢٠٠٦ م

## Group (1)

Q1: Given a p.d.f. of $X$:

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the c.d.f. of $X$ ($F(x)$).

sol.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= 2\int_{0}^{x} t\,dt = \frac{\cancel{2}t^2}{\cancel{2}}\Big|_0^x = x^2$$

$$\therefore F(x) = \begin{cases} x^2 & \text{for } 0 \le x < 1 \\ 0 & \text{o.w.} \end{cases}$$

الصفر دخل ضمن المساحة (وأول حد للتكامل)

Q2: If a r.v. $X \sim unif(0,10)$, Calculate احسب the pr. that:

① $P(X<3)$ ② $P(X>6)$

sol. $\because X \sim uniform(0,10)$

$$\therefore f(x) = \begin{cases} \frac{1}{10} & \text{for } 0 < x < 10 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X<3) = \int_0^3 \frac{1}{10}\,dx = \frac{x}{10}\Big|_0^3 = \frac{1}{10}(3-0) = \frac{3}{10}$$

$$P(X>6) = \int_6^{10} \frac{1}{10}\,dx = \frac{x}{10}\Big|_6^{10} = \frac{1}{10}(10-6) = \frac{4}{10}$$

Q3: Given a c.d.f. of $X$:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{16} & \text{for } 0 \le x < 4 \\ 1 & \text{for } x \ge 4 \end{cases}$$

Find the pr. of: ① $P(X=4)$, ② $P(X>0)$

Sol. $P(X=4) = F(4^+) - F(4^-)$

$= 1 - \frac{4^2}{16} = 1 - \frac{16}{16} = 1-1 = \text{Zero}$

$P(X>0) = 1 - P(X \le 0)$

$= 1 - F(0)$ by def. of $F(x)$

$= 1 - F(0^+)$ by theorem $F(x) = F(x^+)$

$= 1 - \frac{0^2}{16} = 1 - 0 = 1$

---

Q4: Given a c.d.f. of $X$ ($F(x)$), if $F$ is continuous at $x=2$, Is $P(0<X<2) = F(2) - F(0)$ ? why?

Yes, since $X$ is c.r.v. & $P(X=2)=0$

then $P(0<X<2) = P(0<X\le 2) = F(2) - F(0)$ by theorem.

---

Group (2)

Q1: A box has (3) yellow and (2) red Cards. Choose a sample of (3) Cards. Let $X$ be a number of yellow cards in a sample, then find:

① the pr. f. of $X$ ($f(x)$).

② the c.d.f. of $X$ ($F(x)$).

③ the pr. that a sample has (2) yellow Cards.

Sol. 3 y, 2 r $\xrightarrow{\text{at once}}$ sample

$n=5$ $r=3$

$X$ = no. of yellow Cards in a sample.

$x = 1, 2, 3$ (d.r.v.)

$$f(x) = \begin{cases} \dfrac{C^3_x \, C^2_{3-x}}{C^5_3} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

| interval | no. of integer | $f(x)$ | $F(x) = \sum_{x_j \leq x} f(x_j)$ |
|---|---|---|---|
| $x < 1$ | no integer | $0$ | $0$ |
| $1 \leq x < 2$ | $1$ | $f(1) = \frac{C_1^3 C_2^2}{C_3^5}$ | $F(1) = f(1) + 0$ |
| $2 \leq x < 3$ | $2$ | $f(2) = \frac{C_2^3 C_1^2}{C_3^5}$ | $F(2) = 0 + f(1) + f(2)$ |
| $x \geq 3$ | $3$ | $f(3) = \frac{C_3^3 C_0^2}{C_3^5}$ | $F(3) = 0 + f(1) + f(2) + f(3) = 1$ |

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 1 \\ F(1) & \text{for } 1 \leq x < 2 \\ F(2) & \text{for } 2 \leq x < 3 \\ F(3) = 1 & \text{for } x \geq 3 \end{cases}$$

③ $P(X=2) = P(\text{2-yellow cards in a sample}) = f(2) = \frac{C_2^3 C_1^2}{C_3^5}$

---

$Q_2$ Show that there does not exist any number (c) such that the following function $f(x)$ would be a p.f. of $X$ :

$$f(x) = \begin{cases} c\,e^x & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

Sol. by condition (2) : (If $f(x)$ p.f. then it satisfy this condition).

$$c\int_0^\infty e^x\,dx = 1 \Longrightarrow c\int_0^\infty e^x\,dx = c\,e^x\Big|_0^\infty = c\left[e^\infty - e^0\right] = c[\infty - 1] = c(\infty) \neq 1$$

$\therefore$ there does not exist any number (c) such that the following function $f(x)$ would be a p.f. of $X$.

---

$Q_3$: Given a p.d.f. of $X$ :

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{o.w.} \end{cases}$$

Find the value ($t$) such that $P(X\le t)=\frac{1}{16}$.

Sol.

$$P(X\le t)=\frac{1}{8}\int_0^t x\,dx=\frac{1}{8}\cdot\frac{x^2}{2}\Big|_0^t=\frac{1}{16}(t^2-0^2)=\frac{t^2}{16}=\frac{1}{16}$$

$\Rightarrow t^2=1 \Rightarrow t=\pm 1 \Rightarrow t=1$ only since $t\in[0,4]$.

Group (3)

P1: Given the d.f. $F(x)$ of $X$:

$$F(x)=\begin{cases}0 & \text{for } x<-1\\ \frac{x+1}{2} & \text{for } -1\le x<1\\ 1 & \text{for } x\ge 1\end{cases}$$

Find: ① $P(X=1)$, ② $P(X<-1)$, ③ $P(-\frac{1}{2}<X<\frac{1}{2})$

Sol.

① $P(X=1)=F(1^+)-F(1^-)$ (by theorem)

$=1-\left(\frac{1+1}{2}\right)$

$=1-\frac{2}{2}=1-1=$ Zero

② $P(X<-1)=F(-1^-)$ (by theorem)

$=0$

③ $P(-\frac{1}{2}<X<\frac{1}{2})=P(-\frac{1}{2}<X\le\frac{1}{2})$ since $F$ is cont. at $x=\frac{1}{2}$

$=F(\frac{1}{2})-F(-\frac{1}{2})$ by theorem

$=\frac{(\frac{1}{2}+1)}{2}-\frac{(-\frac{1}{2}+1)}{2}$

$=\frac{3}{4}-\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$

P2: Find the value of ($c$) from the following Pr. function:

$$f(x)=\begin{cases}c\left(\frac{1}{4}\right)^x & \text{for } x=1,2,3,\dots\\ 0 & \text{o.w.}\end{cases}$$

Sol. by Cond. (2), we get: $\sum_{x=1}^{\infty} f(x)=1$

$\therefore C \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = 1$

$C\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots\right] = \frac{1}{4} C\left[\underbrace{1+\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\cdots}_{\text{Geometric series}}\right]$

where $r = \frac{1}{4} < 1 \Rightarrow$ this series is convergent (by theorem)

$\therefore \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \frac{4}{3}$ (by theorem)

$\Rightarrow C \frac{1}{\cancel{4}}\left(\frac{\cancel{4}}{3}\right) = 1 \Rightarrow \frac{C}{3} = 1 \Rightarrow C = 3$

$$\therefore f(x) = \begin{cases} 3\left(\frac{1}{4}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

---

$Q_3$: Is the following function is pr. function? why?

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ② $\sum_{x=1}^{5} \frac{1}{5} = \frac{6}{5} > 1$

$\therefore$ Cond. ② doesn't satisfied.

$\therefore f(x)$ is not pr. function.

---

$Q_4$: Find the c.d.f. $F(x)$ from the following pr. f. of $X$:

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \le x < 2 \\ 0 & \text{o.w.} \end{cases}$$

Sol. $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x t\,dt + \int_1^x (2-t)\,dt$

$= \frac{t^2}{2}\Big|_0^x + \left(2t - \frac{t^2}{2}\right)\Big|_1^x$

$= \frac{1}{2}x^2 + \left(2x - \frac{1}{2}x^2\right) - \left(2 - \frac{1}{2}\right)$

$\therefore F(x) = ?$ $\quad = \underbrace{\left(2x - \frac{3}{2} - \frac{1}{2}x^2\right)}_{x \in [1,2]} + \underbrace{\left(\frac{1}{2}x^2\right)}_{x \in [0,1]}$

$$F(X) = \begin{cases} 0 & \text{for } X < 0 \\ \frac{1}{2}X^2 & \text{for } 0 \leq X < 1 \\ 2X - \frac{1}{2}X^2 - \frac{3}{2} & \text{for } 1 \leq X < 2 \\ 1 & \text{for } X \geq 2 \end{cases}$$

---

Group(4)

Q1: Find the Pr. distribution of X from the following Pr.f of X

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

$$\text{Pr. dist. of } X = \{(x_i, f(x_i)) ;\ i = 1, 2, 3\}$$
$$= \{(1, \tfrac{1}{3}), (2, \tfrac{1}{3}), (3, \tfrac{1}{3})\}$$

---

Q2: Show that there does not exist any number (c) such that the following function f(x) would be a p.f.;

$$f(x) = \begin{cases} ce^{-x} & \text{for } X < 0 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

If f(x) is p.f. then Cond.(2) is satisfied:

$$c\int_{-\infty}^{0} e^{-x}\,dx = -ce^{-x}\Big|_{-\infty}^{0} = -c\,[e^{0} - e^{\infty}] = -c\,[1 - \infty] = 1$$
$$= -c + \infty = 1$$
$$\Rightarrow \therefore c \neq 1$$

$\therefore$ There does not exist any number (c) such that the following function f(x) would be a p.f.

---

Q3: Given a set of 3-boys and 3-girls. A sample of (2) students are choosing. If Z be a r.v. represents the number of boys in a sample, then find:

① the Pr. distribution of Z.
② the c.d.f. of Z.
③ the Pr. that a sample has 3-boys.

Sol. $\boxed{\begin{matrix} 3\ b \\ 3\ g \end{matrix}} \xrightarrow{\text{at once}} \boxed{\ \ }$ $r=2$

$n=6$

$Z \equiv$ no. of boys in a sample.

$Z = 0, 1, 2$

① $f(z) = \begin{cases} \dfrac{C_z^3\, C_{2-z}^3}{C_2^3} & \text{for } z = 0,1,2 \\ 0 & \text{o.w.} \end{cases}$

pr. distribution of $Z = \{(z_i, f(z_i))\ ;\ i = 0,1,2\}$

$= \{(0, f(0)), (1, f(1)), (2, f(2))\}$

$= \left\{\left(0, \dfrac{C_0^3 C_2^3}{C_2^3}\right), \left(1, \dfrac{C_1^3 C_1^3}{C_2^3}\right), \left(2, \dfrac{C_2^3 C_0^3}{C_2^3}\right)\right\}$

$= \left\{\left(0, \dfrac{3(1)}{3}\right), \left(1, \dfrac{(3)3}{3}\right), \left(2, \dfrac{3(1)}{3}\right)\right\}$

$= \{(0,1), (1,3), (2,1)\}$

②

$F(z) = \begin{cases} 0 & \text{for } z < 0 \\ f(0) + 0 = F(0) & \text{for } 0 \le z < 1 \\ f(1) + f(0) + 0 = F(1) & \text{for } 1 \le z < 2 \\ f(2) + f(1) + f(0) + 0 = F(2) = 1 & \text{for } z \ge 2 \end{cases}$

③ $P(\text{A sample has 3-boys}) = P(Z = 3) = f(3) = 0$

## Group (5)

$Q_1$: Given a p.d.f. of X:

$$f(x) = \begin{cases} c e^{x} & \text{for } x<0 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value of c.

② Find $F(x)$

Sol: ① by cond.(2), we get: $c\int_{-\infty}^{0} e^{x}\,dx = 1$

$$\Rightarrow \quad c e^{x}\Big|_{-\infty}^{0} = 1 \Rightarrow c[e^{0} - e^{-\infty}] = 1 \Rightarrow c[1-0] = 1$$

$$\therefore c = 1$$

$$\therefore f(x) = \begin{cases} e^{x} & \text{for } x<0 \\ 0 & \text{o.w.} \end{cases} \qquad \left( \begin{matrix} x<0 \\ \Rightarrow -\infty < x < 0 \end{matrix} \right)$$

② $$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} e^{t}\,dt = e^{t}\Big|_{-\infty}^{x} = e^{x} - \underbrace{e^{-\infty}}_{=0}$$

$$= e^{x}$$

$$\therefore F(x) = \begin{cases} e^{x} & \text{for } x<0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

---

$Q_2$: Show that the following function is p.f.:

$$f(x) = \begin{cases} 7\left(\frac{1}{8}\right)^{x} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ①: $7\left(\frac{1}{8}\right)^{1} = f(1) \geq 0$

$7\left(\frac{1}{8}\right)^{2} = f(2) \geq 0$

$\therefore f(x) \geq 0$ & Cond.(1) is satisfied.

Cond. ② $$7\sum_{x=1}^{\infty}\left(\frac{1}{8}\right)^{x} = 7\left[\left(\frac{1}{8}\right)^{1} + \left(\frac{1}{8}\right)^{2} + \left(\frac{1}{8}\right)^{3} + \cdots\right]$$

$$= \left(\frac{7}{8}\right)\Big[\underbrace{1 + \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^{2} + \cdots}_{\text{Geometric Series}}\Big]$$

$\therefore$ by theorem & $r = \left|\frac{1}{8}\right| < 1 \Rightarrow$ this series is convergent

by theorem, we get: $\sum_{x=0}^{\infty} \left(\frac{1}{8}\right)^x = \frac{1}{1-\frac{1}{8}} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$

$\therefore\ 7 \sum_{x=1}^{\infty} \left(\frac{1}{8}\right)^x = \cancel{\left(\frac{7}{8}\right)} \cdot \cancel{\left(\frac{8}{7}\right)} = 1$

$\therefore$ Cond.(2) is satisfied.

---

Q3: If $X \sim$ uniform$(0,1)$, find p.f. $f(x)$ of $X$.

Sol.

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

---

Q4: Given a c.d.f. of $X$

$$F(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{1}{2}(x^3+1) & \text{for } -1 < x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find the p.f. of $X$ ($f(x)$).

Sol.

$$f(x) = F'(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{3}{2}x^2 & \text{for } -1 < x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } -1 \leq x \leq -1 \\ 0 & \text{o.w.} \end{cases}$$ $\longleftarrow$ since $F$ is continuous at $x = -1$

---

Group (6)

Q1: Is the following function is pr. function? why?

$$f(x) = \begin{cases} \frac{x-2}{5} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond.① : $f(1) = \frac{-1}{5} < 0$, $f(2) = 0$, $f(x) \geq 0$ $x = 3, 4, 5$

$\therefore f$ is not pr. function.

Q2: Given a c.d.f. F of X as follows:

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x = 1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

① Find $P(X \le -2)$, $P(X > 4)$

② IS F continuous at $x = 0$ ? why?

Sol.

① $P(X \le -2) = F(-2)$ by def. of F

$= 0$

$P(X > 4) = 1 - P(X \le 4)$

$= 1 - F(4)$

$= 1 - [(1+4)e^{-4}]$

② Yes, since $F(0^+) = F(0^-) =$ Zero.

Q3: State Cauchy distribution.

$$f(x) = \frac{1}{\pi(1+x^2)} \quad \text{for } -\infty < x < \infty$$

Q4: Given a c.d.f. of X:

$$F(x) = \begin{cases} 1-e^{-x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find the pr. f. of X $(f(x))$.

Sol.

$$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

since $F(0^-) = F(0^+)$ & F is conts. at $x = 0$

Group (7)

Consider the c.d.f. of X:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

① Find the p.f. of $X$ ($f(x)$).

② Is $F$ conts. at $x=1$ ? why?

③ Find $P(X=0)$.

sol.

① $$f(x) = F'(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } 0 \le x < 1 \\ 0 & \text{for } x \ge 1 \end{cases} = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{for o.w.} \end{cases}$$

② Yes, since $F(1^+) = F(1^-) = 1$.

③ $$P(X=0) = F(0^+) - F(0^-) = 0 - 0 = \text{Zero}$$

---

Q2: Show that the following function is Pr. f. of $X$ :

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

sol.

Cond. ① $\left(\frac{1}{2}\right)^1 > 0$ , $\left(\frac{1}{2}\right)^2 > 0$, ...

$\therefore f(x) \ge 0 \quad \forall x = 1, 2, 3, \dots$

Cond. ② $$\sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \left[\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$= \left(\frac{1}{2}\right)\underbrace{\left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots\right]}_{\text{Geometric series}} ; \ r = \left|\frac{1}{2}\right| < 1$$

$$= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{\cancel{2}} \cdot \cancel{2} = 1$$

---

Q3: Toss a coin fourth times ($4^{th}$ times). Find the c.d.f. of a r.v. $X$, when $X$ represents the numbers of tail (T) in a sample.

Sol.

$$f(x) = \begin{cases} \frac{C_x^4}{16} & \text{for } x = 0, 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases} \qquad n(s) = 2^4 = 16$$

$$F(x) = \sum_{x_j \le x} f(x_j)$$

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ F(0) = 0 + f(0) & 0 \le x < 1 \\ F(1) = 0 + f(0) + f(1) & 1 \le x < 2 \\ F(2) = 0 + f(0) + f(1) + f(2) & 2 \le x < 3 \\ F(3) = 0 + f(0) + f(1) + f(2) + f(3) & 3 \le x < 4 \\ F(4) = 1 = 0 + f(0) + f(1) + f(2) + f(3) + f(4) & x \ge 4 \end{cases}$$

---

## Group (8)

$Q_1$: Show that the following function is pr. function:

$$f(x) = \begin{cases} \frac{x+2}{25} & \text{for} \quad x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

sol. Cond. ②: $\frac{1}{25} \sum_{x=1}^{5} (x+2) = \frac{1}{25} [3 + 4 + 5 + 6 + 7] = \frac{25}{25} = 1$

Cond. ① $f(1) = \frac{3}{25} > 0$, $f(2) = \frac{4}{25} > 0, \ldots, f(5) = \frac{7}{25} > 0$

$\therefore f(x) \ge 0 \quad \forall x = 1, 2, 3, 4, 5$

$\therefore f(x)$ is pr. f.

---

$Q_2$: Given a c.d.f. of $X$:

$$F(x) = \begin{cases} 1 - (1+x) e^{-x} & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \le 0 \end{cases}$$

Find: ① $P(X \le 2) = F(2)$ by def. of $F$

$= 1 - (1+2) e^{-2}$

② $P(1 < X < 3) = P(1 \le X \le 3)$ since $F$ is conts. at $x = 3$

$= F(3) - F(1)$

$= (1 - (1+3) e^{-3}) - (1 - (1+1) e^{-1})$

③ $P(X > 4) = 1 - P(X \le 4) = 1 - F(4) = 1 - (1 - (1+4) e^{-4})$

④ $P(X = 0) = F(0^+) - F(0^-) = 1 - (1+0) e^{0} - 0$

$= 0$

by theorem & $r=\left|\frac{1}{8}\right|<1 \Rightarrow$ this series is convergent

by theorem, we get: $\sum_{x=0}^{\infty}\left(\frac{1}{8}\right)^x = \frac{1}{1-\frac{1}{8}} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$

$\therefore\ 7\sum_{x=1}^{\infty}\left(\frac{1}{8}\right)^x = \left(\cancel{\frac{7}{8}}\right)\cdot\left(\cancel{\frac{8}{7}}\right) = 1$

$\therefore$ Cond.(2) is satisfied.

---

Q3: If $X \sim$ uniform$(0,1)$, find p.f. $f(x)$ of $X$.

Sol.

$$f(x) = \begin{cases} 1 & \text{for } 0<X<1 \\ 0 & \text{o.w.} \end{cases}$$

---

Q4: Given a c.d.f. of $X$

$$F(x) = \begin{cases} 0 & \text{for } X \leq -1 \\ \frac{1}{2}(X^3+1) & \text{for } -1<X<1 \\ 1 & \text{for } X \geq 1 \end{cases}$$

Find the p.f. of $X$ $(f(x))$.

Sol.

$$f(x) = F'(X) = \begin{cases} 0 & \text{for } X \leq -1 \\ \frac{3}{2}X^2 & \text{for } -1<X<1 \\ 0 & \text{for } X \geq 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{3}{2}X^2 & \text{for } -1<X \leq -1 \\ 0 & \text{o.w.} \end{cases}$$

$\leftarrow$ since $F$ is continuous at $x=-1$

---

Group (6)

Q1: Is the following function is pr. function? Why?

$$f(x) = \begin{cases} \frac{x-2}{5} & \text{for } x=1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond.①: $f(1) = \frac{-1}{5} < 0$, $f(2)=0$, $f(x) \geq 0$ $x=3,4,5$

$\therefore$ $f$ is not pr. function.

Q2: Given a c.d.f. F of X as follows:

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x=1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

① Find $P(X \leq -2)$, $P(X>4)$

② Is $F$ continuous at $x=0$? why?

Sol.

① $P(X \leq -2) = F(-2)$ by def. of $F$

$= 0$

$P(X>4) = 1 - P(X \leq 4)$

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$= 1 - [(1+4)e^{-4}]$

② Yes, since $F(0^+) = F(0^-) = $ Zero.

---

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$$f(x) = \frac{1}{\pi(1+x^2)} \quad \text{for } -\infty < x < \infty$$

---

Q4: Given a c.d.f. of X:

$$F(x) = \begin{cases} 1-e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x<0 \end{cases}$$

Find the pr. f. of X ($f(x)$).

Sol.

$$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x<0 \end{cases} \longrightarrow \text{since } F(0^-) = F(0^+) \text{ \& } F \text{ is conts. at } x=0$$

---

Group (7)

Consider the c.d.f. of X:

$$F(x) = \begin{cases} 0 & \text{for } x<0 \\ x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

① Find the p.f. of $X$ ($f(x)$).
② Is $F$ conts. at $x=1$ ? why?
③ Find $P(X=0)$.

Sol.

① $f(x) = F'(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } 0 \le x < 1 \\ 0 & \text{for } x \ge 1 \end{cases} = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{for o.w.} \end{cases}$

② Yes, since $F(1^+) = F(1^-) = 1$.

③ $P(X=0) = F(0^+) - F(0^-)$
$= 0 - 0 = \text{Zero}$

---

Q2: Show that the following function is Pr. f. of $x$:

$$f(x) = \begin{cases} (\frac{1}{2})^x & \text{for } x = 1,2,3,\ldots \\ 0 & \text{o.w.} \end{cases}$$

Sol.

Cond. ① $(\frac{1}{2})^1 > 0$, $(\frac{1}{2})^2 > 0$, ...

$\therefore f(x) \ge 0 \quad \forall x = 1,2,3,\ldots$

Cond. ② $\sum_{x=1}^{\infty} (\frac{1}{2})^x = [(\frac{1}{2})^1 + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots]$

$= (\frac{1}{2}) \underbrace{[1 + (\frac{1}{2}) + (\frac{1}{2})^2 + \cdots]}_{\text{Geometric series}}$ ; $r = |\frac{1}{2}| < 1$

$= \frac{1}{2} \left( \frac{1}{1-\frac{1}{2}} \right) = \frac{1}{\cancel{2}} \cdot \cancel{2} = 1$

---

Q3: Toss a coin fourth times ($4^{th}$ times). Find the c.d.f. of a r.v. $X$, when $X$ represents the numbers of tail (T) in a sample.

Sol.

$$f(x) = \begin{cases} \frac{C_x^4}{16} & \text{for } x = 0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$$

$n(s) = 2^4 = 16$

$$F(x) = \sum_{x_j \le x} f(x_j)$$

① Find the P.f. of $X$ ($f(x)$).
② Is $F$ conts. at $x=1$ ? Why ?
③ Find $P(X=0)$.

sol.

① $f(x) = F'(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } 0 \le x < 1 \\ 0 & \text{for } x \ge 1 \end{cases} = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{for o.w.} \end{cases}$

② Yes, since $F(1^+) = F(1^-) = 1$.

③ $P(X=0) = F(0^+) - F(0^-)$
$= 0 - 0 = \text{Zero}$

---

Q2: Show that the following function is Pr.f. of $x$ :

$$f(x) = \begin{cases} (\frac{1}{2})^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

sol.

Cond. ① $(\frac{1}{2})^1 > 0$ , $(\frac{1}{2})^2 > 0$ , ...

$\therefore f(x) \ge 0 \quad \forall x = 1, 2, 3, \dots$

Cond. ② $\sum_{x=1}^{\infty} (\frac{1}{2})^x = \left[(\frac{1}{2})^1 + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \dots\right]$

$= (\frac{1}{2}) \underbrace{\left[1 + (\frac{1}{2}) + (\frac{1}{2})^2 + \dots\right]}_{\text{Geometric series}}$ ; $r = |\frac{1}{2}| < 1$

$= \frac{1}{2} \left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{\cancel{2}} \cdot \cancel{2} = 1$

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$$f(x) = \begin{cases} \frac{C^4_x}{16} & \text{for } x = 0, 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases} \qquad n(s) = 2^4 = 16$$

$$F(x) = \sum_{x_j \le x} f(x_j)$$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ F(0) = 0 + f(0) & 0 \le x < 1 \\ F(1) = 0 + f(0) + f(1) & 1 \le x < 2 \\ F(2) = 0 + f(0) + f(1) + f(2) & 2 \le x < 3 \\ F(3) = 0 + f(0) + f(1) + f(2) + f(3) & 3 \le x < 4 \\ F(4) = 1 = 0 + f(0) + f(1) + f(2) + f(3) + f(4) & x \ge 4 \end{cases}$$

---

Group (8)

$Q_1$: Show that the following function is pr. function:

$$f(x) = \begin{cases} \frac{x+2}{25} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

sol. Cond. ②: $\frac{1}{25}\sum_{x=1}^{5}(x+2) = \frac{1}{25}[3+4+5+6+7] = \frac{25}{25} = 1$

Cond. ①: $f(1) = \frac{3}{25} > 0$, $f(2) = \frac{4}{25} > 0, \dots, f(5) = \frac{7}{25} > 0$

$\therefore f(x) \ge 0 \quad \forall x = 1, 2, 3, 4, 5$

$\therefore f(x)$ is pr. f.

---

$Q_2$: Given a c.d.f. of $X$:

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find: ① $P(X \le 2) = F(2)$ by def. of $F$

$= 1-(1+2)e^{-2}$

② $P(1 < X < 3) = P(1 < X \le 3)$ since $F$ is conts. at $x = 3$

$= F(3) - F(1)$

$= (1-(1+3)e^{-3}) - (1-(1+1)e^{-1})$

③ $P(X > 4) = 1 - P(X \le 4) = 1 - F(4) = 1 - (1-(1+4)e^{-4})$

④ $P(X = 0) = F(0^+) - F(0^-) = 1-(1+0)e^{0} - 0$

$= 0$

$Q_3$: Answer by (true) or (False) and given the reason:
If a c.d.f. $F(x)$ Continuous at $x=2$, then $P(X=2)\neq 0$.

Sol. No, since if $F$ conts. at $x=2$, then $F(2^+)=F(2^-)$

and $P(X=2) = F(2^+) - F(2^-)$
$= \text{Zero}$

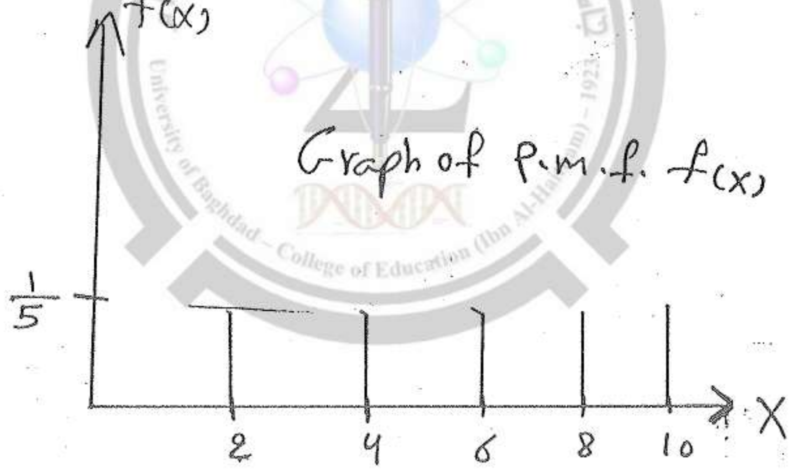
---

$Q_4$: Given a set of integers $\{2,4,6,8,10\}$. Find the p.m.f. $f(x)$ of $X$ and sketch its graph.

Sol.

$$X \sim \text{unif.}(5)$$

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 2,4,6,8,10 \\ 0 & \text{o.w.} \end{cases}$$




Graph of p.m.f. $f(x)$

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