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المرحلة الثالثة

Chapter Four

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CHAPTER FOUR

Expectation and Variance:

Def: Let x be a. r. v either d.r.v. or c.r.v."E(x)" is called the "Expectation of x" or "expected Value of x" or "mean of x" And denoted by μ.

Defined as followes;

1- $E(x) = \sum_{\forall x} x\, f(x)$, when x is a d.r.v.

2- $E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$ when x is a c.r.v.

Note: 1- the value of E (x) is constant.

2- E (x) exists if $\sum_{\forall x} |x|\, f(x) < \infty$, when x is d.r.v.

3- E (x) exists if $\int_{-\infty}^{\infty} |x|\, f(x)\, dx < \infty$, when x is c.r.v.

Ex "1": Given p.m.f $f(x) = \begin{cases} \dfrac{\binom{3}{x}}{8} & for\ x = 0,2,3,1 \\ 0 & O.W \end{cases}$

Find E (X)?

Sol: $E(x) = \sum_{x=0}^{3} x\, f(x)$

X	$f(x)=\dfrac{\binom{3}{x}}{8}$	x.f(x)
0	$\binom{3}{0}\Big/8 = \frac{1}{8}$	0
1	$\binom{3}{1}\Big/8 = \frac{3}{8}$	$\frac{3}{8}$

2	$\binom{3}{2}\Big/8=\frac{3}{8}$	$\frac{6}{8}$
3	$\binom{3}{3}\Big/8=\frac{1}{8}$	$\frac{3}{8}$
	$\sum_{x=0}^{3} f(x)=1$	$\sum_{x=0}^{3} x f(x)=\frac{12}{8}$

$$\therefore E(x)=\sum_{x=0}^{3} x f(x)=\frac{3}{2}$$

H.W: given ap. M.f $f(x)=\begin{cases}\frac{x}{15} & \text{for } x=1,2,3,4,5\\ 0 & 0.w.\end{cases}$

Find the expected Value of x.

Ex "2": Given a p.d.f $f(x)=\begin{cases}\frac{x}{8} & \text{for } 0<x<4\\ 0 & 0.w.\end{cases}$ Find E(x) ?

Sol: $E(x)=\int_{-\infty}^{\infty} x f(x)\,dx=\int_{0}^{4} x\frac{x}{8}\,dx=\int_{0}^{4}\frac{x^2}{8}\,dx$

$$=\frac{1}{8}\,\frac{x^3}{3}\Big|_0^4=\frac{1}{24}[64-0]=\frac{64}{24}=\frac{8}{3}$$

H.W: Given ap. d.f $f(x)=\begin{cases}3x^2 & \text{for } 0<x<1\\ 0 & 0.w.\end{cases}$

Find E(x)?

تراجع ←

$+1 < X < e$

Ex: Given ap. d.f $f(x)=\begin{cases}\frac{1}{x} & for\ \cancel{0<x<1} \\ 0 & o.w\end{cases}$ $\quad -1 < X < 2.72 \simeq e^1$

Dose E(x) exist?

Sol: $E(x)=\int_1^e x\frac{1}{x}dx=\int_1^e dx = 1-e$

$\therefore E(x)$ is exist.

$X>1$

Ex: Given ap .d.f $f(x)=\begin{cases}\frac{1}{x2} & \cancel{0<x<1} \\ 0 & o.w.\end{cases}$

Dose E(x) exist?

Sol: $E(x)=\int_1^\infty x\frac{1}{x^2}dx=\int_1^\infty \frac{1}{x}dx = \ln x\Big|_1^\infty=[\infty-0]=+\infty \not< \infty$

$\therefore E(x)$ is not exist.

Expectation of afunction of x:

Def: Let x be ar.v. and let g(x) be afunction of x then

$E[g(x)]=\sum_{\forall x} g(x)f(x)$, if x is d.r.v.

$=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx$, if x is c.r.v.

Ex "1": Given ap.m.f $f(x)=\begin{cases}\frac{x}{10} & \text{for } x=1,2,3,4 \\ 0 & o.w.\end{cases}$

find $E(x^2)$?

sol: $g(x)=x^2$

$E[g(X)]=\sum_{\forall x} g(X)f(x)$

$$E[x^2]=\sum_{x=1}^{4}(x)^2\frac{x}{10}=\sum_{x=1}^{4}\frac{x^3}{10}=\left[\frac{1}{10}+\frac{8}{10}+\frac{27}{10}+\frac{64}{10}\right]$$

$$=\frac{100}{10}=10$$

Ex"2": Given ap.d.f $f(x)=\begin{cases}\frac{x}{8} & \text{for } 0<x<4\\ 0 & \text{o.w.}\end{cases}$

find $E(\sqrt{x})$?

sol: $g(x)=\sqrt{x}=x^{\frac{1}{2}}$

$$E[g(x)]=E(x^{\frac{1}{2}})=\int_0^4 x^{\frac{1}{2}}\frac{x}{8}dx=\frac{1}{8}\int_0^4 x^{\frac{3}{2}}dx=\frac{1}{8}\cdot\frac{2}{5}x^{\frac{5}{2}}\Big|_0^4$$

$$\frac{1}{20}\left[(\sqrt{4})^5-0\right]=\frac{1}{20}[32]=\frac{32}{20}$$

Note: if $f(x)$ be a p. d.f of a r.v. x. then $E(b)=b$, where b is constant.

Proof: case "1": If x is a d.r.v. with p.m.f. f(x)

$E(b)=\sum_{x} b\, f(x)=b\sum f(x)=b$.

Case "2": If x is a c.r.v. with p.d.f. f(x).

$$E(b)=\int_{-\infty}^{\infty} b f(x)\,dx=b\int_{-\infty}^{\infty} f(x)\,dx=b\,.$$

Properties of Expectation :

Theorem "1" If x is ar.v. have ap.f. f(x), and E(x) exists.

Let $y=ax+b$, $a\ b\in R$, then $E(y)=aE(x)+b$.

Sol:case :"1" If x is a c. r.v. With P.d.f f(x)

$Y=g(x)=ax+b$

$$E(y)=E[g(x)]=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx$$

$$E[ax+b]=\int_{-\infty}^{\infty}(ax+b)\ f(x)dx=\int_{-\infty}^{\infty} ax\ f(x)\ dx+\int_{-\infty}^{\infty} b\ f(x)\ dx$$

$$=a\int_{-\infty}^{\infty} x f(x)\ dx+b\int_{-\infty}^{\infty} f(x)\ dx=a\,E(x)+b$$

case"2": If x is ad.r.v. with p.m.f f(x).

y=g(x)=ax+b

$$E(y)=E[g(x)]=\sum_{\forall x} g(x)\ f(x)$$

$$E[ax+b]=\sum_{\forall x}(ax+b)\ f(x)=\sum\ ax\ f(x)+\sum b\,f(x)$$

$$=a\sum x\,f(x)+b\sum f(x)=aE(x)+b$$

theorem "2" : let x be a r.v. if u(x) and v(x) are two functions of x. then:

$$E[u(x)\mp v(x)]=E[u(x)]\mp E[v(x)]$$

proof: " case "1": If x is ad.r.v. with p.m.f f(x) let $g(x)=u(x)\mp v(x)$

$$E[u(x)\mp v(x)]=E[g(x)]=\sum_{\forall x} g(x)\ f(x)$$

$$=\sum_{\forall x}[u(x)\mp v(x)]f(x)=\sum_{\forall x}u(x)f(x)\mp\sum_{\forall x}v(x)\,f(x)$$

$$=E[u(x)]\mp E[v(x)]$$

case "2": If x is ac.r.v. with p.d.f f(x).

let $g(x)=u(x)\mp v(x)$

$$E[u(x)\mp v(x)]=E[g(x)]=\int_{-\infty}^{\infty} g(x)\ f(x)dx$$

$$=\int_{-\infty}^{\infty}[u(x)\mp v(x)]f(x)dx=\int_{-\infty}^{\infty} u(x)f(x)dx\mp\int_{-\infty}^{\infty} v(x)\ f(x)\ dx$$

$$=E[u(x)]\mp E[v(x)]$$

- Ex "1" Given ap.d.f $f(x)=\begin{cases} \dfrac{x+2}{18} & \text{for } -2<x<4 \\ 0 & 0.w \end{cases}$

find $E\left[2x^3-1\right]$ & $E\left[(x+2)^2\right]$

Sol: $E\left[2x^3-1\right] \Rightarrow 1.\ g(x)=2x^3-1 \Rightarrow E\left[g(x)\right]=\int_{-2}^{4} g(x)\ f(x)\,dx$

Or 2. $E\left[2x^3-1\right]=2E(x^3)-1$

$$E(x^3)=\int_{-2}^{4} x^3\left(\frac{x+2}{18}\right)dx=\frac{1}{18}\int_{-2}^{4}(x^4+2x^3)\,dx$$

$$=\frac{1}{18}\left[\frac{x^5}{5}+\frac{x^4}{2}\right]_{-2}^{4}=\frac{1}{18}\left[\left(\frac{1024}{5}+\frac{256}{2}\right)-\left(\frac{-32}{5}+\frac{16}{2}\right)\right]$$

$$=\frac{1}{18}\left(\frac{1056}{5}+\frac{240}{2}\right)=\frac{1}{18}\left(\frac{1056}{5}+120\right)$$

$E\left[(x+2)^2\right] \Rightarrow 1.\ g(x)=(x+2)^2 \Rightarrow E\left[g(x)\right]=\int_{-2}^{4} g(x)\ f(x)\,dx$

Or 2. $E\left[x^2+4x+4\right]=E(x^2)+4E(x)-4$

$$=\int_{-2}^{4} x^2\ f(x)\,dx+4\int_{-2}^{4} x\ f(x)\,dx+4$$

$$= \cdots$$

H.W: 1 Given ap.d.f., $f(x)=\begin{cases} \dfrac{x}{2} & 0<x<4 \\ 0 & o.w. \end{cases}$

Find $E\left[(x+2)^2\right]$?

2. Given ap. m.f, $f(x)=\begin{bmatrix} \dfrac{x}{10} & \text{for } x=1,2,3,4 \\ 0 & \text{o.w.} \end{bmatrix}$ find $E\left[2x^3-1\right], E\left[(x-1)^3\right]$?

$(x-1)^3=(x^3-3x^2+3x-1)$

theorem "3": let x be ar.v.

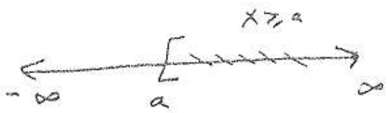
a. If $\exists(a)$ such that $p(x \geq a) = 1$, then $E(x) \geq a$.

b. If $\exists(b)$ such that $p(x \leq b) = 1$, then $E(x) \leq b$.

c. If $p(a \leq x \leq b) = 1$, then $a \leq E(x) \leq b$.

Proof: a. case "1": If x is a c.r.v. with p.d.f f(x)

$$P(x \geq a) = 1 \Rightarrow \int_a^\infty f(x)\,dx = 1$$



$\because f(x) > 0 \quad \text{for } a \leq x < \infty$

$\quad = 0 \quad \text{o.w.}$

$Rx = \{x : a \leq x < \infty\}$

$$E(x) = \int_a^\infty x\, f(x)\,dx \geq \int_a^\infty a\, f(x)\,dx = a \underbrace{\int_a^\infty f(x)\,dx}_{=1} = a$$

$$\therefore E(x) \geq a$$

Case (2) if x is d.r.v. with p.m.f f(x)

$$E(x) = \sum_{x=a}^\infty x\, f(x) \geq \sum a\, f(x) = a \underbrace{\sum f(x)}_{=1} = a$$

$\therefore E(x) \geq a$.

b. Similary of (a).

c. Case "1": If x is a d.r.v. have a p.m.f f(x)

$$P(a \leq x \leq b) = 1 \Rightarrow \sum_{x=a}^{b} f(x) = 1$$

$\therefore f(x) > 0 \quad \text{for } a \leq x \leq b$

$\quad = 0 \quad \text{o.w}$

$$E(x) = \sum_{x=a}^{b} x f(x) \geq \sum a f(x) = a \underbrace{\sum f(x)}_{=1}$$

$\therefore E(x) \geq a$ .... (1)

$$E(x) = \sum_{x=a}^{b} x f(x) \leq \sum b f(x) = b \underbrace{\sum f(x)}_{=1}$$

$\therefore E(x) \leq b$ .... (2)

$\therefore$ by "1" & "2" we get $a \leq E(x) \leq b$

theorem 4": If $p(x \geq a) = 1$ and $E(x) = a$ then $p(x = a) = 1$ and $p(x > a) = 0$.

proof: case "1": If x is ac.r.v. from ap.d.f f(x)

$$\therefore p(x \geq a) = 1 \Rightarrow \int_a^\infty f(x)\,dx = 1$$

$\therefore f(x) > 0$ for $a \leq x < \infty$

$=0$ ow

$E(x) = \int_a^\infty x \; f(x)\,dx = a = a.1 = a\int_a^\infty f(x)\,dx = \int_a^\infty af(x)\,dx$

$\int_a^\infty x\,f(x)\,dx = a\int_a^\infty f(x)\,dx \Rightarrow \int_a^\infty x\,f(x)\,dx = \int_a^\infty a\,f(x)\,dx$

..this inquality hold only when x=a

i.e. $(x>a)=\phi \Rightarrow P(X > a) = 0$

$p(x \geq a) = p[(x = a) \cup (x > a)] = p(x = a) + p(x > a) \Rightarrow 1 = p(x = a) + 0 \Rightarrow p(x = a) = 1$

case "2": If x is ad.r.v from ap.m.f f(x)

$\therefore p(x \geq a) = 1 \Rightarrow \sum_{x=a}^{\infty} f(x) = 1$

$\therefore f(x) > 0$ for $a \leq x < \infty$

$=0$ ow

$E(x) = \sum_{x=a}^{\infty} xf(x) = a = a.1 = a.\sum_{x=a}^{\infty} f(x) = \sum_{x=a}^{\infty} af(x)$

$\sum_{x=a}^{\infty} xf(x) = \sum_{x=a}^{\infty} af(x)$

This inequality hold only when x =a

i.e. $(x>a)=\phi \Rightarrow p(x > a) = 0$

$p(x \geq a) = [(x = a) \cup (x > a)] = p(x = a) + p(x > a)$

$\Rightarrow 1 = p(x = a) + 0 \Rightarrow p(x = a) = 1$

variance of random variable:

Def: let x be ar.v. the variance of x denoted by v(x) or δ^2 is defined as

: $v(x) = E\{[x - E(x)]^2\}$

$\because E(x) = \mu$, then $v(x) = E[(x-\mu)^2]$

note: since $(x-\mu)^2 \geq 0$ then $E[(x-\mu)^2] \geq 0$

$\therefore v(x) \geq 0$ alwayes.

properties of variance:

theorem "5" let x be ar.v.,then $v(x) = E(x^2) - [E(x)]^2$

Proof: $v(x)=E\{[x-E(X)]^2\}$

$\because V(x)=E\{x^2-2E(x)x+[E(x)]^2\}$, $E(x)$ constant.

$v(x)=E(x^2)-2E(x)E(x)+[E(x)]^2$

$v(x)=E(x^2)-2[E(x)]^2+[E(x)]^2 \Rightarrow v(x)=E(x^2)-[E(x)]^2$

note: 1. $v(x)=0 \Leftrightarrow E(x^2)=E[(x)]^2$

2. $\because v(x)\ge 0 \Rightarrow \therefore E(x^2)\ge [E(x)]^2$

3. $v(b)=0$, b is constant.

Theorem "6" let x be a r.v. and $v(x)$ exist , If $y=ax+b$; $a,b\in R$ then $v(y)=a^2v(x)$.

Proof: $y=ax+b \Rightarrow E(y)=aE(x)+b$

$V(y)=E\{[y-E(y)]^2\}=E\{[(ax+b)-(aE(x)+b)]^2\}$

$=E\{[ax-aE(x)]^2\}=E\{a^2[x-E(x)]^2\}$

$=a^2E\{[X-E(x)]^2\}=a^2V(x)$

Ex: Given a p.d.f $f(x)=\begin{cases}3x^2 & for\ 0<x<1\\ 0 & o.w.\end{cases}$

a. find $E(x)$ & $v(x)$. b. If $y=1-2x$, then find $E(y)$, $v(y)$.

SOL.

$$a.\ E(x)=\int_0^1 X(3X^2)dx=3\frac{X^4}{4}\Big|_0^1=\frac{3}{4}$$

$v(x)=E(x^2)-[E(x)]^2$

$$E=(x^2)=\int_0^1 x^2(3x^2)dx=\frac{3}{5}x^5\Big|_0^1=\frac{3}{5}$$

$$\therefore v(x)=\frac{3}{5}-\frac{9}{16}=\frac{48-45}{80}=+\frac{3}{80}$$

b. $\because y=(-2)x+1 \Rightarrow \therefore E(y)=(-2)E(x)+1$

$$\therefore E(y)=(-2)(\frac{3}{4})+1=\frac{-3}{2}+1=-\frac{1}{2}$$

$$v(y)=(-2)^2v(x)=4.(\frac{3}{80})=\frac{12}{80}$$

H.w.: Given a p.d.f. $f(x)=f(x)=\begin{cases}6x(1-x) & for\ 0<x<1\\ 0 & 0.w.\end{cases}$

If $y=2-3x$, then find $E(y)$ & $v(y)$

Theorem "7" $v(x)=0$ iff $\exists K$ where k is constant such that $p(x=k)=1$

$\Leftarrow$ suppose that $p(x=k)=1$, t.p $V(X)=0$

Poof: $f(x)>0 \quad for\ x=k$

$=0 \quad 0.w.$

$v(x)=v(x=k)=0$ by note.

$\Rightarrow$ suppose $V(x)=0$

,t.p $p(x=k)=1$

$\because v(x)=0 \Rightarrow E\{[x-E(x)]^2\}=0$

$Y=[x-E(x)]^2 \Rightarrow E(Y)=0$

by th. "4" $\Rightarrow [E(x)=a \rightarrow p(x=a)=1]$

$\therefore p(y=0)=1 \Rightarrow p\{[x-E(x)]^2=0\}=1$

$\therefore p\{(x-E(x))=0\}=1 \Rightarrow \ \therefore p[x=E(x)]=1$

$\therefore x=k \Rightarrow E(x)=E(k)=k \Rightarrow \therefore p[x=k]=1$

* ***Existence of Mean and Variance:***

E(x) exists iff $E(|x|)<\infty$

$\because V(x)=E(x^2)-[E(x)]^2 \Rightarrow \therefore V(x)$ exists iff $E(x)$ & $E(x^2)$ exist

Ex. "1": Given acauchy p.d.f $f(x)=\dfrac{1}{\pi(1+x^2)}$ for $-\infty<x<\infty$

Show that E(x) dose not exist?

Sol.: $E(|x|)=\int_{-\infty}^{\infty}|x|\dfrac{1}{\pi(1+x^2)}dx=2\int_0^{\infty}x\dfrac{1}{\pi(1+x^2)}dx$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{2x}{1+x^2}dx=\frac{1}{\pi}\ln(1+x^2)\Big|_0^{\infty}=\frac{1}{\pi}[\ln\infty-\ln 1]\not<\infty$$

$E(|x|)\not<\infty \Rightarrow E(x)$ *dose not exist*

Exercises: 1. Let x be a c.r.v. have a p.d.f f(x) where

$f(x)>0$ for $0<x<b<\infty$, Show that $E(x)=\int_0^b[1-F(x)]dx$

$=0$ o.w.

Hint.: $f(x)=\dfrac{dF(x)}{dx} \leftrightarrow f(x)dx=dF(x)$

2. If x is d.r.v. have p.M.f $f(x)>0$ for $x=-1, 0, 1$

$=0$ o.w.

a. If $f(0)=\frac{1}{2}$, Find $E(x^2)$. $\sum f(x)=1 \Rightarrow f(-1)+f(0)+f(1)=1$

b. If $f(0)=\frac{1}{2}$, and $E(x)=\frac{1}{6}$, Find f(-1), f(1)

3. Given a p.d.f $f(x)=\begin{cases}1-|x| & \text{for } |x|<1 \\ 0 & \text{o.w.}\end{cases}$

a. Find E(x) & V(x), b. If y= 2-3 x, find E(y) & V(y).

Hint $$f(x)=\begin{cases}1-|x| & -1<x<1 \\ 0 & \text{o.w.}\end{cases} = \begin{cases}1+x & -1<x<0 \\ 1-x & 0\leq x<1 \\ 0 & \text{o.w.}\end{cases}$$

Moments of Random Variables:

Def.: Let x be a r.v. either d.r.v. or c.r.v., let $K \in I^+$ then $E(x^k)$ is called "the k^{th} moment of x" or "the moment of order k of x".

when $k = 1 \Rightarrow E(x^1) = 1^{st}$ moment of $x = \mu$

$E(x^2) = 2^{nd}$ moment of x

Note: $E(x^k)$ exists iff $E(|x|^k) < \infty$

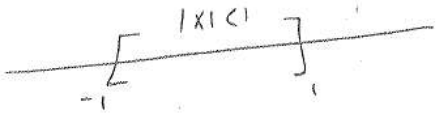
بدون برهان Theorem "8": If $E(x^k)$ exists then $E(x^j)$ exists, $j < k$ and $j, k \in I^+$

Proof: case "1" If x is c.r.v. with p.d.f f(x)

$\because E(x^k)$ exists $\Rightarrow \therefore E(|x|^k) < \infty$

T.p $E(x^j)$ exists, T.p $E(|x|^j) < \infty$

$$E(|x|^j) = \int_{-\infty}^{\infty} |x|^j f(x)dx$$



$$E(|x|^j) = \int_{|x|\leq 1}^{-\infty} |x|^j f(x)dx + \int_{|x|>1} |x|^j f(x)dx$$

2^{nd} central moment of R.V.X is equal to V(x).

Ex.: Let x be a r.v.s.t $E(x) = 1$, $E(x^2) = 2$ and $E(x^3) = 5$. Find the 3^{rd} central moment of x.

Sol.: $E[(x-\mu)^3] = E\{x^3 - 3\mu x^2 + 3\mu^2 x - \mu^3\}$

$= E(x^3) - 3\mu E(x^2) + 3\mu^2 E(x) - \mu^3$

$= 5 - 3 \cdot 1 \cdot 2 + 3 \cdot 1 \cdot 1 - 1 = 1$

Exercises: 1. If x ~ uniform (a, b); a, b $\in$ R. Find the valued of 1^{st} central moment of x and also find the 2^{nd} central moment of x.

2. Let $\mu = E(x)$ and $\delta^2 = V(x)$ show that $E[(x-\mu)^4] \geq \delta^4$

i.e. 4^{th} central moment of x is greater than or equal the square of variance of x.

Moment Generating Function (M.g.f).

Def.: A moment generating function (M.g.f) of a r.v.x is a function that determines all moments of x, denoted by $M_x(t)$ suppose that $t \in (-h, h)$, $h > 0$.

If $E[e^{tx}]$ exists $\forall t \in (-h, h)$ then $M_x(t) = E[e^{tx}]$, $-h < t < h$

There are two cases of $M_x(t)$.

Case "1": If x is a d.r.v. from a p.M.f f(x)

$$M_x(t) = E[e^{tx}] = \sum_{\forall x} e^{tx} f(x)$$

Case "2": If x is a c.r.v. have a p.d.f f(x)

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Ex.: Given a p.d.f $f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w} \end{cases}$

Find $M_x(t)$ and sketch it's graph.

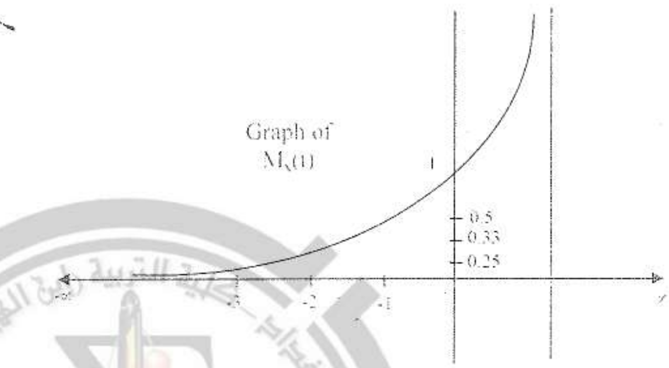
Sol.: $M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \cdot e^{-x} dx = \int_0^{\infty} e^{-(1-t)x} dx$

This integration exist only when (1-t) > 0, i.e. t < 1

$$M_x(t)=\frac{-1}{1-t}e^{-(1-t)x}\Big|_0^{\infty}=\frac{-1}{1-t}\left[\underset{=0}{e^{-\infty}}-\underset{=1}{e^{0}}\right]=\frac{1}{1-t}$$

$$M_x(t)=\frac{1}{1-t}\qquad for\quad t<1$$

t	$M_x(t)$
1	∞
0	1
-1	0.5
-2	0.33
-3	0.25
.	
$-\infty$	0



Ex.: Given a p.M.f. $f(x)=\begin{cases}\dfrac{x}{15} & \text{for } x=1,2,3,4,5\\ 0 & \text{o.w}\end{cases}$

Find $M_x(t)$.

Sol.: $M_x(t)=E(e^{tx})=\sum_{x=1}^{5}e^{tx}f(x)=\sum_{x=1}^{5}\frac{x}{15}e^{tx}$

$$=\frac{1}{15}\sum_{x=1}^{5}xe^{tx}=\frac{1}{15}[1\cdot e^{t}-2\cdot e^{2t}+3\cdot e^{3t}+4\cdot e^{4t}+5e^{5t}]\quad for\ -\infty<t<\infty$$

Theorem "9": Let x be a r.v. have a M.g.f. $M_x(t)$, then

$$M_x(0)=1,\ M'_x(0)=E(x),\ M''_x(0)=E(x^2),\ M'''_x(0)=E(x^3),\ \ldots,$$
$$M_x^{(k)}(0)=E(x^k)$$

Proof: $*\,M_x(t)=E(e^{tx})$ by Macclaurin series

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^k}{k!}+\ldots$$

$$\therefore e^{tx}=1+tx+\frac{t^2x^2}{2!}+\frac{t^3x^3}{3!}+\ldots+\frac{t^kx^k}{k!}+\ldots$$

$$M_x(t)=E[1+tx+\frac{t^2x^2}{2!}+\frac{t^3x^3}{3!}+\ldots+\frac{t^kx^k}{k!}+\ldots]$$

$$M_x(t)=1+tE(x)+\frac{t^2}{2!}E(x^2)+\frac{t^3}{3!}E(x^3)+\ldots+\frac{t^k}{k!}E(x^k)+\ldots$$

$M_x(t=0)=M_x(0)=1$

$$* M'_x(t)=\frac{dM_x(t)}{dt}=E(x)+\frac{2t}{2!}E(x^2)+\frac{3t^2}{3!}E(x^3)+\ldots+\frac{kt^{k-1}}{k!}E(x^k)+\ldots$$

$M'_x(0)=E(x)$

$$* M''_x(t)=E(x^2)+\frac{6t}{3!}E(x^3)+\ldots+\frac{k(k-1)t^{k-2}}{k!}E(x^k)+\ldots$$

$M''_x(0)=E(x^2)$

Simillary we can find $E(x^3)$, $E(x^4)$, ... $E(x^k)$, ...

Note: $M_x(t)=M_x(0)+tM'_x(0)+\frac{t^2}{2!}M''_x(0)+\ldots+\frac{t^k}{k!}M_x^{(k)}(0)$

This series is called the M. g.f. by Macclaurin series.

Ex.: Given $M_x(t)=\frac{1}{1-2t}$, $t<\frac{1}{2}$ Find $E(x)$ and $V(x)$

Sol.: $M_x(t)=(1-2t)^{-1}$, $t<\frac{1}{2}$

$M'_x(t)=-(1-2t)^{-2}\cdot(-2)=2(1-2t)^{-2}$

$\therefore E(x)=M'_x(0)=2(1-0)^{-2}=2$

$V(x)=E(x^2)-[E(x)]^2$

$M''_x(t)=8(1-2t)^{-3}$

$\therefore E(x^2)=M''_x(0)=8(1-0)^{-3}=8$

$\therefore V(x)=8-2^2=4\geq 0$

Theorem "10": Let x be a r.v. have a M.g.f $M_x(t)$. If $Y=ax+b$, a, b, $\in R$ then $M_y(t)=e^{bt}\cdot M_x(at)$

Proof: $Y=ax+b$

$M_y(t)=E(e^{ty})\,(by\ def.)=E[e^{t(ax+b)}]=E[e^{(at)x+bt}]$

$M_y(t)=[e^{(at)x}\cdot e^{bt}]=e^{bt}E[e^{(at)x}]=e^{bt}\cdot M_x(at)$

$Since\ M_x(t)=E(e^{tx})\Rightarrow M_x(at)=E[e^{(at)x}]$

Ex: Given $M_x(t)=\frac{1}{1-3t}$ for $t<\frac{1}{3}$, If y= 1-2x , find $M_y(t)$.

Sol: y= (-2)x+1 , a=-2 , b=1

By th. (10) $\Rightarrow M_y(t) = e^{bt} M_x(at) = e^t M_x(-2t)$

$\because M_x(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$

$M_x(-2t) = \frac{1}{1-3(-2t)} = \frac{1}{1+6t}$ for $t > \frac{-1}{6}$

$M_y(t) = e^t \cdot \frac{1}{1+6t}$ for $t > \frac{-1}{6}$

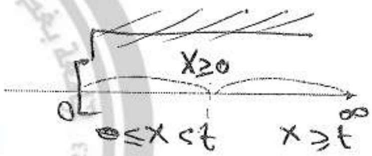
The bounded of probability:

Theorem "11": ((Markov inquality))

If x is a r.v. and if $P(x \geq 0) = 1$ then $P(x \geq t) \leq \frac{E(x)}{t}$, for $t > 0$

Proof: case "1": if x is a c.r.v with a p.d.f f(x)

$$\because p(x \geq 0) = 1 \Rightarrow \int_0^\infty f(x)dx = 1$$



, (t>0)

$$f(x) > 0 \quad \text{for } x \geq 0$$
$$= 0 \quad \text{O.W}$$

$$E(x) = \int_0^\infty x f(x)dx = \int_0^t x f(x)dx + \int_t^\infty x f(x)dx \geq \int_t^\infty x f(x)dx \geq \int_t^\infty t f(x)dx \quad [x \geq t]$$

$$\therefore E(x) \geq \int_t^\infty t f(x)dx \Rightarrow E(x) \geq t \int_t^\infty f(x)dx$$

$$E(x) \geq t \cdot p(x \geq t)$$

$$\therefore p(x \geq t) \leq \frac{E(x)}{t}$$

Case "2": If x is a d.r.v. with a p.m.f. f(x) by the same method.

Theorem "12": "Chebyshev inqualities"

Let x be a r.v. where V(x) exists,then:

1. $p(|x-\mu| \geq t) \leq \frac{V(x)}{t^2}$, $t > 0$, 2. $p(|x-\mu| < t) \geq 1 - \frac{V(x)}{t^2}$, $t > 0$,

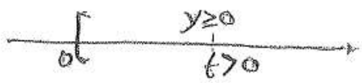
Note: 1. $\frac{V(x)}{t^2}$ *is called the upper buond of* $p(|x-\mu| \geq t)$

2. $1 - \frac{V(x)}{t^2}$ *is called the lower buond of* $p(|x-\mu| < t)$

Proof: 1. $V(x) = E[(x-\mu)^2] \geq 0$

$Let\ y=(x-\mu)^2 \geq 0 \Rightarrow E(y)=E[(x-\mu)^2 = V(x) \geq 0$

$\therefore$ by Markov inquality, we get



$p(y \geq t^2) \leq \dfrac{E(y)}{t^2} \Rightarrow p[(x-\mu)^2 \geq t^2] \leq \dfrac{V(x)}{t^2}$

$p[|x-\mu| \geq t] \leq \dfrac{V(x)}{t^2}$

2. $p(A^c) = 1 - p(A)$

$p[|x-\mu| \geq t^2] = 1 - p[|x-\mu| < t^2] \leq \dfrac{V(x)}{t^2}$

$p[|x-\mu| < t^2] \geq 1 - \dfrac{V(x)}{t^2}$

Ex. "1": If $x \sim$ unif. $(-\sqrt{3}, \sqrt{3})$ then:

a. Find the upper bound of $p(|x-\mu| \geq \frac{3}{2})$

b. Find the value of $p(|x-\mu_x| \geq \frac{3}{2})$

Sol.:

a. $f(x) = \begin{cases} \dfrac{1}{\sqrt{3}+(\sqrt{3})} = \dfrac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < x < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$

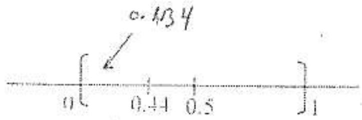
$u.b = \dfrac{V(x)}{t^2}, \qquad t = \dfrac{3}{2}$

$E(x) = \int_{\sqrt{3}}^{\sqrt{3}} X \dfrac{1}{2\sqrt{3}} dx = \dfrac{1}{4\sqrt{3}} X^2 \Big|_{-\sqrt{3}}^{\sqrt{3}} = 0$

$E(x^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \dfrac{1}{2\sqrt{3}} dx = \dfrac{1}{6\sqrt{3}} x^3 \Big|_{-\sqrt{3}}^{\sqrt{3}} = \dfrac{1}{6\sqrt{3}}[(\sqrt{3})^3 - (-\sqrt{3})^3] = \dfrac{1}{6\sqrt{3}}[3\sqrt{3}+3\sqrt{3}] = 1$

$\therefore V(x) = E(x^2) - [E(x)]^2 = 1 - 0 = 1$

$\therefore u.b = \dfrac{V(x)}{t^2} = \dfrac{1}{(\frac{3}{2})^2} = \dfrac{4}{9} = 0.44$



b. $p(|x - \mu| \geq \frac{3}{2}) = p(|x - 0| \geq \frac{3}{2}) = p(|x| \geq \frac{3}{2})$

$= 1 - p(|x| < \frac{3}{2}) = 1 - p(-\frac{3}{2} < x < \frac{3}{2}) = 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} dx$

$= 1 - \frac{1}{2\sqrt{3}} x \Big|_{-\frac{3}{2}}^{\frac{3}{2}} = 1 - \frac{1}{2\sqrt{3}}(\frac{3}{2} + \frac{3}{2}) = 1 - \frac{3}{2\sqrt{3}} = 1 - \frac{\sqrt{3}}{2}$

$= 1 - 0.866 = 0.134$

Ex. "2": Given a p.d.f $f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$

a. Find the lower bound of $p(\frac{5}{4} < x < \frac{11}{4})$

b. Find the value of $p(\frac{5}{4} < x < \frac{11}{4})$

Sol.: a. $L.b = 1 - \frac{V(x)}{t^2}$

$E(x) = \int_0^3 x \cdot \frac{2x}{9} dx = \frac{2}{27} x^3 \Big|_0^3 = \frac{2}{27}(27 - 0) = 2$

$E(x^2) = \int_0^3 x^2 \cdot (\frac{2x}{9}) dx = \frac{2}{36} x^4 \Big|_0^3 = \frac{1}{18}(81) = \frac{9}{2} = 4.5$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$= 4.5 - 4 = 0.5 = \frac{1}{2}$

$p(\frac{5}{4} < x < \frac{11}{4}) = p(\frac{5}{4} - 2 < x - 2 < \frac{11}{4} - 2) = p(-\frac{3}{4} < x - 2 < \frac{3}{4})$

$= p(|x - 2| < \frac{3}{4})$

$\therefore t = \frac{3}{4}$

$\therefore L.b = 1 - \dfrac{V(x)}{t^2} = 1 - \dfrac{\frac{1}{2}}{\frac{9}{16}} = 1 - \frac{1}{2} \times \frac{16}{9} = 1 - \frac{8}{9} = \frac{1}{9}$

b. $p(\frac{5}{4} < x < \frac{11}{4}) = \int_{\frac{5}{4}}^{\frac{11}{4}} \frac{2x}{9} dx$

$= \frac{1}{9} x^2 \Big|_{\frac{5}{4}}^{\frac{11}{4}} = \frac{1}{9}[\frac{121}{16} - \frac{25}{16}] = \frac{1}{9}[\frac{96}{16}] = \frac{6}{9}$

Median of Distribution of r.v.:

Def.: The median (m) is a value of x such that satisfing the two following inqualities:

$p(x < m) < \frac{1}{2}$ & $p(x \le m) \ge \frac{1}{2}$

By properties of c.d.f F(x)

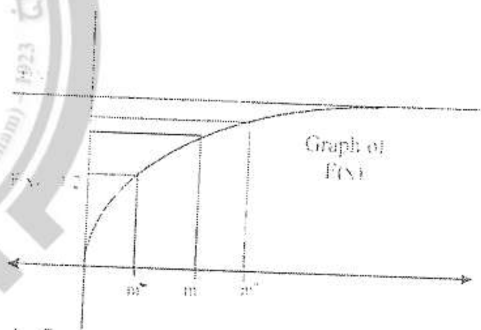
$p(x < m) = F(\bar{m})$, $p(x \le m) = F(m) = F(m^{-})$

$i.e.: p(x < m) = F(\bar{m}) < \frac{1}{2}$, $p(x \le m) = F(m^{-}) \ge \frac{1}{2}$

Note ① If $\bar{m} = m = \bar{m}$ then

$p(x \le m) = F(m) = \frac{1}{2}$

② The value of median is unique



Ex.: Given a p.m.f $f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$

Find the median of x.

Sol.: $p(x < m) < \frac{1}{2}$ & $p(x \le m) \ge \frac{1}{2}$

Suppose that m = 1

$p(x < 1) = 0 < \frac{1}{2}$, $p(x \le 1) = f(1) = \frac{1}{15} \not\ge \frac{1}{2}$

$\therefore m \ne 1$

Suppose that m = 2

$p(x<2)=f(1)=\frac{1}{15}<\frac{1}{2}$

$p(x\le 2)=f(1)+f(2)=\frac{1}{15}+\frac{2}{15}=\frac{1}{15}\ngeq\frac{1}{2}$

$\therefore m\neq 2$

Suppose that m=3

$p(x<3)=f(1)+f(2)=\frac{1}{15}+\frac{2}{15}=\frac{1}{5}<\frac{1}{2}$

$p(x\le 3)=f(1)+f(2)+f(3)=\frac{1}{15}+\frac{2}{15}+\frac{3}{15}=\frac{6}{15}\ngeq\frac{1}{2}$

$\therefore m\neq 3$

Suppose that m = 4

$\frac{12}{30}<\frac{15}{30}$

$p(x<4)=f(1)+f(2)+f(3)=\frac{6}{15}<\frac{1}{2}$

$p(x\le 4)=f(1)+f(2)+f(3)+f(4)=\frac{10}{15}\ge\frac{1}{2}$

$\therefore m=4$ is median

<u>Ex.:</u> Given a p.d.f $f(x)=\begin{cases}\frac{1}{x^2} & \text{for } x>1\\ 0 & \text{o.w.}\end{cases}$

Find the median of x?

Sol.: $p(x<m)<\frac{1}{2}$ & $p(x\le m)\ge\frac{1}{2}$

$$p(x<m)=\int_1^m\frac{1}{x^2}\,dx=-\frac{1}{x}\Big|_1^m=-[\frac{1}{m}-1]<\frac{1}{2}$$

$$-\frac{1}{m}+1<\frac{1}{2}\Rightarrow-\frac{1}{m}<-\frac{1}{2}\Rightarrow m<2\quad .....(1)$$

$$p(x\le m)=\int_1^m\frac{1}{x^2}\,dx\ge\frac{1}{2}$$

$$-[\frac{1}{m}-1]\ge\frac{1}{2}\Rightarrow m\ge 2\quad ....(2)$$

From (1) & (2) $\Rightarrow$ median = 2

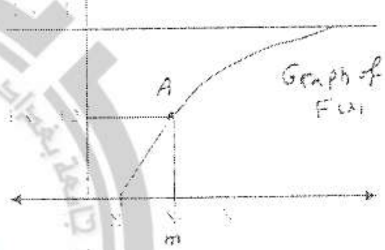
Note: We can also find the value of (Median) from the graph of F(x) such that. At the point $F(x)=\frac{1}{2}$, draw a line to cut the curve of F(x) at A, draw a line to cut the X-axis at m (median) from ex. "2" above

$$f(x)=\begin{cases}\frac{1}{x^2} & \text{for } x>1\\ 0 & \text{o.w.}\end{cases}$$

$$F(x)=p(X\le x)=\int_1^x\frac{1}{t^2}\,dt=1-\frac{1}{x}\Rightarrow F(x)=\begin{cases}1-\frac{1}{x} & \text{for } x\ge 1\\ 0 & \text{o.w.}\end{cases}$$

or : $F(x)=\frac{1}{2}$

$$1-\frac{1}{x}=\frac{1}{2}\Rightarrow-\frac{1}{x}=-\frac{1}{2}\Rightarrow x=2$$



To Find the Mode of Dist. Of R.V.X

Def.: Mode is a value of a r.v.x that maximize f(x)

i.e. If x_1 = mode. ⟶ then $f(x_1)$ is a Max.

Note: $F(x_1)$ is Max. $\Leftrightarrow f''(x_1)<0$

Ex.: Given a p.d.f. $f(x)=\begin{cases}12x^2(1-x) & 0<x<1\\ 0 & \text{o.w.}\end{cases}$

Find the mode of x?

Sol.: 0 < mode < 1

$f(x)=12x^2-12x^3\Rightarrow\quad f'(x)=24x-36x^2$

$\Rightarrow\quad 12x(2-3x)=0\qquad$ either $x=0\qquad$ or $\quad x=\frac{2}{3}$

$f''(x)=24-72x\Rightarrow f''(0)=24>0\qquad\Rightarrow f(0)$ is min.

$$f''(\tfrac{2}{3})=24-72(\tfrac{2}{3})=-24<0\Rightarrow f(\tfrac{2}{3})\quad is\ \max$$

$x_1=\frac{2}{3}=$ mode

Def.: Percentile

It is a value of x say (x_0) such that $p(x \le x_0) = \frac{t}{100}$ $\quad 0 < t < \infty$

denoted by P_t, $\quad 0 < t < 100$

i.e.: $P_t = x_0$

Ex.: Given a p.d.f $f(x) = \begin{cases} \frac{x}{2} & , 0 < x < 2 \\ 0 & , \quad \text{o.w.} \end{cases}$

Find P_{40}, P_{65}

Sol.: let $p_{40} = x_0$, $t = 40$

$p(x \le x_0) = \frac{t}{100}$, $\quad p(x \le x_0) = \frac{40}{100} = 0.4$

$\int_0^{x_0} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^{x_0} = 0.4$

$[x^2 - 0^2] = 4(0.4) \Rightarrow x^2 = 1.6 \Rightarrow x_0 = \sqrt{1.6} = 1.26$

Exercises

① Given a p.f.

$$f(-1)=\frac{1}{8},\ f(0)=\frac{6}{8},\ f(1)=\frac{1}{8}$$

Find the u.b. of $P(|X|\geq 2\sigma)$.

sol.

$$E(X)=\sum_{x=-1}^{1} x f(x)=(-1)f(-1)+(0)f(0)+(1)f(1)$$
$$=-\frac{1}{8}+0+\frac{1}{8}=0$$

$$\boxed{E(X)=\mu=0}$$

$$E(X^2)=\sum_{x=-1}^{1} x^2 f(x)=(-1)^2 f(-1)+(0)^2 f(0)+(1)^2 f(1)$$
$$=\frac{1}{8}+0+\frac{1}{8}=\frac{1}{4}$$

$$\delta^2=V(X)=E(X^2)-[E(X)]^2=\frac{1}{4}-0=\frac{1}{4}$$

$$\boxed{\sigma=\frac{1}{2}}$$

$$P(|X|\geq 2\sigma)=P\left(|X-0|\geq 2\cdot\frac{1}{2}\right)=P(|X-\mu|\geq 1)$$
$$=P(|X-\mu|\geq t)\leq\frac{V(X)}{t^2}\Rightarrow t=1$$

$$u.b=\frac{V(X)}{t^2}=\frac{\left(\frac{1}{4}\right)}{1}=\frac{1}{4}$$

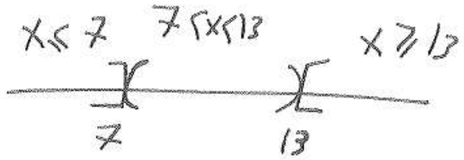
Note If we have to find the exact value of above pr. :

$$P(|X|\geq 2\cdot\sigma)=P\left(|X|\geq 2\cdot\frac{1}{2}\right)=P(|X|\geq 1)=1-P(|X|<1)$$
$$=1-P(-1<X<1)=1-P(X=0)=1-f(0)$$
$$=1-\frac{6}{8}=\frac{2}{8}$$

② If X is a r.v. with $E(X)=10$, $P(X\geq 7)=0.1$, $P(X\geq 13)=0.3$, then show that $V(X)\geq\frac{9}{2}$.

sol.

$$P(|X-\mu|<t)\geq 1-\frac{V(X)}{t^2}$$
$$P(|X-\mu|\geq t)\leq\frac{V(X)}{t^2}$$



$$P[(X\leq t)\cup(7<X<13)\cup(X\geq 13)]=P(S)$$
$$P(X\leq 7)+P(7<X<13)+P(X\geq 13)=1$$
$$0.2+P(7<X<13)+0.3=1$$
$$P(7<X<13)=0.5=\frac{1}{2}$$
$$\therefore E(X)=\mu=10$$

④ Let $X \sim b(2,p)$ & $Y \sim b(4,p)$. If $P(X \geq 1) = \frac{5}{9}$, Find $P(Y \geq 1)$.

Sol. $\because X \sim b(2,p)$

$$\therefore f(x,2,p) = \begin{cases} \binom{2}{x} p^x (1-p)^{n-x} & \text{for } x = 0,1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \geq 1) = \frac{5}{9} \Rightarrow 1 - P(X < 1) = \frac{5}{9} \Rightarrow 1 - P(X=0) = \frac{5}{9}$$

$\therefore P(X=0) = \frac{4}{9} = f(0)$

$\binom{2}{0} p^0 (1-p)^2 = \frac{4}{9}$

$1 \cdot 1 \cdot (1-p)^2 = \frac{4}{9}$

$(1-p)^2 = \frac{4}{9}$

$(1-p) = \mp \frac{2}{3}$

If $1-p = \frac{2}{3}$

then $\boxed{p = \frac{1}{3}}$

If $1-p = -\frac{2}{3}$

Since $p = 1 + \frac{2}{3} = \frac{5}{3} > 1$

$0 < p < 1$

$\therefore \boxed{p = \frac{1}{3}}$

$\because Y \sim b(4,p)$

$$\therefore f(y,4,p) = \begin{cases} \binom{4}{y} p^y (1-p)^{4-y} & ; y = 0, \dots, 4 \\ 0 & \text{o.w.} \end{cases}$$

$\because p = \frac{1}{3} \Rightarrow 1-p = \frac{2}{3}$

$P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y=0)$

$= 1 - f(0)$

$= 1 - \binom{4}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4$

$\therefore P(Y \geq 1) = 1 - \left(\frac{2}{3}\right)^4 = 1 - \frac{16}{27} = \frac{11}{27} < 1$

⑤ If $X \sim G(\frac{1}{2})$, find Median of X

Sol. $X \sim G(p)$; $p = \frac{1}{2}$, $q = 1 - \frac{1}{2} = \frac{1}{2}$

$$f(x;p) = \begin{cases} \frac{1}{2} \left(\frac{1}{2}\right)^{x-1} & \text{for } x = 1,2,3,\dots \\ 0 & \text{o.w.} \end{cases}$$

$P(X < m) \leq \frac{1}{2}$ & $P(X \leq m) \geq \frac{1}{2}$, $m \equiv$ median

If $m = 1$, $P(X < 1) = P(X = 0) = 0 < \frac{1}{2}$

$P(X \leq 1) = P(X = 1) = \frac{1}{2}$

then $\boxed{m = 1}$

Can $m = 2$?

$P(X<2) = P(X=1) = \frac{1}{2}$

$P(X \leq 2) = P(X=1,2) = P(X=1)+P(X=2) = \frac{1}{2} + \frac{1}{2}(\frac{1}{2})$

$= \frac{3}{4} > \frac{1}{2}$.

$\therefore \boxed{m=2}$

There are two values of Median 1, 2.

⑥ Given $M_X(t) = (\frac{2}{3} + \frac{1}{3}e^t)^9$, Show that :

$P(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} \binom{9}{x}(\frac{1}{3})^x(\frac{2}{3})^{9-x} = P(0 < X < 6)$

Sol.

$M_X(t) = (\frac{2}{3} + \frac{1}{3}e^t)^9 \Longrightarrow P = \frac{1}{3}, n=9, 1-b = \frac{2}{3}$

$\therefore \boxed{X \sim b(9, \frac{1}{3})}$

$M_X(t) = (1-P+Pe^t)^n$

$$f(x, 9, \tfrac{1}{3}) = \begin{cases} \binom{9}{x}(\frac{1}{3})^x(\frac{2}{3})^{9-x} & \text{for } x = 0, 1, 2, \ldots, 9 \\ 0 & \text{o.w.} \end{cases}$$

$E(X) = \mu = nP$, $\sigma^2 = V(X) = nP(1-P)$

$E(X) = 9(\frac{1}{3})$; $V(X) = 3(\frac{2}{3})$

$= 3$; $= 2$

$\mu = 3$; $\sigma^2 = 2$

$\sigma = \sqrt{2}$

$\mu - 2\sigma = 3 - 2\sqrt{2} = 3 - 2(1.4) = 3 - 2.8 = 0.2$ $\boxed{\sqrt{2} = 1.4}$

$\mu + 2\sigma = 3 + 2.8 = 5.8$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(.2 < X < 5.8) = P(1 \leq X \leq 5)$

$= \sum_{x=1}^{5} f(x)$

$= \sum_{x=1}^{5} \binom{9}{x}(\frac{1}{5})^x(\frac{2}{3})^{9-x}$

$= P(0 < X < 6)$

(7) If $X \sim b(n,p)$, and $Y = \frac{X}{n}$, Prove that :

$\forall \varepsilon > 0$, $\lim_{n\to\infty} P(|Y-p| \geq \varepsilon) = 0$

Sol.

$P(|X-\mu| \geq t) \leq \frac{V(X)}{t^2} = u.b.$

$P(|X-\mu| < t) \geq 1 - \frac{V(X)}{t^2} = L.b.$

$P(|\mu-p| \geq \varepsilon) = P(|\frac{X}{n} - p| \geq \varepsilon) = P(|\frac{X-np}{n}| \geq \varepsilon)$

$= P(|X-np| \geq n.\varepsilon)$

$\boxed{\mu = np}$

$P(|X-np| \geq n.\varepsilon) \geq \frac{V(X)}{t^2}$, $\boxed{V(X) = np(1-p)}$

$= \frac{np(1-p)}{n^2\varepsilon^2}$

$= \frac{p(1-p)}{n\varepsilon}$

a^2 $n \to \infty$; $P(|X-np| \geq n.\varepsilon) \to 0$

$\therefore \lim_{n\to\infty} P(|X-\mu| \geq t) = \lim_{n\to\infty} P(|X-np| \geq n.\varepsilon)$

$= 0$ By $(0 \leq P(A) \leq 1)$

$P(|Y-p| < \varepsilon) = P(|\frac{X}{n} - p| \geq \varepsilon) = P(|\frac{X-np}{p}| < \varepsilon)$

$= P(|X-np| < n.\varepsilon)$; $\mu = np$, $t = n.\varepsilon$

$< 1 - \frac{V(X)}{t^2}$; $V(X) = np(1-p)$

$= 1 - \frac{np(1-p)}{n^2\varepsilon^2} = 1 - \frac{p(1-p)}{n\varepsilon^2}$

$\therefore \lim_{n\to\infty} (|Y-p| < \varepsilon) = \lim_{n\to\infty} (1 - \frac{p(1-p)}{n\varepsilon^2}) = 1-0$

$= 1$

(8) Suppose that $X_1, X_2, \dots, X_n$ are n- independent random variables, and for $i = 1, 2, \dots, n$ let ψ_i be the M.g.f. of X_i. Let $Y = \sum_{i=1}^{n} x_i$ and let ψ be the M.g.f. of Y. Then for any t s.t. $\psi_i(t)$ exist,

$\psi(t) = \prod_{i=1}^{n} \psi_i(t)$.

Sol. X_i $(i = 1, 2, \dots, n)$ is indep. r.v.s

ψ_i $(i = 1, 2, \dots, n)$ be M.g.f. of X_i

6

If $y=\sum_{i=1}^{n} X_i$, then if ψ be the M.g.f. of ψ, then forevery t s.t. ($\psi_i(t)$ exists), then

$$\psi(t)=\prod_{i=1}^{n}\psi_i(t) \quad \text{s.t. } y=\sum_{i=1}^{n}X_i$$

$$\psi_y(t)=E(e^{ty})=E[e^{(\Sigma x_i)t}]=E[e^{tx_1+tx_2+\cdots+tx_n}]$$

$$=E(e^{tx_1})\cdot E(e^{tx_2})\cdots E(e^{tx_n})$$

$$=(\psi_1(t))\,(\psi_2(t))\cdots(\psi_n(t))$$

$$=\prod_{i=1}^{n}\psi_i(t)$$

(9) Let X be a r.v. with M.g.f. ψ_1, Let $Y=ax+b$; where $a,b\in R$ and let ψ_2 denote the M.g.f. of Y, then for any value of t such that $\psi_1(at)$ exists, $\psi_2(t)=\exp(bt)\cdot\psi_1(at)$.

sol.

$$\psi_2(t)=E(e^{ty})=E[e^{(ax+b)t}]=E[e^{axt}\cdot e^{bt}]=e^{bt}E(e^{axt})$$

$$=e^{bt}\cdot\psi_1(at)=\exp(bt)\,\psi_1(at)$$

since $\psi_1(t)=E(e^{tx})$

$\therefore \quad \psi_1(at)=E(e^{atx})$

نظرية
4 درجة

(10) If X is a r.v. with $M_X(t)$ exists for $t\in(-h,0)$, then

$$P(X\leq a)\leq \exp(-at)\cdot M_X(t)$$

sol. $M_X(t)=E(e^{tx})$

$$y-e^{tx}>0$$

$$e^{at}>0$$

$$\therefore P(X\geq t)\leq\frac{E(X)}{t} \quad \text{by Markave}$$

$$P(Y\geq e^{at})\leq\frac{E(Y)}{e^{at}}$$

$$P(e^{tx}\geq e^{at})\leq \exp(at)\cdot M_X(t)$$

$$\therefore P(X\leq a)\leq \exp.(at)\cdot M_X(t)$$

X
صفر

(11) Given a m.g.f. $M_X(t) = e^{3t^2+2t}$ for $-\infty < t < \infty$
Find the L.b. of $P(-\frac{1}{2} < X < \frac{9}{2})$.

Sol.

$$M'_X(t) = \left(e^{3t^2+2t}\right)(6t+2)$$

$$E(X) = M'_X(0) = e^0(0+2)$$

$$M''_X(t) = \left(e^{3t+2t}\right)(6) + (6t+2)^2 e^{3t^2+2t}$$

$$E(X^2) = (e^0)(6) + (2)^2(e^0) = 6+4 = 10$$

$$V(X) = E(X^2) - [E(X)]^2 = 10 - (2)^2 = 6$$

$$\therefore P(-\tfrac{1}{2} < X < \tfrac{9}{2}) = P(-\tfrac{1}{2} - 2 < X-2 < \tfrac{9}{2} - 2)$$

$$= P(-2.5 < X-2 < 2.5)$$

$$= P(|X-2| < 2.5) \Rightarrow t = \tfrac{5}{2} = 2.5$$

$$L.b. = 1 - \frac{V(X)}{t^2} = 1 - \frac{6}{(\frac{5}{2})^2} = 1 - 6 \cdot \frac{4}{25} = 1 - \frac{24}{25}$$

$$= \frac{1}{25} \simeq 0.04$$

(12) Given a p.m.f. $f(x) = \begin{cases} \frac{x}{10} & \text{for } x = 0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$
Find the pr. Percentite at the point $x = 3.5$.

Sol.

$$P(X \le x_0) = \frac{t}{100}$$

$$P(X \le 3.5) = \frac{t}{100} \Rightarrow \sum_{x=0}^{3} f(x) = \frac{t}{100}$$

$$f(0) + f(1) + f(2) + f(3) = \frac{t}{100}$$

$$\frac{0}{10} + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = \frac{t}{100} \Rightarrow \frac{6}{10} = \frac{t}{100} \Rightarrow 10t = 600$$

$$\boxed{t = 60\,\%}$$

(13) Find the mean , median and mode of Cauchy dist. if exists.

Sol. $f(x) = \dfrac{1}{\pi(1+x^2)}$ $\quad -\infty < x < \infty$ $\quad$ (Cauchy dist.)

Mean: $E(X)$ exist only when $E(|X|) < \infty$

$$E(|X|) = \int_{-\infty}^{\infty} |x| \frac{1}{\pi(1+x^2)} dx = \int_0^{\infty} \frac{2x}{\pi(1+x^2)} dx = \frac{1}{\pi} Ln(1+x^2)\Big|_0^{\infty}$$

$$= \frac{1}{\pi}[Ln(\infty) - Ln(0)] = \infty$$

$E(X)$ not exist.

Median $P(X \le m) \ge \frac{1}{2}$ & $P(X < m) \le \frac{1}{2}$

$P(X < m) \le \frac{1}{2}$

$$\int_{-\infty}^{m} \frac{1}{\pi(1+x^2)}\,dx \le \frac{1}{2}$$

$$\frac{1}{\pi}\int_{-\infty}^{m} \frac{1}{1+x^2}\,dx \le \frac{1}{2}$$

$$\frac{1}{\pi} \tan^{-1}x \Big|_{-\infty}^{m} \le \frac{1}{2}$$

$$\frac{1}{\pi}\Big[\tan^{-1}(m) - \underbrace{\tan^{-1}(\infty)}_{-\pi/2}\Big] \le \frac{1}{2}$$

$$\tan^{-1}(m) + \frac{\pi}{2} \le \frac{\pi}{2}$$

$$\tan^{-1}(m) \le 0$$

$$m \le \tan(0)$$

$$\therefore m \le 0$$

$P(X \le m) \ge \frac{1}{2}$

$$\int_{-\infty}^{m} \frac{1}{\pi(1+x^2)}\,dx \ge \frac{1}{2}$$

$$\frac{1}{\pi}\int_{-\infty}^{m} \frac{1}{(1+x^2)}\,dx \ge \frac{1}{2}$$

$$\frac{1}{\pi} \tan^{-1}x \Big|_{-\infty}^{m} \ge \frac{1}{2}$$

$$\frac{1}{\pi}\big[\tan^{-1}(m) - \tan^{-1}(\infty)\big] \ge \frac{1}{2}$$

$$\tan^{-1}(m) \ge 0$$

$$m \ge \tan(0)$$

$$\therefore m \ge 0$$

$\therefore \boxed{m_e = 0}$ median.

Mode $f(x) = \frac{1}{\pi}(1+x^2)^{-1} \Rightarrow f'(x) = -\frac{1}{\pi}(1+x^2)^{-2}(2x) = 0$

$$\frac{-2x}{\pi(1+x^2)^2} = 0 \Rightarrow x = 0$$

$$f''(x) = \frac{2}{\pi}(1+x^2)^{-3}(2x)(2x) - \frac{1}{\pi}(1+x^2)^{-2}(2)$$

$$f''(0) = 0 - \frac{2}{\pi} < 0 \Rightarrow mode = 0$$

$\therefore \boxed{M_o = 0}$ mode.

(14) Given a p.d.f. $f(x) = \begin{cases} e^x & \text{for } x < 0 \\ 0 & \text{o.w.} \end{cases}$

Find $M_X(t)$ and sketch it's graph. Also, find $E(x)$ by two methods.

Sol.

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{0} e^{tx} e^{x}\,dx = \int_{-\infty}^{0} e^{x(1+t)}\,dx$$

This integration exists only when $(1+t) > 0 \Rightarrow \boxed{t > -1}$

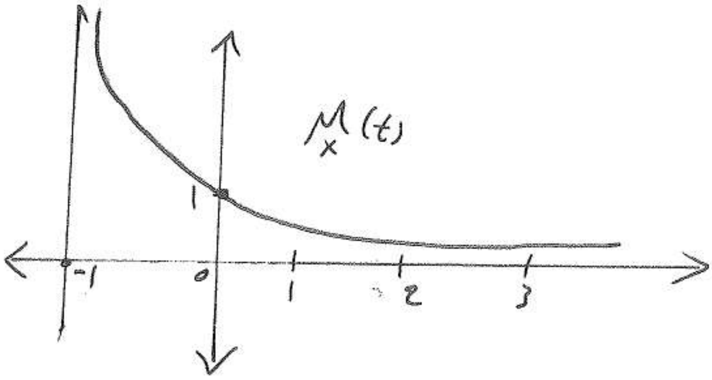
$$M_x(t) = \frac{1}{1+t} e^{x(1+t)} \Big|_{-\infty}^{0} = \frac{1}{1+t}\left[e^{0} - e^{-\infty}\right] = \frac{[1-0]}{1+t} = \frac{1}{1+t}$$

$$M_x(t) = \begin{cases} \frac{1}{1+t} & \text{for } t > -1 \\ 0 & \text{o.w.} \end{cases}$$

must $1+t>0$

$t>-1$

t	$M_x(t)$
-1	∞
0	1
1	$1/2$
2	$1/3$
3	$1/4$
$\vdots$	
∞	0



① $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cdot e^{x}\,dx$

$$= uv\Big|_{-\infty}^{0} - \int_{-\infty}^{0} v\,du \qquad u = x, \; dv = e^{x}, \quad du = 1, \; v = e^{x}$$

$$= xe^{x}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}\,dx$$

$$= \left[0 \cdot e^{0} - (-\infty)e^{-\infty}\right] - e^{x}\Big|_{-\infty}^{0} = [0-0] - \left[e^{0} - e^{-\infty}\right] = -1$$

$$M_x(t) = \frac{1}{1+t} = (1+t)^{-1} \quad ; t > -1$$

$$M_x'(t) = -1(1+t)^{-2}(1)$$

$$E(X) = M_x'(0) = (-1)[1+0]^{-2} = -1$$

⑮ If $X \sim unif(0,2)$, Find the u.b. of $P(|X-\mu| \geq 2)$.

Sol: $u.b. = \frac{V(X)}{t^2}, \quad t = 2$

$$f(x) = \begin{cases} \frac{1}{2-0} = \frac{1}{2} & 0 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu = E(X) = \int_0^2 x\, f(x)\,dx = \int_0^2 x \cdot \frac{1}{2}\,dx = \frac{1}{2} \cdot \frac{x^2}{2}\Big|_0^2 = \frac{1}{4}[4-0] = \frac{4}{4} = 1$$

$$\boxed{\mu = E(X) = 1}$$

$$E(X^2) = \int_0^2 x^2 \cdot \frac{1}{2}\,dx = \frac{1}{2}\frac{x^3}{3}\Big|_0^2 = \frac{1}{6}[8-0] = \frac{8}{6} = \frac{4}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$
$$= \frac{4}{3} - (1)^2$$
$$= \frac{4}{3} - \frac{3}{3} = \frac{1}{3}$$
$$U.b. = \frac{V(X)}{t^2} = \frac{(\frac{1}{3})}{(2)^2} = \frac{1/3}{4} = \frac{1}{12}$$

(16) Find the Pr. percentite at the point $x=1$, if Given a p.d.f.

$$f(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

$$P(X\le 1) = \frac{t}{100} \Rightarrow \int_0^1 f(x)\,dx = \frac{t}{100} \Rightarrow \frac{3}{8}\int_0^1 x^2\,dx = \frac{t}{100}$$
$$\frac{3}{8}\cdot\frac{x^3}{3}\Big|_0^1 = \frac{1}{8} = \frac{t}{100} \Rightarrow t = \frac{100}{8} = 12.5\,\% = 0.125$$

(17) A Coin is tossed 4-times, X = number of heads. Find the mean, median and mode of X (if exists).

Sol.

$$f(x) = \begin{cases} \frac{\binom{4}{x}}{16} & x = 0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$$

X	$f(x) = P(X=x)$
0	$f(0) = \frac{1}{16}$
1	$f(1) = \frac{4}{16}$
2	$f(2) = \frac{6}{16}$
3	$f(3) = \frac{4}{16}$
4	$f(4) = \frac{1}{16}$

$$\sum_{x=0}^{4} f(x) = \frac{16}{16} = 1$$

$f(2)$ is a Maximam

$\therefore$ mode $= 2$

Median

~~If $m=0$~~

$P(X<0) \le \frac{1}{2}$; $P(X\le 0) \ge \frac{1}{2}$

$0 \le \frac{1}{2}$ $\qquad f(0) = \frac{1}{16} \ngeq \frac{1}{2}$

$\therefore \boxed{m \ne 0}$

If $m=1$

$P(X<1) \le \frac{1}{2}$; $P(X\le 1) \ge \frac{1}{2}$

$f(0) = \frac{1}{16} < \frac{1}{2}$; $f(0)+f(1) \ge \frac{1}{2}$

$\frac{1}{16} + \frac{4}{16} \ge \frac{1}{2}$

$\frac{15}{16} \ngeq \frac{1}{2}$

$\boxed{\therefore m \ne 1}$

if $m = 2$

$P(X<2) \leq \frac{1}{2}$; $P(X \geq 2) \geq \frac{1}{2}$

$f(0)+f(1) \leq \frac{1}{2}$ $\quad$ $f(0)+f(1)+f(2) \geq \frac{1}{2}$

$\frac{1}{16} + \frac{4}{16} \leq \frac{1}{2}$ $\quad$ $\frac{1}{16}+\frac{4}{16}+\frac{6}{16} \geq \frac{1}{2}$

$\frac{5}{16} \leq \frac{1}{2}$ $\quad$ $\frac{11}{16} \geq \frac{1}{2}$

$\therefore$ median $= \boxed{m=2}$

Mean

$$E(x) = \sum_{x=0}^{4} x\, f(x) = 0\cdot f(0) + 1\cdot f(1) + 2\cdot f(2) + 3\cdot f(3) + 4\cdot f(4)$$

$$= 1\cdot\frac{4}{16} + 2\cdot\frac{6}{16} + 3\cdot\frac{4}{16} + 4\cdot\frac{1}{16}$$

$$= \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16} = \frac{32}{16} = 2 = \mu$$

$\therefore \boxed{\mu = 2}$

(18) If $X \sim b(n,p)$, then find the value of (n) s.t.

$$P\left(\left|\frac{X}{n} - p\right| < 0.1\right) \geq 0.95$$

Sol. By theorem (chebyshive theorem)

$$P\left(\left|\frac{X}{n} - p\right| < 0.1\right) \geq 0.95 \xleftrightarrow{\text{مقارنة}} P\left(\left|\frac{X}{n} - \mu\right| < t\right) \geq 1 - \frac{V(x)}{t^2}$$

$$= P(|X - np| < (0.1)n) = P\left(|X-\mu| < \frac{n}{10}\right) \Rightarrow \boxed{t = \frac{n}{10}}, \ \mu = np$$

$$L.b. = 1 - \frac{V(x)}{t^2}$$

$$V(x) = np(1-p)$$

$$L.b. = 1 - \frac{np(1-p)}{\left(\frac{n}{10}\right)^2} = 1 - \left[np(1-p)\frac{100}{n^2}\right]$$

$$= 1 - \frac{100\,p(1-p)}{n} \geq 0.95$$

$$\frac{100\,p(1-p)}{n} \leq 0.05 = \frac{5}{100} = \frac{1}{20}$$

$$2000\,p(1-p) \leq n$$

$$\boxed{n \geq 2000\,p(1-p)} \quad \forall p \quad 0<p<1$$

6

(19) If X has a M.g.f. as follows:

$$M_X(t) = \frac{2e^t}{5(1-\frac{3}{5}e^t)} \text{ for } t < \ln(\tfrac{5}{3}), \text{ then find } P(X>7 \mid X>3).$$

Sol.

$$M_X(t) = \frac{(\frac{2}{5})e^t}{(1-\frac{3}{5}e^t)} = \frac{pe^t}{1-qe^t}$$

memoryless

$\therefore X \sim G(p = \frac{2}{5})$

$\therefore P(X>7 / X>3) = P(X>4) = (q)^4 = (\frac{3}{5})^4.$

(20) If X has a p.m.f.

$$f(x) = \begin{cases} (\frac{2}{3})(\frac{1}{3})^{x-1} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \end{cases}$$

6

Find $P(X>5 \mid X>2)$

(1) which distribution that X have

(2) write the m.g.f. of X, $E(x)$ & $Var(x)$

Sol. $X \sim G(\frac{2}{3})$, $p = \frac{2}{3}$, $q = \frac{1}{3}$

$\therefore P(X>5 / X>2) = P(X>3) = (q)^3 = (\frac{1}{3})^3 = \frac{1}{27}$

(3) Compute $P(X>5 \mid X>2)$

(21) If $X \sim P(\lambda)$, then show that:

$E(x) = V(x) = \lambda$

derive the Expected value of X and the variance of X

$\therefore M_X(t) = e^{\lambda(e^t-1)}$

6

$M'_X(t) = e^{\lambda(e^t-1)}(\lambda e^t)$

$E(x) = M'_X(0) = e^0(\lambda)e^0 = \lambda$

$M''_X(t) = e^{\lambda(e^t-1)}(\lambda e^t) + (\lambda e^t)\, e^{\lambda(e^t-1)}(\lambda e^t)$

$E(x^2) = M''_X(t=0) = 1.(\lambda) + (\lambda).(1).(\lambda)$

$E(x^2) = \lambda + \lambda^2$

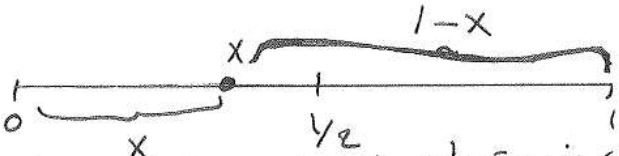
$V(x) = E(x^2) - [E(x)]^2$

$= \lambda + \lambda^2 - \lambda^2 = \lambda$

$$\boxed{E(X) = V(X) = \lambda}$$

(22) A point X is chosen at random from a stick of length one unite, and then the stick is broken at the chosen point into two unequal parts, find the expected value of longer part.

sol.



مسطرة (stick)

تنقسم المسطرة الى قسمين غير متساويين ، فاحد الاقسام صغير X والآخر كبير $(1-X)$

$\because$ X is shorter part, $(1-X)$ is longer part.

$\therefore \quad X \in (0, 1/2)$

$\therefore \quad X \sim unif.(0, \frac{1}{2})$

$$f(x) = \begin{cases} \frac{1}{1/2 - 0} = 2 & \text{for } 0 < x < \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

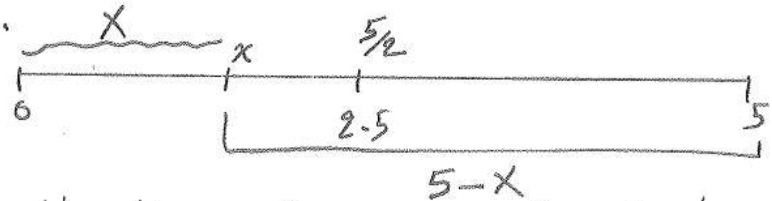
Find $E[\text{longer part}]$

i.e Find $E(1-X)$?

$$E(1-X) = 1 - E(X)$$
$$= 1 - \int_0^{1/2} x \cdot (2)\, dx = 1 - \frac{2}{2} x^2 \Big|_0^{1/2} = 1 - [(\tfrac{1}{2})^2 - 0]$$
$$= 1 - 1/4 = 3/4$$

$\therefore E(1-X) = \frac{3}{4}$ (القيمة المتوقعة للجزء الأكبر)

(23) A point X is chosen on a line of long 5 cm. This chosen point divides the line into two unequal parts. Find the expectation of shorter part.



sol.

Let X divide the line into unequal two parts.

$\because$ X is **the shorter** part, $(5-X)$ is the longer part.

$\therefore \quad X \in (0, \frac{5}{2})$

$\therefore \quad X \sim unif(0, \frac{5}{2})$

$$f(x) = \begin{cases} \frac{1}{\frac{5}{2} - 0} = \frac{2}{5} & \text{for } 0 < x < \frac{5}{2} \\ 0 & \text{o.w.} \end{cases}$$

Find $E(\text{shorter part}) = E(x)$?

$$E(X) = \int x f(x)\,dx = \int_0^{5/2} x \left(\tfrac{2}{5}\right) dx = \frac{\cancel{2}}{5}\cdot\frac{x^2}{\cancel{2}}\Big|_0^{5/2}$$
$$= \frac{1}{5}\left[\left(\tfrac{5}{2}\right)^2 - 0\right] = \frac{1}{\cancel{5}}\cdot\frac{\cancel{25}}{4} = \frac{5}{4} = 1.25$$

(24) Given ~~a~~ p.d.f.

$$f(x) = \begin{cases} x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Find mode of X.

Sol. $f(x) = 12x^2 - 2x^3$

$f'(x) = 24x - 36x^2 = 0$

$12x(2-3x) = 0$

$(12x = 0)$ or $(2 - 3x = 0)$

$x_1 = 0$ or $x_2 = \frac{2}{3}$

$f''(x) = 24 - 72x$

if $(x=0) \Rightarrow f''(0) = 24 \Rightarrow f(0)$ is min. then mode $\neq 0$

if $(x = \frac{2}{3}) \Rightarrow f''(\frac{2}{3}) = 24 - 72(\frac{2}{3}) = -24 < 0$ then $f(\frac{2}{3})$ is Max.

$\therefore$ mode $= \frac{2}{3}$

لتسهيل العمليات

$\int_0^1 12x^2 = \frac{12}{3}x^3\Big|_0^1 = 4[1-$

$\int_0^1 ax^2 = 1$

$\frac{a}{3}x^3\Big|_0^1 = 1$

$\frac{a}{3} = 1$

$a = 3$

(25) If X has a poisson dist. with parameter m, then $E(X) = V(X) = m$.

Theorem

derive

Sol. $M_X(t) = e^{m(e^t-1)}$ (by theorem)

$M'_X(t) = e^{m(e^t-1)} \cdot me^t$

$\therefore E(X) = M'_X(0) = m e^{m(e^0-1)} = m$

$M''_X(t) = e^{m(e^t-1)} \cdot me^t + me^t \cdot e^{m(e^t-1)} \cdot (me^t)$

$E(X^2) = e^0 m e^0 + m e^0 \cdot e^0 (m e^0)$

$= m + m^2$

$V(X) = E(X^2) - [E(X)]^2$

$V(X) = (m + \cancel{m^2}) - \cancel{m^2}$

$V(X) = m = E(X)$

(26) Given a m.g.f. of X, $M_x(t) = \frac{1}{1-2t}$ for $t < \frac{1}{2}$
Find the mean and variance of X.

Sol. $M_x(t) = \frac{1}{1-2t} = (1-2t)^{-1}$

$M'_x(t) = -(1-2t)^{-2}(-2) = 2(1-2t)^{2}$

$M''_x(t) = -4(1-2t)^{-3}(-2) = 8(1-2t)^{-3}$

$M'_x(0) = 2$, $M''_x(0) = 8$

$\therefore E(X) = 2$, $E(X^2) = 8$ (by theorem)

$V(X) = E(X^2) - [E(X)]^2$
$= 8 - (2)^2 = 4$

4

(27) If X has a uniform dist. on $(-\sqrt{3}, \sqrt{3})$, then find the upper bound of $P(|X-\mu| \geq \frac{3}{2})$.

Sol. $U.b. = \frac{V(x)}{t^2}$

$f(x) = \frac{1}{\sqrt{3}-(-\sqrt{3})} = \begin{cases} \frac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < x < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$

$E(x) = \int_{-\sqrt{3}}^{\sqrt{3}} x \left(\frac{1}{2\sqrt{3}}\right) dx = 0$

$E(X^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \left(\frac{1}{2\sqrt{3}}\right) dx = 1$

$\therefore V(X) = E(X^2) - (E(X))^2$
$= 1 - 0^2 = 1 \Rightarrow V(X) = 1$

$U.b. = \frac{V(X)}{t^2} = \frac{1}{(\frac{3}{2})^2} = \frac{4}{9} = 0.44$

(28) If X has a geometric dist. with $p = \frac{1}{4}$, then find $P(X>8 | X>3)$.

Sol. $p = \frac{1}{4} \Rightarrow q = \frac{3}{4}$

$P(X>8|X>3) = P(X>5) = [P(A^c)]^5 = q^5 = (\frac{3}{4})^5$

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(29) If $X \sim b(n,p)$, then the M.g.f. of X is

$$M_X(t) = (1-p+pe^t)^n$$

Sol.

$\because X \sim b(n,p)$

$$\therefore f(x,n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0,1,2,\dots,n \\ 0 & \text{o.w.} \end{cases}$$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

since $(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}$

Let $b = pe^t$ and $a = 1-p$

then $(1-p+pe^t)^n = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = M_X(t)$

$$\therefore \boxed{M_X(t) = (1-p+pe^t)^n}$$

(30) Given a p.d.f. $f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$

Find and sketch graph of $M_X(t)$.

(1) which distribution & find X have
(2) derive $M_X(t)$
(3) $E(X)$ $V(X)$

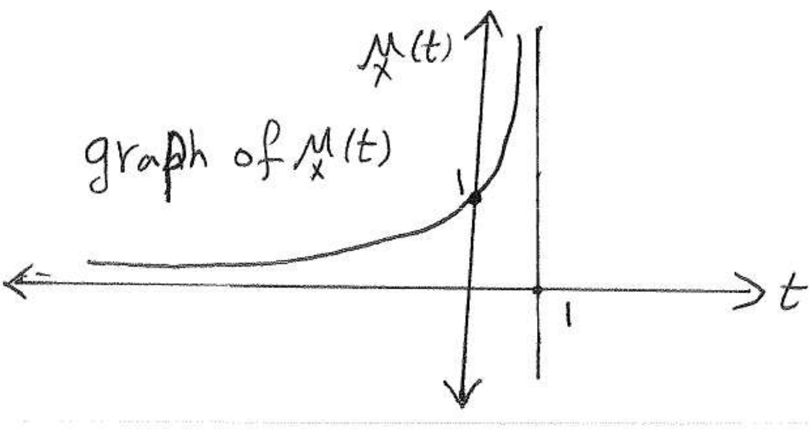
Sol.

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}(e^{-x})\,dx = \int_0^\infty e^{-(1-t)x}\,dx$$

$e^{-(1-t)x}$ exists when $t < 1$

$$M_X(t) = \frac{-1}{1-t}\int_0^\infty e^{-(1-t)x}(-(1-t))\,dx = \frac{1}{1-t} \quad \text{for } t<1$$

t	$M_X(t) = \frac{1}{1-t}$
1	∞
0	1
-1	.5
⋮	
$-\infty$	0



(31) Given $M_X(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$, if $Y = 1-2X$, then find $M_Y(t)$.

Sol. $M_Y(t) = e^{bt} M_X(at)$, $Y = aX+b$

$M_X(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$

$M_Y(t) = e^{t} M_X(-2t)$, $Y = 1-2X$

where $M_X(-2t) = \frac{1}{1+6t}$ for $t > \frac{-1}{6}$

(32) If a r.v. X with $M_X(t)$ for $-h<t<h$

show that $P(X \geq a) \leq e^{-at} M_X(t)$ for $0<t<h$

and $P(X \leq a) \leq e^{-at} M_X(t)$ for $-h<t<0$

Proof $M_X(t) = E(e^{tX})$, $Y = e^{tX}$

$e^{at} > 0$; $a>0$ or $t>0$

$P(X \geq t) \leq \frac{E(X)}{t}$ (by theorem)

$P(Y \geq e^{at}) \leq \frac{E(Y)}{e^{at}}$

$P(e^{tX} \geq e^{at}) \leq \frac{E(e^{tX})}{e^{at}}$

$P(\ln e^{tX} \geq \ln e^{at}) \leq \frac{M_X(t)}{e^{at}} = e^{-at} M_X(t)$

$P(tX \geq at) \leq e^{-at} M_X(t)$ for $0<t<h$

$P(X \geq a) \leq e^{-at} M_X(t)$ —— ①

$P(tX \geq at) \leq e^{-at} M_X(t)$

$P(X \leq a) \leq e^{-at} M_X(t)$ for $-h<t<0$ —— ②

(33) If X has a geometric dist. with parameter p, then the m.g.f. of X is $M_X(t) = \frac{pe^t}{1-qe^t}$ for $t < \ln(\frac{1}{q})$.

Sol. $X \sim G(p,q) \Rightarrow f(x) = \begin{cases} p\,q^{x-1} & , x = 1,2,3,\dots \\ 0 & \text{o.w.} \end{cases}$

$M_X(t) = E(e^{tx})$

$= \sum_{x=1}^{\infty} e^{tx} f(x)$

$= \sum_{x=1}^{\infty} e^{tx}(pq^{x-1}) = \frac{p}{q}\sum_{x=1}^{\infty}(qe^t)^x$

$= \frac{p}{q}[qe^t + (qe^t)^2 + \dots + (qe^t)^n + \dots]$

$= \frac{p}{q}\cdot qe^t[1 + qe^t + q^2e^{2t} + \dots + q^{n-1}e^{(n-1)t} + \dots]$

$\therefore r = \frac{qe^t}{1} = qe^t$

$|r|<1 \Longrightarrow qe^t<1 \Rightarrow e^t < \frac{1}{q} \Rightarrow \ln e^t < \ln(\frac{1}{q})$

$\Rightarrow t < \ln(\frac{1}{q})$

$S = \underbrace{1 - qe^t + (qe^t)^2 + \dots}_{\text{Geometriq series}}$, $S = \frac{1}{1-r} = \frac{1}{1-qe^t}$

$M_X(t) = \frac{pe^t}{1-qe^t}$ for $t < \ln(\frac{1}{q})$

(34) $X \sim N(\mu, \sigma^2)$, $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$; find $E(Z)$, $V(Z)$.

Sol. $E(Z) = E(\frac{X-\mu}{\sigma})$

$= E(\frac{X}{\sigma} - \frac{\mu}{\sigma})$

$= E(\frac{X}{\sigma}) - \frac{\mu}{\sigma}$

$= \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma}$

$= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$

$Var(Z) = V(\frac{X}{\sigma} - \frac{\mu}{\sigma})$

$= \frac{V(X)}{\sigma^2} - V(\frac{\mu}{\sigma})$

$= \frac{\sigma^2}{\sigma^2} - 0$

$= 1$

$M_Z(t) = e^{(tZ)} = E[e^{(\frac{X}{\sigma} - \frac{\mu}{\sigma})t}]$

$= e^{-\frac{\mu}{\sigma}t} E[e^{\frac{X}{\sigma}t}]$

$= e^{-\frac{\mu}{\sigma}t} M_{\frac{X}{\sigma}}(t)$

$= \cancel{e^{-\frac{\mu}{\sigma}t}}\, \cancel{e^{\frac{\mu}{\sigma}t}}{}^{+ \frac{\sigma^2 t}{2\sigma}}$

$= e^{\frac{\sigma^2 t^2}{2\sigma^2}} = e^{t^2/2} = e^{0+\frac{t^2}{2}}$

$\therefore Z \sim N(0,1)$

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(35) If a r.v. X has Gamma dist. with parameters (a) & (b), then show that $V(x) = b \cdot E(x)$

Sol. $\because X \sim G(a,b)$

$\therefore E(x) = a \cdot b$ & $V(x) = a \cdot b^2$

$\therefore V(x) = (a \cdot b) \cdot b$

$= E(x) \cdot b$

$= bE(x)$

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OR $\because X \sim G(a,b)$

$\therefore M_x(t) = (1-bt)^{-a}$

$M'_x(t) = (-a)(1-bt)^{-a-1} \cdot (-b) = (ab)(1-bt)^{-(a+1)}$

$M'_x(0) = (a \cdot b)(1-0)^{-(a+1)} = a \cdot b = E(x)$; (Since $E(x) = M'_x(0)$)

$E(x^2) = ?$

$E(x^2) = M''_x(0)$

$M''_x(t) = (a \cdot b)(a+1)(1-bt)^{-(a+2)} \cdot (-b) = (ab^2)(a+1)(1-bt)^{-(a+2)}$

$E(x^2) = M''_x(0) = ab^2(a+1)(1-0) = a^2b^2 + ab^2$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$= (a^2b^2) + (ab^2) - (ab)^2 = ab^2$

$\therefore V(x) = (a \cdot b) \cdot b = E(x) \cdot b \Rightarrow V(x) = bE(x)$

(36) If a m.g.f. of X is as follows:

$M_x(t) = (1-2t)^{-7}$, then find the p.d.f. of x, E(x) & V(x).

Sol. $\beta = 2$, $\alpha = 7 \rightarrow r = 14$

$\therefore X \sim \chi^2(14)$ chi-squar with (14) d.f.

$$\therefore f(x) = \begin{cases} \dfrac{x^6 e^{-\frac{x}{2}}}{\Gamma(7) 2^7} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

3

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$E(x) = r = 14$, $\sigma^2 = V(x) = 2r = 2(14) = 28$.

Find $P(X \leq a) = 0.95$
the value of a

(37) Given a p.d.f.

$$f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0<x<3 \\ 0 & \text{o.w.} \end{cases}$$

Find the lower bounded of $P(\frac{5}{4} < X < \frac{11}{4})$.

Sol. $l.b = 1 - \frac{V(X)}{t^2}$

we must find $E(X)$, $E(X^2)$:

$$E(X) = \int_0^3 x f(x)dx = \int_0^3 x\left(\frac{2}{9}x\right)dx = \frac{2}{9}\left[\int_0^3 x^2 dx\right]$$

$$= \frac{2}{9}\left[\frac{x^3}{3}\right]_0^3 = \frac{2}{27}[27-0] = 2$$

$$E(X^2) = \int_0^3 x^2 f(x)dx = \int_0^3 x^2\left(\frac{2}{9}x\right)dx = \frac{2}{9}\left[\int_0^3 x^3 dx\right]$$

$$= \frac{2}{36} x^4\Big]_0^3 = \frac{1}{\cancel{18}_2}[\cancel{81}^9] = \frac{9}{2} = 4.5$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= 4.5 - (2)^2$$

$$= 4.5 - 4 = 0.5 = \frac{1}{2}$$

$$P\left(\frac{5}{4} < X < \frac{11}{4}\right) = P\left(\frac{5}{4} - 2 < X-2 < \frac{11}{4} - 2\right)$$

$$= P\left(-\frac{3}{4} < X-2 < \frac{3}{4}\right)$$

$$= P\left(|X-2| < \frac{3}{4}\right)$$

$\therefore t = \frac{3}{4}$

$$l.b = 1 - \frac{V(X)}{t^2} = 1 - \frac{\frac{1}{2}}{(\frac{3}{4})^2} = 1 - \frac{1/2}{9/16} = 1 - \frac{8}{9}$$

$$= \frac{1}{9}$$

$$\therefore \boxed{l.b = \frac{1}{9}}$$

(38) Given a p.d.f. of X

$$f(x) = \begin{cases} \frac{1}{4} e^{-\frac{x}{4}} & \text{for } x>0 \\ 0 & \text{o.w.} \end{cases}$$

① Find $M_x(t)$ ② If $y = 2 - x$, then find $M_y(t)$

Sol.

Sol.

① $M_X(t) = E(e^{tX}) = \frac{1}{4}\int_0^\infty e^{tx}\cdot e^{-\frac{1}{4}x}\,dx$

$$= \frac{1}{4}\int_0^\infty e^{-(\frac{1}{4}-t)x}\,dx$$

This integration exist only when $(\frac{1}{4}-t) > 0$

i.e $\frac{1}{4} > t \rightarrow t < \frac{1}{4}$

$\therefore M_X(t) = \frac{1}{4}\cdot\frac{-1}{(\frac{1}{4}-t)}\, e^{-(\frac{1}{4}-t)x}\Big|_0^\infty = \frac{-1}{4(\frac{1}{4}-t)}[e^{-\infty}-e^{0}]$

$$= \frac{-1}{4(\frac{1}{4}-t)}[0-1]$$

$$= \frac{1}{4(\frac{1}{4}-t)} \quad \text{for } t < \frac{1}{4}$$

② $y = 2-x$, $b = 2$, $a = -1$

$M_y(t) = e^{bt}\cdot M_x(at)$

$= e^{2t}\cdot M_x(t)$

$M_x(-t) = \frac{1}{4(\frac{1}{4}+t)}$ for $-t < \frac{1}{4}$

$M_x(-t) = \frac{1}{4(\frac{1}{4}+t)}$ for $t > \frac{-1}{4}$

$\therefore M_y(t) = e^{2t}\cdot\frac{1}{4(\frac{1}{4}+t)}$ for $t > \frac{-1}{4}$