وزارة التعليم العالي والبحث والعلمي
جامعة ديالى
كلية تربية المقداد / قسم الرياضيات

محاضرات مادة
الإحصاء والاحتمالية

للعام الدراسي (2023-2024)

المرحلة الثالثة

Chapter Five

مدرس المادة: م.م هند إبراهيم محمد

Note: (Pr. dist. = Pr. fun. = f)

Probability Distribution of Two Random Variables

(I) Joint Probability Density Function (J.P.d.f.)

Def. Let X and Y be two c.r.v.'s. A function f defined over XY-plane is a joint p.d.f. of X and Y if for a subset $R \in XY$-plane, then:

$$\boxed{P[(x,y)\in R] = \iint_R f(x,y)\,dx\,dy}$$

if $R = \{(x,y) ;\ a<x<b \ \&\ c<y<d\}$

Then

$$P[(x,y)\in R] = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b\int_c^d f(x,y)\,dy\,dx$$

$\boxed{\text{Review}}$

ch.5 (Pr. dist. of X & Y)

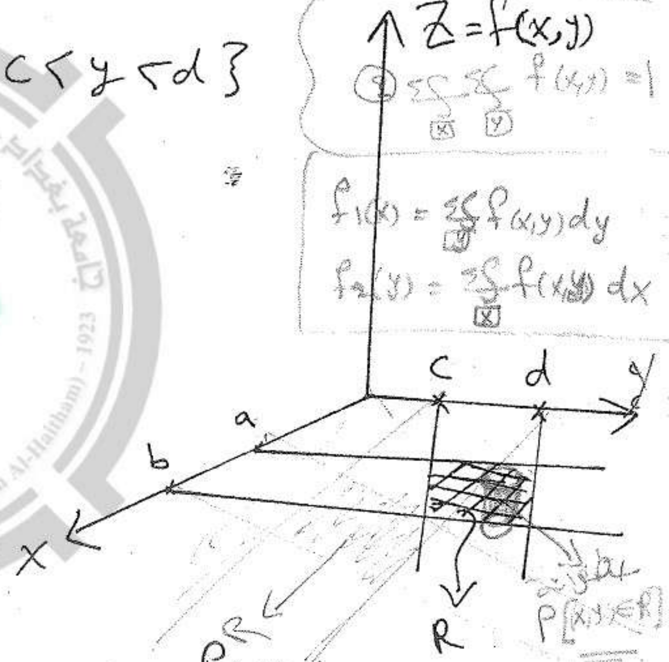
f(x,y) is J.p.m.f. / J.p.d.f.

① $f(x,y) \geq 0 \ \forall (x,y) \in R$

② $\sum_{x}\sum_{y} f(x,y) = 1$

$f_1(x) = \sum_y \int f(x,y)\,dy$

$f_2(y) = \sum_x \int f(x,y)\,dx$



Note: (مقارنة بين J.Pr.f. و Pr.f.)

① f(x,y) is a Solid (مجسم) over XY-plane. ⟶ f(x) is a curve

② R can be □, ▭, △, ○, ⬭, ... ⟶ (a,b)
R ≡ Domain of func. f(x,y)

③ P[(x,y)∈R] is the Volum (حجم) inside (داخل) Solid (الجسم) of f(x,y). ⟶ مساحة تحت المنحني Area under the curve (X c.r.v.)

Properties of Joint p.d.f.

① A joint p.d.f. f(x,y) satisfies two conditions:

ⓐ $f(x,y) \geq 0 \quad \forall (x,y) \in R$

ⓑ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

② Also:

ex. (انواع Region)

① $R = \{(x,y) ;\ 0<x<1, 0<y<1\}$

② $R = \{(x,y) ;\ 0<x<y<1\}$

③ $R = \{(x,y) ;\ x^2+y^2 \leq 1\}$

∵ $f(x,y)$ is a function of X and Y, from $f(x,y)$, we can find a function of X alone $f_1(x)$ and a function of Y alone $f_2(y)$.
s.t.

حدية

$f_1(x)$ is called Marginal p.d.f. of X
where $f_1(x) = \int_{\Box}^{\Box} f(x,y)\,dy$; limits of integral follows limits of y

$f_2(y)$ is called Marginal p.d.f. of Y
where $f_2(y) = \int_{\Box}^{\Box} f(x,y)\,dx$; limits of integral follows limits of x

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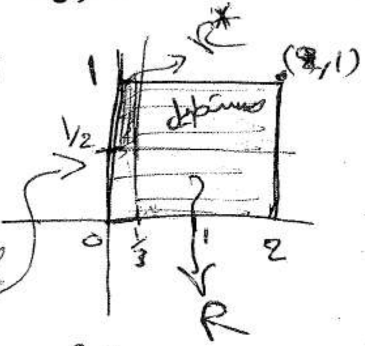
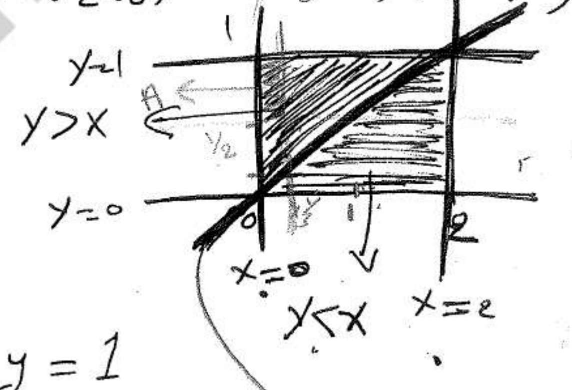
Example: Given a joint p.d.f. (J.p.d.f.) $f(x,y)$:

$$f(x,y) = \begin{cases} Kx^2y & \text{for } 0<x<2,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$



ملاحظة: في هذا المثال $\iint dx\,dy = \iint dy\,dx$ لأن X و Y محدودة ومفصولة

(a) Find the value of K, معزولة Isolation X, Y
(b) Find $P(0<X<\frac{1}{3}, \frac{1}{2}<Y<2)$, (f) $P(X>\frac{1}{2})$, $P(Y<\frac{1}{3})$
$\int_{1/2} f_1(x)\,dx$ $\quad \int_0^{1/3} f_2(y)\,dy$
(c) Find $P(X+Y \geq 1)$, (d) Find $f_1(x)$ and $f_2(y)$



Sol. By cond. (2)

(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$R = \{(x,y) : 0<x<2,\ 0<y<1\}$

$\int_0^1\int_0^2 K x^2 y\,dx\,dy = 1 \Rightarrow \int_0^1 Ky\left(\frac{x^3}{3}\Big|_0^2\right)dy = 1$

$\frac{1}{3}\int_0^1 Ky\,(8-0)\,dy = 1 \Rightarrow \frac{8K}{3}\int_0^1 y\,dy = 1$

$\frac{8K}{3}\,\frac{y^2}{2}\Big|_0^1 = 1 \Rightarrow \frac{8K}{6}[1^2-0] = 1 \Rightarrow \frac{4}{3}K = 1 \Rightarrow \boxed{K = \frac{3}{4}}$

الخط المساواة لا يعطي المنطقة R لهذا يجب تحديد الأكبر والأصغر

(b) $f(x,y)=\begin{cases}\frac{3}{4}x^2y & 0<x<2,\ 0<y<1\\ 0 & \text{o.w}\end{cases}$

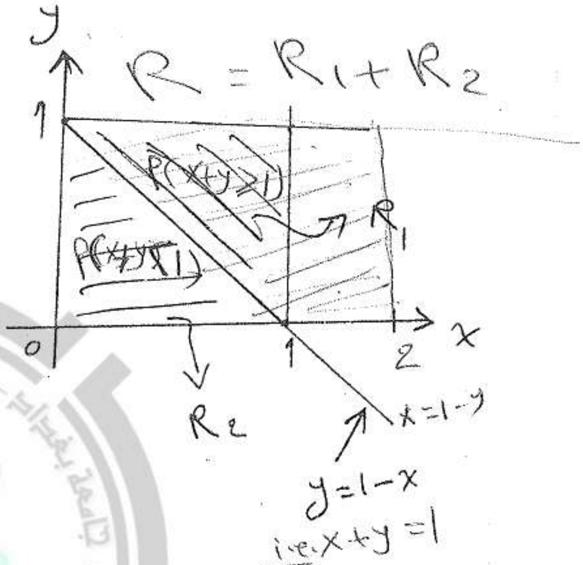
$$P(0<x<\tfrac{1}{3},\ \tfrac{1}{2}<y<2)=\int_{1/2}^{1}\int_{0}^{1/3}\frac{3}{4}x^2y\,dx\,dy=\cdots=\frac{1}{32(9)}=\frac{1}{288}$$

(c) $P(x+y\geq 1)=?$ (d) $P(x+y<1)=?$

Consider $(x+y)=1 \Rightarrow y=1-x$

x	$y=1-x$
0	1
1	0

للرسم $\longrightarrow$



$P(x+y\geq 1)=$ Volum of $f(x,y)$ over R_1

$P(x+y<1)=$ Volum of $f(x,y)$ over R_2

$P(x+y\geq 1)=1-P(x+y<1)$

$$P(x+y<1)=\int_0^1\int_0^{1-x}f(x,y)\,dy\,dx$$

$$=\int_0^1\left[\int_0^{1-x}\frac{3}{4}x^2y\,dy\right]dx=\int_0^1\frac{3}{4}x^2\,\frac{y^2}{2}\Big|_0^{1-x}dx$$

$$=\int_0^1\frac{3}{8}x^2[(1-x)^2-0]\,dx$$

$$=\frac{3}{8}\int_0^1 x^2[1-2x+x^2]\,dx$$

$$=\frac{3}{8}\left[\frac{x^3}{3}-2\frac{x^4}{4}+\frac{x^5}{5}\right]\Big|_0^1$$

$$=\frac{3}{8}\left[\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)-(0-0+0)\right]=\frac{3}{8}\left[\frac{10-15+6}{30}\right]$$

$$=\frac{3}{8}\left(\frac{1}{30}\right)=\frac{1}{80}$$

$$P(x+y<1)=\frac{1}{80}\Rightarrow P(x+y\geq 1)=1-\frac{1}{80}=\frac{79}{80}\in[0,1]$$

(d) To find $f_1(x)$ & $f_2(y)$

$$f_1(x)=\int_0^1 f(x,y)\,dy$$

$f_1(x) = \int_0^1 \frac{3}{4}x^2 y\,dy = \frac{3}{4}x^2\frac{y^2}{2}\Big|_0^1 = \frac{3}{8}x^2[1-0] = \frac{3}{8}x^2$

$$f_1(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

$f_2(y) = \int_0^2 f(x,y)\,dx$

$= \int_0^2 \frac{3}{4}x^2 y\,dx = \frac{3}{4}y\frac{x^3}{3}\Big|_0^2 = \frac{1}{4}y[8-0] = \frac{8}{4}y$

$= 2y$

$$f_2(y) = \begin{cases} 2y & \text{for } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$, $P(y<\frac{1}{3}) = \int_0^{1/3} 2y\,dy = y^2\Big|_0^{1/3} = \frac{1}{9}$

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* (يسمى اعتمادا على حسب عدد المتغيرات)

Univariat	احادي one
Bivariat	ثنائي two
multivariat	متعدد n

(لم يشرح)

Example 10 Given a J.p.d.f.

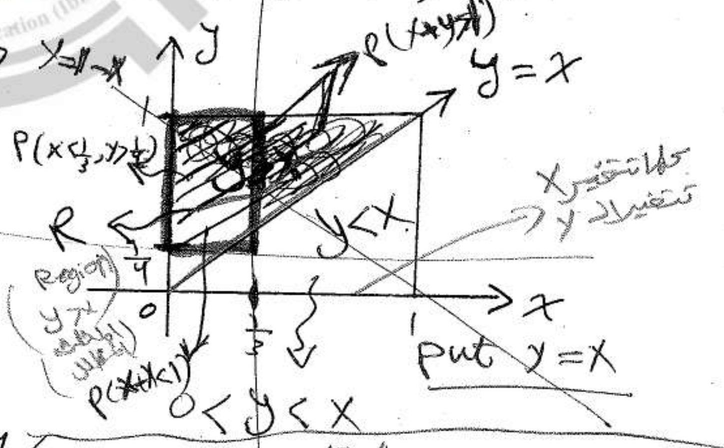
$$f(x,y) = \begin{cases} 2 & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

Find $f_1(x)$, $f_2(y)$, $P(X>\frac{1}{3})$, $P(Y<\frac{1}{2})$, $P(X<\frac{1}{3})$, Find (1) $P(X>\frac{1}{2})$ (2) $P(Y<\frac{1}{2})$ (3) $P(X>\frac{1}{2}, Y<\frac{1}{2})$

Sol. $P(X>1/2, Y<1/2) = \int_{1/2}\int_0^{1/2} f(x,y)\,dy\,dx =$

$R = \{(x,y);\ 0<x<y<1\}$, Consider $y=x$, then:

$$R = \begin{Bmatrix} 0<x<y & ; & x<y<1 \\ 0<x<1 & ; & 0<y<1 \end{Bmatrix}$$



$f_1(x) = \int_x^1 2\,dy$ for $0<x<1$

$= 2y\Big|_x^1 = 2[1-x]$

$$f_1(x) = \begin{cases} 2(1-x) & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

*(H.W. Prove that f_1 & f_2 are pr. fun.)

$f_2(y) = \int_0^y 2\,dx$ for $0<y<1$

$= 2x\Big|_0^y = 2y$

$$f_2(y) = \begin{cases} 2y & \text{for } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

2010/2/10

Find $P(X+Y<1) = ?$ → $X+Y=1$, $y=1-x$

تقاطع المستقيمات :

$y=x$, $y=1-x$

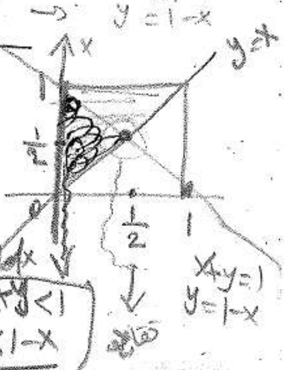
$x=1-x$ → $\boxed{x=\frac{1}{2}}$

$(x,y)=(\frac{1}{2},\frac{1}{2})$

$\therefore P(X+Y<1) = \int_0^{1/2}\int_0^{1-x} f(x,y)\,dy\,dx$

$X+Y<1$

$Y<1-X$



$$P(X>\tfrac{1}{3}) = \int_{\frac{1}{3}}^{1} f_1(x)\,dx$$

$$= \int_{\frac{1}{3}}^{1} 2(1-x)\,dx = \frac{8}{18}$$

$$P(Y<\tfrac{1}{2}) = \int_{0}^{\frac{1}{2}} 2y\,dy = \cdots = \frac{1}{4}$$

$$P(X<\tfrac{1}{3}, Y>\tfrac{1}{4}) = \int_{\frac{1}{4}}^{1}\underbrace{\int_{0}^{\frac{1}{3}} 2\,dx}_{x}\,dy = \cdots = \frac{1}{2}$$

* لو لم يطلب f_1 و f_2 ويطلب $\left(\begin{matrix} P(X<\frac{1}{3}) \\ P(Y>\frac{1}{2}) \end{matrix}\right)$ نسحب ال f ضمنياً implicit

Find $P(X+Y\geq 1) = ?$

(II) Joint Probability Mass Function (J.P.m.f.)

Def. Let X and Y be two d.r.v.. A function $f(x,y)$ is a joint p.m.f. of X and Y iff

$$\boxed{f(x,y) = P(X=x, Y=y) \quad \forall (x,y)\in R}$$

X is d.r.v.
$f(x)$ is p.m.f.
$f(x) = P(X=x)$
$f(2) = P(X=2)$

and satisfies two conditions:

(a) $f(x,y)\geq 0 \quad \forall (x,y)\in R$

(b) $\sum\limits_{\forall y}\sum\limits_{\forall x} f(x,y) = 1$
$\forall R$

Also,

$f_1(x) = \sum\limits_{\forall y} f(x,y)$, limit of sumation followes limit of y.

$f_2(x) = \sum\limits_{\forall x} f(x,y)$, limit of Sumation followes limit of X.

where $f_1(x)$ and $f_2(y)$ are called Marginal p.m.f. of X and Y respectively.

بالتعاقب

Example: A box has (3) balls (1), (2), (3). Choose two balls one by one without replac.; find $f(x,y)$.

Sol.

$n=3 \xrightarrow[\text{without rep.}]{(1-1)} k=2$

$$S = \left\{ \begin{matrix} (1,2) & (2,1) & (3,1) \\ (x_1,y_1) & (x_2,y_1) & (x_3,y_1) \\ (1,3) & (2,3) & (3,2) \\ (x_1,y_3) & (x_2,y_3) & (x_3,y_2) \end{matrix} \right\} ; \text{ S has "6" elts.}$$

OR A box has (3) balls which are numbered 1,2,3.
مرقمة

كم حالة لسحب 2 من 3 بدون ارجاع

$P_2^3 = \frac{3!}{1!} = 6 = n(S)$ (النوع)

سؤال مقترح 2

نفس السؤال مع تغيير طريقة السحب (1-1) with rep. $n(S) = (3)^2 = 9$

Graph of $f(x,y)$

$X = 1^{st}$ chosen ball ; $x = 1, 2, 3$

$Y = 2^{nd}$ chosen ball ; $y = 1, 2, 3$ $\Rightarrow X \neq Y$

تمثل الاختيار الاول

تمثل الاختيار الثاني

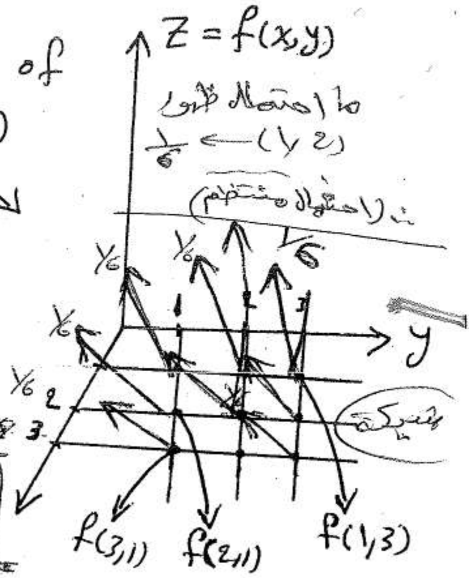
$P(X=2, Y=1) = \frac{1}{6} = f(2,1)$ (عنصر من عناصر)

$P(X=3, Y=2) = \frac{1}{6} = f(3,2)$

لا توجد هذه الحالة ex. $f(1,1) = 0$

$f(x,y) = P(X=x; Y=y) = \frac{1}{6}$

$\boxed{P(X \geq 2, Y \leq 2) = \sum_{x=2,3} \sum_{y=1,2} f(x,y)}$



قيمة الدالة ثابتة uniform(K)

$f(x,y) = \begin{cases} \frac{1}{6} & \text{for } x = 1,2,3 \,;\, y = 1,2,3 \text{ \& } x \neq y \\ 0 & \text{o.w.} \end{cases}$

الرسم الهندسي بالبعد (3) (3-dim.)

$f(X=2) = f_1(2)$ $\boxed{P(X=1) = f_1(1)}$

Find: $f_1(x) = \sum_{\forall y} f(x,y) = \begin{cases} \frac{2}{6}, & x = 1,2,3 \\ 0 \end{cases}$ & $f_2(y) = \sum_{\forall x} f(x,y) = \sum_{x=1}^{3} \frac{1}{6} = \frac{3}{6}$ $\begin{cases} \frac{3}{6}, & y = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$

$P(X \geq 2) = \sum_{x=2}^{3} f_1(x) = f_1(2) + f_1(3) = \frac{3}{6} + \frac{3}{6} = 1$

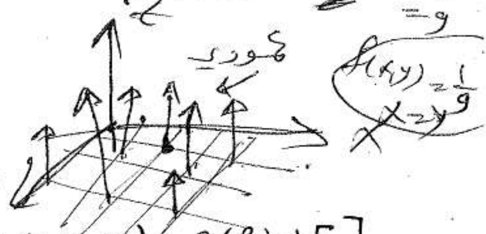
$P(Y \leq 2) = ?$

Example (4) Given a J.p.m.f. $f(x,y)$ (لم يُعطى)

$f(x,y) = \begin{cases} \frac{1}{21}(x+y) & \text{for } x = 1,2,3 \\ & \quad y = 1,2 \\ 0 & \text{o.w.} \end{cases}$

نفس المثال اعلاه ولكن مع ارجاع $n(S) = 3^2 = 9$

Find $P(X \geq 2, Y \leq 2)$, $f_1(x)$, $f_2(y)$



Sol.

$P(X \geq 2, Y \leq 2) = \sum_{y=1}^{2} \sum_{x=2}^{3} \frac{1}{21}(x+y)$

$= \frac{1}{21} \sum_{y=1}^{2} [(2+y) + (3+y)]$

$= \frac{1}{21} \sum_{y=1}^{2} [2y+5] = \frac{1}{21} [(2(1)+5) + 2(2)+5]$

$= \frac{1}{21} [7+9] = \frac{16}{21}$

$P(X \geq 2, Y \leq 2) = \sum_{x=2}^{3} \sum_{y=1}^{2} \frac{1}{6} = \sum \left(\frac{1}{6} + \frac{1}{6}\right) = \left(\frac{2}{6}\right)\left[\frac{1}{6} + \frac{1}{6}\right] = \frac{4}{6}$

$P(X=3, Y=1) = f(3,1) = \frac{1}{21}(3+1) = \frac{4}{21}$

$f_1(x) = \sum_{y} f(x,y)$

$f_1(x) = \sum_{y=1}^{2} \frac{1}{21}(x+y) = \frac{1}{21}[(x+1) + (x+2)]$

$= \frac{1}{21}[3+2x]$

$f_1(x) = \begin{cases} \frac{1}{21}[3+2x] & \text{for } x = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$

$$f_2(y) = \sum_x f(x,y)$$

$$= \sum_{x=1}^{3} \frac{1}{21}(x+y)$$

$$= \frac{1}{21}\left[(1+y)+(2+y)+(3+y)\right]$$

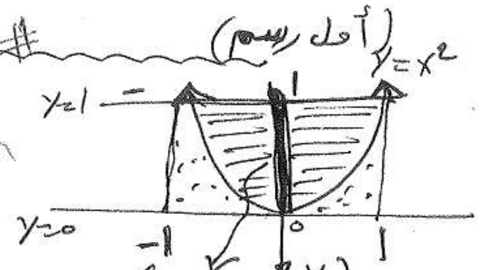
$$= \frac{1}{21}(6+3y) = \frac{3}{21}(y+2) = \frac{1}{7}(y+2)$$

$$f_2(y) = \begin{cases} \frac{1}{7}(y+2) & \text{for } y=1,2 \\ 0 & \text{o.w.} \end{cases}$$

Example (5) Given a J.p.d.f.

$$f(x,y) = \begin{cases} c x^2 y & \text{for } x^2 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$ Region

(أول رسم)



ⓐ Find the value of c ⓑ Find $P(X \le y)$

Sol.

$$R = \{(x,y) ; x^2 \le y \le 1\} = \begin{cases} 0 \le x^2 \le y, & x^2 \le y \le 1 \\ 0 \le x^2 \le 1, & 0 \le y \le 1 \end{cases}$$

تشبيك R من النهاية لمعالم الدوال

suppose $y = x^2 \Rightarrow x = \mp\sqrt{y}$

$0 \le x^2 \le y$

$|x| \le \sqrt{y}$

$-\sqrt{y} \le x \le \sqrt{y}$

$x^2 \le 1$

$|x| \le 1$

$-1 \le x \le 1$

$0 \le x^2 \le y \le 1$

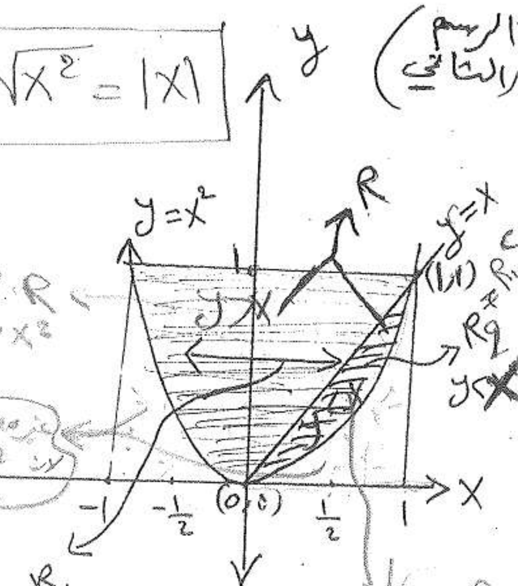
$0 \le y \le 1$

$y = x^2$

$x = \mp\sqrt{y}$

$\boxed{\text{by } \sqrt{x^2} = |x|}$

(الرسم الثاني)



ومن الصعوبة التحكم بها لان R $x > x^2$

$$R = \begin{cases} -\sqrt{y} \le x \le \sqrt{y} & ; \; x^2 \le y \le 1 \\ -1 \le x \le 1 & ; \; 0 \le y \le 1 \end{cases}$$

x	$y = x^2$
0	0
$\mp\frac{1}{2}$	$\frac{1}{4}$
∓ 1	1

Consider $y = x^2$

$y=x^2$, $y=x$: لايجاد نقاط التقاطع

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x=0, x=1$

$\Rightarrow (0,0)$, $(1,1)$

$\underline{R}$

$0 \le x^2 \le y \le 1$

لذلك

$P(Y \le X) = 1 - P(X \le Y)$

(a) By Cond.(2)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\int_{-1}^{1}\int_{x^2}^{1} c\,x^2 y\,dy\,dx = 1$$

$$1 = \frac{c}{2}\int_{-1}^{1} x^2 y^2 \Big|_{x^2}^{1} dx = \frac{c}{2}\int_{-1}^{1} x^2(1-x^4)\,dx$$

$$1 = \frac{c}{2}\int_{-1}^{1}[x^2 - x^6]\,dx = \frac{c}{2}\left[\frac{x^3}{3} - \frac{x^7}{7}\right]_{-1}^{1}$$

$$1 = \frac{c}{2}\left[\left(\frac{1}{3}-\frac{1}{7}\right) - \left(\frac{-1}{3} - \frac{-1}{7}\right)\right]$$

$$1 = \frac{c}{2}\left[\left(\frac{1}{3}-\frac{1}{7}\right) + \left(\frac{1}{3}-\frac{1}{7}\right)\right]$$

$$1 = \frac{c}{2}\left(\frac{2}{3} - \frac{2}{7}\right) = c\left(\frac{1}{3}-\frac{1}{7}\right) = c\left(\frac{7-3}{21}\right) = \frac{4}{21}c$$

$$\Rightarrow \boxed{c = \frac{21}{4}}$$

$$\therefore f(x,y) = \begin{cases} \frac{21}{4}x^2 y & \text{for } x^2 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

(الرسم الثاني)

(b) To find $P(X \le y)$:

$$R = \begin{Bmatrix} -\sqrt{y} \le x \le \sqrt{y} \;, & x^2 \le y \le 1 \\ -1 \le x \le 1 \;, & 0 \le y \le 1 \end{Bmatrix}$$

$P(X \le y) =$ volum of $f(x,y)$ over R_1

but $P(y \ge X) = 1 - P(y < X)$

$P(y < X) =$ Volum of $f(x,y)$ over R_1^c

$$P(y<X) = \int_0^1\int_{x^2}^{x} \frac{21}{4} x^2 y\,dy\,dx$$

$\begin{pmatrix} x=0 \\ \& x=1 \end{pmatrix}$ لأن

$$= \frac{21}{8}\int_0^1 x^2 y^2 \Big|_{x^2}^{x} dx$$

$$= \frac{21}{8}\int_0^1 x^2[x^2 - x^4]\,dx = \frac{21}{8}\int_0^1 [x^4 - x^6]\,dx$$

$$= \frac{21}{8}\left[\frac{x^5}{5}-\frac{x^7}{7}\right]\Big|_0^1$$

$$= \frac{21}{8}\left[\left(\frac{1}{5}-\frac{1}{7}\right)-(0-0)\right]$$

$$= \frac{21}{8}\left[\frac{7-5}{35}\right] = \frac{\cancel{21}^{3}}{\cancel{8}_{4}}\cdot\frac{\cancel{2}}{\cancel{35}_{5}} = \frac{3}{20}$$

$$\therefore\ P(X \le Y) = 1 - P(X > Y) \cong \int_0^1 (x^2 - x)\,dx \quad \left(\text{المساحة بين منحنيين}\right)$$

$$= 1 - P(Y < X)$$

$$= 1 - \frac{3}{20} = \frac{17}{20}$$

(c) To find $f_1(x)$ and $f_2(y)$

$$R = \left\{\begin{array}{ll} -\sqrt{y} \le x \le \sqrt{y}, & x^2 \le y \le 1 \\ -1 \le x \le 1, & 0 \le y \le 1 \end{array}\right\}$$

$$f_1(x) = \int_{x^2}^{1} \frac{21}{4} x^2 y\,dy \quad \text{for } -1 \le x \le 1$$

$$= \frac{21}{4} x^2 \frac{y^2}{2}\Big|_{x^2}^{1} = \frac{21}{8} x^2 (1 - x^4)$$

$$f_1(x) = \begin{cases} \frac{21}{8} x^2 (1 - x^4) & \text{for } -1 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4} x^2 y\,dx \quad \text{for } 0 \le y \le 1$$

$$= \frac{21}{4} y \frac{x^3}{3}\Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{7}{4} y \left[y^{\frac{3}{2}} + y^{\frac{3}{2}}\right] = \frac{7}{2} y^{\frac{5}{2}}$$

$$f_2(y) = \begin{cases} \frac{7}{2} y^{\frac{5}{2}} & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$\boxed{P(X+Y<1) = ?}$ (H.w.)

Stochastic Independence

الاستقلال التصادفي

Def: X and Y are stochastically independence iff :

$$\boxed{f_1(x)\, f_2(y) = f(x,y) \quad \forall (x,y)}$$

مقارنة مع
A & B are indep. events: $P(A)P(B) = P(AB)$

شرح بـ(½ ساعة)
الوقت للتدريس

Denoted by (S. indep.)

Example: Given a J.p.m.f

x \ y	1	2	3	4	$f_1(x)$ للاوان
1	$f(1,1) = .1$	$f(1,2) = 0$	$f(1,3) = .1$	$f(1,4) = 0$	$f_1(1) = .1 + .1 = .2$
2	$f(2,1) = .3$	$f(2,2) = 0$	$f(2,3) = .1$	$f(2,4) = .2$	$f_1(2) = .3 + .1 + .2 = .6$
3	$f(3,1) = 0$	$f(3,2) = .2$	$f(3,3) = 0$	$f(3,4) = 0$	$f_1(3) = .2$
$f_2(y)$ بالاوان	$f_2(1) = .4$	$f_2(2) = .2$	$f_2(3) = .2$	$f_2(4) = .2$	$\sum_{x=1}^{3} f_1(x) = \sum_{y=1}^{4} f_2(y) = 1$

$$f(x,y) \equiv P(X = x, Y = y)$$

a. T.P. $f_1(X) \cdot f_2(Y) = f(X,Y) \quad \forall (x,y)$

Note: $\sum_x f_1(x) = 1$, $f_1(X)$ is p.m.f. of X

$\sum_y f_2(y) = 1$, $f_2(y)$ is p.m.f. of Y

Prove that

$f_1(1) \cdot f_2(1) = (.2)(.4) = .08$

$f(1,1) = .1$

$\therefore f_1(1) f_2(1) \neq f(1,1)$ & $f_1(2) f_2(2) \neq f(2,2)$

$\therefore$ X and Y are <u>not</u> S. indep.

b. Find $P(X \geq 2, Y \leq 3)$

$$P(X \geq 2, Y \leq 3) = \sum_{y=1}^{3} \sum_{x=2}^{3} f(x,y) = \sum_{y=1}^{3} [f(2,y) + f(3,y)]$$

$$= [f(2,1) + f(2,2) + f(2,3)] + [f(3,1) + f(3,2) + f(3,3)]$$

$$= .03 + 0 + .1 + 0 + .2 + 0$$

$$= .6$$

c. Find $P(X \geq 2) = \sum_{x=2}^{3} f_1(x)$

$$= f_1(2) + f_1(3)$$

$$= 0.6 + 0.2 = 0.8$$

& $P(Y \leq 3) = \sum_{y=1}^{3} f_2(y)$

$$= f_2(1) + f_2(2) + f_2(3)$$

$$= .4 + .2 + .2 = .8$$

Uniform Distribution of two R.V.'s

الوقت شرح بـ 3 ساعة للتدريس

<u>Def.</u> If a c.r.v. X and Y have a uniform distribution on a region R, then the joint p.d.f. of X and Y is:

$$\boxed{f(x,y) = \frac{1}{\text{Area}(R)} \quad \text{for } (x,y) \in R}$$

(مساحة المنطقة)

Example 6 X and Y have a uniform dist. on a triangle Δ with vertices $(0,0)$, $(0,1)$ and $(1,1)$. Find $f(x,y)$.

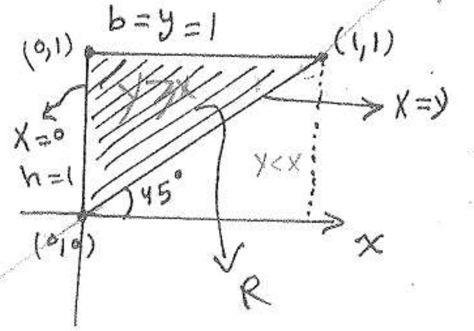
<u>Sol.</u>

R is a triangle Δ

$y \geq x \qquad x \geq 0$

$y \leq 1 \qquad \Rightarrow \quad 0 \leq x \leq y \leq 1$

dependent معتمد



$R = \{(x,y);\ 0 \leq x \leq y \leq 1\}$

$$\text{Area}(\Delta) = \frac{1}{2} \cdot h \cdot b$$

$$= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

X, Y are dep. $\Rightarrow f_1(x) f_2(y) \neq f(x,y)$

$$f(x,y) = \frac{1}{\text{Area}(R)} = \frac{1}{\frac{1}{2}} = \begin{cases} 2 & \text{for } 0 \le x \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Example 7: A point (x,y) is chosen from a region R which is bounded by a curve $y = x^2$ and a line $y = x$, find $f(x,y)$.

Sol. Sketch $y = x$ and $y = x^2$

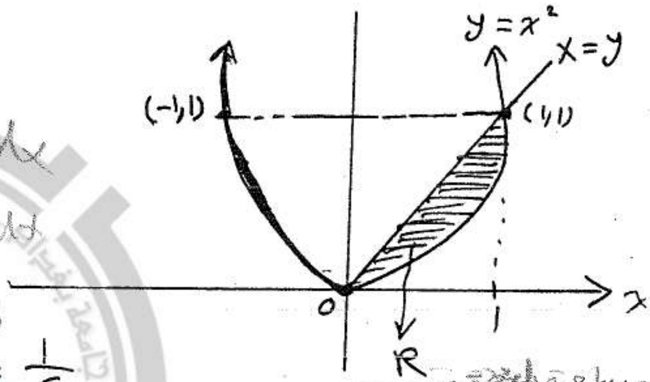
A point $(x,y) \in R$ must be a bove the curve $y = x^2$ and under the line $y = x$, then $x^2 \le y \le x$. ($x^2 = x$ نقاط التقاطع $(0,0)$, $(1,1)$)

$\rightarrow x = 0, x = 1$ و $y = 0, y = 1$

Also that point must be between $0 \le x \le 1$, $0 \le y \le 1$

$0 \le x^2 \le y \le x \le 1$

$R = \{(x,y);\ 0 \le x^2 \le y \le x \le 1\}$



$$\text{Area}(R) = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 |x - x^2|\, dx = \int_0^1 x - x^2\, dx$$

$$= \int_0^1 |(x - x^2)|\, dx$$

$x^2 \le x$ موجبة

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$f(x,y) = \frac{1}{1/6} = \begin{cases} 6 & \text{for } 0 \le x^2 \le y \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

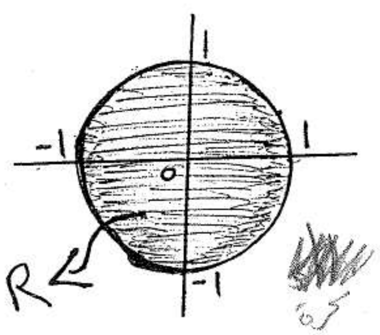
مساحة المنطقة $\int_0^1 |x^2 - x|\, dx = \int_0^1 (x - x^2)\, dx$

Example 8: A point (x,y) is chosen from a region R where

$R = \{(x,y);\ x^2 + y^2 \le 1\}$

a. find $f(x,y)$

b. find $f_1(x)$ & $f_2(y)$



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Sol.

a. Let $x^2 + y^2 = 1 \rightarrow r = 1$ نصف القطر

$\text{Area}(R) = \pi r^2 = \pi$

radius $\leftarrow r = 1$ نصف القطر

then $f(x,y) = \begin{cases} \frac{1}{\pi} & \text{for } x^2 + y^2 \le 1 \\ 0 & \text{o.w.} \end{cases}$

b. We must find $f_1(x)$ & $f_2(y)$:

$$f_1(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy \quad \text{for } -1 \le x \le 1$$

$$= \frac{1}{\pi}\, y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \frac{1}{\pi}\left[\sqrt{1-x^2} + \sqrt{1-x^2}\right] = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{for } -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{1}{\pi}\,dx \quad \text{for } -1 \le y \le 1$$

$$= \frac{1}{\pi}\, x \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2} & \text{for } -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ 0 & \text{o.w.} \end{cases}$$

$$f_1(x)\cdot f_2(y) = \frac{2}{\pi}\cdot\frac{2}{\pi}\sqrt{(1-y^2)(1-x^2)} = \frac{4}{\pi^2}\sqrt{(1-y^2)(1-x^2)}$$

$$f(x,y) \ne f_1(x)\, f_2(y) \qquad \left(\text{since } f(x,y) = \frac{1}{\pi}\right)$$

---

* Conditional Function and Conditional probability

Def. Let X and Y be two r.v.'s, $f(y|x)$ denoted the conditional p.d.f or p.m.f. of Y given $X=x$.

$f(x|y)$ denoted the conditional p.d.f. or p.m.f. of X given $Y=y$.

where

$$\boxed{f(y|x) = \frac{f(x,y)}{f_1(x)} \; ; \quad f_1(x) \ne 0}$$

$$\boxed{f(x|y) = \frac{f(x,y)}{f_2(y)} \; ; \quad f_2(y) \ne 0}$$

$$\left(P(A|B) = \frac{P(AB)}{P(B)} \right)$$

Note:

$$P(a<x<b \,/\, y=c) = \int_a^b f(x|y=c)\,dx$$

$$= \sum_{x=a}^{b} f(x|y=c)$$

$$P(c<y<d \mid x=a) = \int_c^d f(y \mid x=a)\,dy$$
$$= \sum_{y=c}^{d} f(y \mid x=a)$$

Example 8: Given a J.p.d.f.

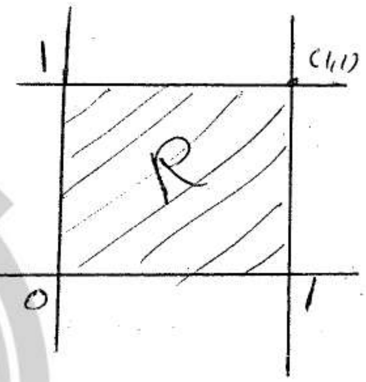
$$f(x,y) = \begin{cases} 8xy & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(1) find $f(x \mid y)$ & $f(y \mid x)$

(2) find $P(x>\frac{1}{4} \mid y=\frac{1}{2})$, $P(y<\frac{1}{4} \mid x=\frac{1}{2})$.

Sol. $R = \{(x,y) ; 0<x<y<1\}$

$$R = \begin{cases} 0<x<y & x<y<1 \\ 0<x<1 & 0<y<1 \end{cases}$$



(1) we must find $f_1(x)$ and $f_2(y)$

$$f_1(x) = \int_x^1 8xy\,dy = 8x \frac{y^2}{2}\Big|_x^1 = 4x(1-x^2)$$

$$= 4x(1-x^2) \quad \text{for } 0<x<1$$

$$f_2(y) = \int_0^y 8xy\,dx = \frac{8}{2} y x^2 = 4y^3 \quad \text{for } 0<y<1$$

$$f(x \mid y) = \frac{f(x,y)}{f_2(y)} = \frac{8xy}{4y^3} = \begin{cases} \frac{2x}{y^2} & \text{for } \begin{cases} 0<x<y \\ 0<y<1 \end{cases} \\ 0 & \text{o.w.} \end{cases}$$

$$f(y \mid x) = \frac{f(x,y)}{f_1(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$$

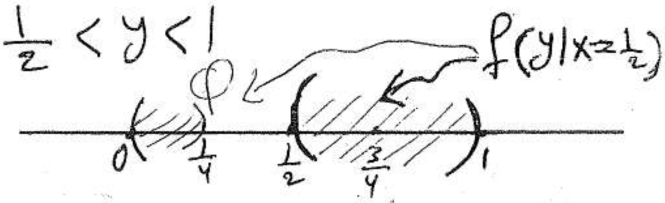
$$\therefore f(y \mid x) = \begin{cases} \frac{2y}{1-x^2} & \text{for } x<y<1 \ \& \ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$x \neq 1$ لان مقام $f(y|x)$ = صفر

دالة تعتمد على y (مقدار y) ومعتمدة على x

(2) $P(y<\frac{1}{4} \mid x=\frac{1}{2}) = \int_0^{\frac{1}{4}} f(y \mid x=\frac{1}{2})\,dy = \int_0^{\frac{1}{4}} 0\,dy = 0$

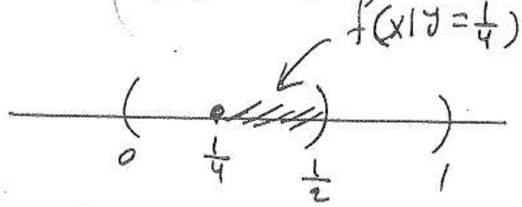
$$x<y<1 \ , \ x=\frac{1}{2} \implies \frac{1}{2}<y<1$$



المنطقة التي تعمل فيها الدالة $f(y|x=\frac{1}{2})$

$P(X>\frac{1}{4} \mid y=\frac{1}{2}) = \int_0^y f(x|y=\frac{1}{2})\,dx$ ⟵ (نعوض او y بـ $\frac{1}{2}$ $f(x|y)$ ثم ناخذ الاحتمالية)

$\boxed{\begin{array}{l} P(X>\frac{1}{4} \mid y=\frac{3}{4}) \\ 0<x<y,\ y=\frac{3}{4} \\ \therefore 0<x<\frac{3}{4} \Rightarrow \frac{1}{4}<x<\frac{3}{4} \end{array}}$ $= \int_{1/4}^{y} \frac{2x}{y}\,dx$



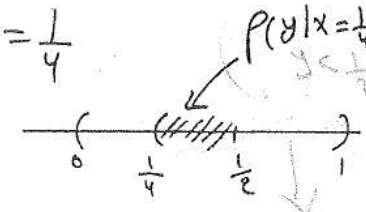
$0<x<y\ ,\ y=\frac{1}{2} \Rightarrow 0<x<\frac{1}{2}$

$$= \int_{1/4}^{1/2} \frac{2x}{(\frac{1}{2})^2}\,dx = 4\cdot 2\cdot \frac{x^2}{2}\Big|_{1/4}^{1/2}$$

$$= 4\left[\frac{1}{4}-\frac{1}{16}\right] = 4\cdot\left(\frac{3}{16}\right) = \frac{3}{4}$$

$P(y<\frac{1}{2} \mid X=\frac{1}{4}) = ?$ $\qquad x<y<1\ ,\ x=\frac{1}{4}$

$\frac{1}{4}<y<1$



$P(y<\frac{1}{2} \mid x=\frac{1}{4}) = \int_x^{1/2} f(y/x=\frac{1}{4})\,dy$ ⟵ ملاحظة

نعوض بـ $f(y|x=\frac{1}{4})$ ونعوض بـ $x=\frac{1}{4}$ ثم ناخذ الاحتمالية

$$= \int_x^{1/2} \frac{2y}{1-x^2}\,dy = \int_{1/4}^{1/2} \frac{2y}{1-(\frac{1}{4})^2}\,dy$$

$$= \int_{1/4}^{1/2} \frac{2y}{15/16}\,dy = \frac{16}{15}\cdot\cancel{2}\cdot\frac{y^2}{\cancel{2}}\Big|_{1/4}^{1/2}$$

$$= \frac{16}{15}\left(\frac{1}{4}-\frac{1}{16}\right) = \frac{\cancel{16}}{\cancel{15}_{5}}\cdot\frac{\cancel{3}}{\cancel{16}} = \frac{1}{5}$$

\# Example 6: If $X \sim unif(0,1)$ and

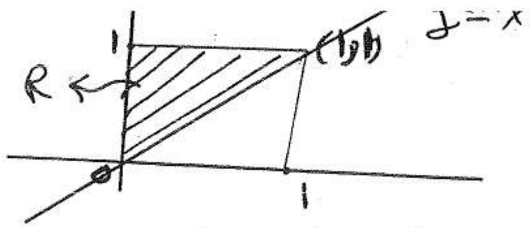
$$f(y|x) = \begin{cases} \frac{1}{1-x} & \text{for } x<y<1,\ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

(a) find $f(x,y)$?

$\because X \sim unif(0,1)$

$f_1(x) = \frac{1}{1-0} = 1 \Longrightarrow f_1(x) = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$



$$f(x,y) = f_1(x) \cdot f(y|x)$$

$$= (1)\left(\frac{1}{1-x}\right) = \begin{cases} \frac{1}{1-x} & \text{for } x<y<1,\ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore \boxed{f(y|x) = f(x,y)}$$

b. Find $P(X>\frac{1}{2} \mid y=\frac{3}{4}) = \int_0^y f(x|y=\frac{3}{4})\,dx$

$$f(x|y) = \frac{f(x,y)}{f_2(y)}$$

$$f_2(y) = \int_0^y f(x,y)\,dx = \int_0^y \frac{1}{1-x}\,dx = -Ln(1-x)\Big|_0^y$$

$$= -[Ln(1-y) - \underbrace{\cancel{Ln(1-0)}}_{\text{Zero}}]$$

$$= -Ln(1-y) \quad \text{for } 0<y<1$$

$$\therefore f_2(y) = -Ln(1-y) \quad \text{for } 0<y<1$$

$$\therefore f(x|y) = \frac{\frac{1}{1-x}}{-Ln(1-y)}$$

$$f(x|y) = \begin{cases} \frac{1}{-Ln(1-y)} \cdot \frac{1}{1-x} & \text{for } 0<x<y \ \&\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X>\tfrac{1}{2} \mid y=\tfrac{3}{4}) = \int_{1/2}^{y} f(x|y=\tfrac{3}{4})\,dx$$

$(0<X<y,\ y=\frac{3}{4})$

$\Rightarrow 0<X<\frac{3}{4}$

يتكامل

$X>\frac{1}{2}$ ثم

$\Rightarrow \frac{1}{2}<X<\frac{3}{4}$

$$= \int_{1/2}^{y} \frac{-1}{Ln(1-y)} \cdot \frac{1}{1-x}\,dx = \int_{1/2}^{3/4} \frac{-1}{Ln(1-\frac{3}{4})} \cdot \left(\frac{1}{1-x}\right) dx$$

$$= \frac{-1}{Ln\,1 - Ln\,4} \cdot \Big[-Ln(1-x)\Big]_{1/2}^{3/4}$$

$$= \frac{+1}{-Ln\,4}\left[Ln(1-\tfrac{3}{4}) - Ln(1-\tfrac{1}{2})\right]$$

$$= \frac{-1}{Ln\,4}\left[Ln(\tfrac{1}{4}) - Ln(\tfrac{1}{2})\right]$$

$$= \frac{-1}{Ln\,2^2}\left[(\underbrace{\cancel{Ln\,1}}_{\text{Zero}} - Ln\,4) - (\underbrace{\cancel{Ln\,1}}_{\text{Zero}} - Ln(2))\right]$$

$$= \frac{-1}{2\ln 2}\left(-2\ln 2 + \ln 2\right) = \frac{-1}{2\ln 2}\left(-\ln 2\right) = \frac{1}{2}$$

H.W Find $P\left(Y < \frac{3}{4} \mid X = \frac{1}{2}\right)$

الموضوع الثاني

Expectation of two Random Variables

Review

① **Def.** $E(XY)$ is the expectation or the mean of both X and Y such that:

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy \quad \text{if } X \text{ and } Y \text{ are c.r.v.s}$$

$$= \sum_{y}\sum_{x} xy\, f(x,y) \quad \text{if } X \text{ and } Y \text{ are d.r.v.s}$$

Note: ① $E(X) \equiv$ mean of $X = \mu_x$

s.t. $E(X) = \int_{-\infty}^{\infty} x f_1(x)\, dx$ if X is c.r.v.

$= \sum_{x} x f_1(x)$ if X is d.r.v.

② $E(Y) \equiv$ mean of $Y = \mu_y$

s.t. $E(y) = \int_{-\infty}^{\infty} y f_2(y)\, dy$ if Y is c.r.v.

$= \sum_{y} y f_2(y)$ if Y is d.r.v.

③ (a) $V(X) = E(X^2) - [E(X)]^2$

(b) $V(Y) = E(Y^2) - [E(Y)]^2$

④ Covariance of both X & Y

We use the symbole $Cov(X,Y)$ (it's mean Covariance), s.t

$$Cov(X,Y) = E\{[X - E(X)][Y - E(Y)]\}.$$

Theorem (1) If X and Y are random variables, then:

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Proof

$$\begin{aligned} Cov(X,Y) &= E\{[X-E(X)][Y-E(Y)]\} \\ &= E\{XY - YE(X) - XE(Y) + E(X)\cdot E(Y)\} \\ &= E(XY) - E(X)E(Y) - E(X)E(Y) + E(Y)E(X) \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

Theorem (2) If X and Y are independent, then

$$E(XY) = E(X)\,E(Y)$$

or $Cov(X,Y) = 0$

Proof **Case ①** If X and Y are c.r.v.s

$\because$ X & Y are indep. $\Rightarrow f_1(x) f_2(y) = f(x,y)$

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy)\, f(x,y)\, dx\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_1(x) f_2(y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} x f_1(x)\, dx \cdot \int_{-\infty}^{\infty} y f_2(y)\, dy \quad \text{by Comm. Property} \\ &= E(X)\cdot E(Y) \end{aligned}$$

Case ② By the same method.

Note By Th. 2 above, if X & Y are indep., then

$$Cov(X,Y) = 0$$

Example: Given a J.p.d.f.

$$f(x,y) = \begin{cases} 2 & \text{for } 0<y<x<1 \\ 0 & \text{o.w.} \end{cases}$$

هناك ٣ طرق للاجابة

Are X & Y indep.? (If $Cov(X,Y) = 0$ Then X & Y are indep.)

① By using $Cov(X,Y)$
or ② By using $E(X,Y)$
③ $f_1(x) f_2(y) = f(x,y)$

$$E(xy) = \int_0^1 \int_0^x xy\,(2)\,dy\,dx$$

$$= \int_0^1 \cancel{2}x \frac{y^2}{\cancel{2}}\Big]_0^x dx$$

$$= \int_0^1 x \cdot x^2\,dx$$

$$= \frac{x^4}{4}\Big]_0^1$$

$$= \frac{1}{4}.$$

Rho

$$\rho_{x,y} = \frac{Cov(x,y)}{\sqrt{V(x) \cdot V(y)}}$$

$$\rho = \frac{\sigma_{x,y}}{\sigma_x\,\sigma_y}$$

$\rho_{x,y}$

Correlation coefficient

$$E(x^2) = \int_0^1 x^2(2x)\,dx = \frac{2}{4}x^4\Big]_0^1 = \frac{2}{4} = \frac{1}{2}$$

$$V(x) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

$$E(y^2) = \int_0^1 y^2\,[2(1-y)]\,dy = 2\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$$

$$= 2\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= 2\left[\frac{4-3}{12}\right]$$

$$= \frac{2}{12} = \frac{1}{6}$$

$$V(y) = \frac{1}{6} - \left[\frac{2}{6}\right]^2$$

$$= \frac{1}{6} - \frac{4}{36}$$

$$= \frac{6}{36} - \frac{4}{36}$$

$$= \frac{2}{36} = \frac{1}{18}$$

$$R = \begin{Bmatrix} 0 < y < x & , & y < x < 1 \\ 0 < y < 1 & , & 0 < x < 1 \end{Bmatrix}$$

$$E(XY) = \int_0^1 \int_y^1 xy \cdot f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_y^1 xy(2)\, dx\, dy = \int_0^1 \int_0^X (XY)\, f(X,Y)\, dy\, dx$$

$$= \int_0^1 2y \left.\frac{x^2}{2}\right|_y^1 dy = \int_0^1 y(1-y^2)\, dy$$

$$= \int_0^1 [y - y^3] = \left.\frac{y^2}{2} - \frac{y^4}{4}\right|_0^1$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E(X) = \int_0^1 x\, f_1(x)\, dx \quad , \quad E(y) = \int_0^1 y\, f_2(y)\, dy$$

$$f_1(x) = \int_0^x 2\, dy = \int_0^x 2\, dy = 2y\Big|_0^x = 2x \Rightarrow f_1(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_y^1 f(x,y)\, dx = \int_y^1 2\, dx = 2x\Big|_y^1 = 2[1-y] \Rightarrow f_2(y) = \begin{cases} 2(1-y) & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \int_0^1 x\, f_1(x)\, dx = \int_0^1 x(2x)\, dx = \int_0^1 2x^2\, dx = \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3}$$

$$E(y) = \int_0^1 y\, f_2(y)\, dy = \int_0^1 2y(1-y)\, dy = \int_0^1 (2y - 2y^2)\, dy = y^2 - \frac{2}{3}y^3 \Big|_0^1$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{4} - \left(\frac{2}{3}\right)\cdot\left(\frac{1}{3}\right) = \frac{9-8}{36} = \frac{1}{36}$$

$\operatorname{Cov}(X,Y) \neq 0 \Longrightarrow X$ and Y are dependent.

~~$\forall$ $\operatorname{Cov}(X,Y) > 0$ all ways~~

(تصحيح)

$$\left.\begin{matrix} V(X) = 1/18 \\ V(Y) = 1/18 \end{matrix}\right] \rho_{X,Y} = \frac{\operatorname{Cov}(X,Y)}{\sqrt{V(X)\cdot V(Y)}} = \frac{\frac{1}{36}}{\sqrt{\frac{1}{18}\cdot\frac{1}{18}}} = \frac{\frac{1}{36_2}}{\frac{1}{18_1}} = \frac{1}{2}$$

معامل الارتباط Correlation Coefficient

Def. when X and y are dependent and we want to know how much X depends on y we use the correlation to measure it, denoted by "$\rho_{x,y}$" read (Rho) where:

$-1 \le \rho_{x,y} \le 1$

$$\boxed{\rho_{x,y} = \frac{Cov(X,y)}{\sqrt{V(X)V(y)}} = \frac{E(Xy) - E(X)E(y)}{\sqrt{V(X)\,V(y)}}} = \frac{Cov(X,y)}{\sigma_x\,\sigma_y}$$

$Cov(X,y) \neq 0$ if X & y are dependent

where $-1 \le \rho \le 1$

H.W. Find $\rho_{x,y}$ from above example. $\longrightarrow Cov(X,y) = \frac{1}{2} \neq 0$

② Conditional Mean and Conditional Variance

① $f(x|y) = \frac{f(x,y)}{f_2(y)}$

$f(y|x) = \frac{f(x,y)}{f_1(x)}$

$E(y|X)$ denote to the conditional mean of y given $X = x$.

$E(x|y)$ denote to the conditional mean of X given $Y = y$.

where $\underbrace{E(y|X)}_{g(X) = \text{func. of } X} = \int_{\square}^{\square} y\, f(y|X)\, dy$, limit of integral follows, limit of y.

$\underbrace{E(X|y)}_{g(y) = \text{func. of } y} = \int_{\square}^{\square} x\, f(X|y)\, dx$, limit of integral follows, limit of X.

Note Since $f(y|X)$ defined for interval of y interm of X, then $E(y|X)$ is a function of X. Also $E(X|y)$ is a function of y.

$V(y|X)$ denote to the conditional variance of y given $X = x$.

$V(X|y)$ denote to the Conditional Variance of X given $Y = y$.

where

$$V(y|X) = E\{[y-E(y|X)]^2 | X\}$$

$$= \int_{[y]} [y-E(y|x)]^2 f(y|x)\,dy \quad \Big\}\ \text{by def. of } E(X|x)$$

$$\cancel{\sum_{[y]} [y-E(y|x)]^2}\ f(y|x)$$

$$V(X|y) = E\{[X-E(X|y)]^2 | y\}$$

$$\cancel{= \sum_{[x]} [X-E(X|y)]^2 f(x|y)} \quad \Big\}\ \text{by def. of } E(X|x)$$

$$= \int_{[x]} [X-E(X|y)]^2 f(x|y)\,dx$$

Example: Given a J.p.d.f $f(x,y)$

$$f(x,y) = \begin{cases} 2 & \text{for } 0 \le x+y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(X|y), E(y|X), V(X|y), V(y|X)$

Sol.

$$R = \{(x,y);\ 0 \le x+y \le 1\} = \left\{ \begin{matrix} 0 \le x \le 1-y, & 0 \le y \le 1-x \\ 0 \le x \le 1, & 0 \le y \le 1 \end{matrix} \right\}$$

$$E(X|y) = \int_x x f(x|y)\,dx$$

$$f(X|y) = \frac{f(x,y)}{f_2(y)}$$

$$f_2(y) = \int_0^{1-y} f(x,y)\,dx = \int_0^{1-y} 2\,dx = 2x\Big|_0^{1-y} = 2(1-y)$$

$$f_2(y) = \begin{cases} 2(1-y) & \text{for } 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f(X|y) = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

$$f(x|y) = \begin{cases} \frac{1}{1-y} & \text{for } 0 \le x \le 1-y \\ 0 & \text{o.w.} \end{cases}$$ (هي دالة لـ X وقيمة y معلومة)

$$E(X|y) = \int_0^{1-y} x \cdot \frac{1}{(1-y)}\,dx = \frac{1}{(1-y)} \cdot \frac{x^2}{2}\Big|_0^{1-y} = \frac{(1-y)^2}{2(1-y)} = \left(\frac{1-y}{2}\right) \quad 0 \le y \le 1$$ دالة لـ X

$$V(X|y) = \int_0^{1-y} [x - E(X|y)]^2 f(x|y)\,dx = \int_0^{1-y} \left[x - \frac{(1-y)}{2}\right]^2 dx$$

$$= \int_0^{1-y} \left[x^2 - (1-y)x + \frac{(1-y)^2}{4}\right] dx = \frac{1}{12}(1-y)^2 \text{ for } 0 \le y \le 1$$

طريقة ثانية للحل

Find $E(y|X)$ & $V(y|X)$ H.W

or

$$V(X|Y) = E(X^2|Y) - [E(X|Y)]^2 = \frac{(1-y)^2}{3} - \left(\frac{1-y}{2}\right)^2$$

$$E(X^2|y) = \int_0^{1-y} x^2 f(x|y)\,dx = \frac{x^3}{3(1-y)}\Big|_0^{1-y} = \frac{(1-y)^3}{3(1-y)} = \frac{(1-y)^2}{3}$$

Theorem (3) If X and Y are two random variables, then:

$$a.\ E[E(Y|X)] = E(Y) \quad , \quad b.\ E[E(X|Y)] = E(X)$$

Proof Case (1): If X and Y are c.r.v. with p.d.f $f(x)$.

$E(Y|X)$ is a function of X.

suppose that $E(Y/X) = g(X)$ = as a func of x

$$E[E(Y/X)] = E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_1(x)\,dx = \int_{-\infty}^{\infty} [E(Y/X)]\, f_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y f(y|x)\,dy\right] f_1(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(y|x) f_1(x)\,dy\,dx$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} \Rightarrow f(y/x) \cdot f_1(x) = f(x,y)$$

$$\int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} y\, f_2(y)\,dy = E(Y)$$

Case (2): If X and Y are d.r.v. with p.m.f. $f(x)$.

$E(Y|X)$ is a function of X.

Suppose that $E(Y/X) = g(X)$

$$E(E(Y/X)) = E(g(X)) = \sum_x g(x) \cdot f_1(x) = \sum_x [E(Y/X)] f_1(x)$$

$$= \sum_x \left[\sum_y y f(y|x)\right] f_1(x) = \sum_x \sum_y y\, f(y|x) f_1(x)$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} \Rightarrow f(y|x) \cdot f_1(x) = f(x,y)$$

$$\sum_y y \sum_x f(x,y) = \sum_y y\, f_2(y) = E(Y)$$

b. By the same method.

Example: Given a J. p.d.f. $f(x,y)$

$$f(x,y) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E[E(Y|X)]$.

Sol.

$E(E(Y|X)) = E(Y)$ (by Theorem 3)

$$E(Y) = \int_{-\infty}^{\infty} y\, f_2(y)\,dy \qquad f_2(y) = \int_0^y f(x,y)\,dx = \int_0^y 2\,dx = 2x\Big|_0^y = 2y$$

$$f_2(y) = \begin{cases} 2y & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore E(Y) = \int_0^1 y(2y)\,dy = 2\int_0^1 y^2\,dy = \frac{2}{3} y^3 \Big|_0^1 = \frac{2}{3}$$

Joint Distribution function (J.d.f) of two aR.v.s X and Y

Def. The J.d.f of X and Y is defined to a function such that for all values of X and Y $(-\infty < x < \infty, -\infty < y < \infty)$ then:

③

$$F(x,y) = P(X \le x, Y \le y)$$

$F(x) = P(X \le x)$ Rev.

① $\int_{-\infty}^{x} f(t)\,dt$ if X is c.r.v.

② $\sum_{x_j \le x} f(x_j)$ if X is d.r.v.

Remarks

1. $P(a < X \le b, c < Y \le d) = [F(b,d) - F(a,d)] - [F(b,c) - F(a,c)]$

نقارن اعلاه للمتغير العشوائي الواحد(X) :

$$\left\{\begin{array}{l} 1.\ F(x) = P(X \le x) \\ 2.\ P(a < X \le b) = F(b) - F(a) \end{array}\right\}$$

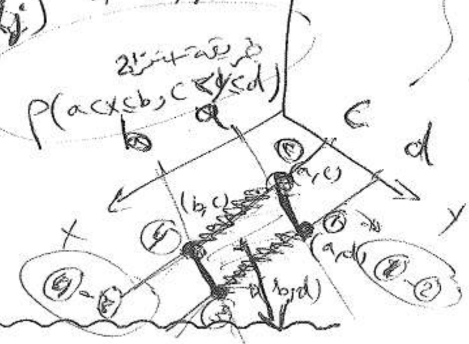
2. (i) $F_1(x) = P(X \le x) = \lim_{y \to \infty} P(X \le x \text{ and } Y \le y) = \lim_{y \to \infty} F(x,y)$

(ii) $F_2(y) = \lim_{x \to \infty} F(x,y)$

3. If X and Y have cont. J.d.f $F(x,y)$ with J.P.d.f $f(x,y)$, then for any values of X and Y the J.d.f. is

① $F(x,y) = \sum_{x_i \le x} \sum_{y_j \le y} f(x_i, y_j)$ if x, y d.r.v.

② $F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(r,s)\,dr\,ds$ therefore



Note

$$* \quad f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \, \partial y}, \quad f(x,y) \text{ diff.}$$

$F(x,y) = P(X \le x, Y \le y)$

Example: Given a J.d.f $F(x,y)$

$$F(x,y) = \begin{cases} 0 & \text{for } x < 0, y < 0 \\ \frac{1}{16} xy(x+y) & \text{for } 0 \le x \le 2, 0 \le y \le 2 \\ 1 & \text{for } x > 2, y > 2 \end{cases}$$

Find $F_1(x)$, $F_2(y)$, $f(x,y)$.

Sol. $F_1(x) = \lim_{y \to \infty} F(x,y) = \lim_{y > 2} F(x,y) = F(x,2)$

$$= \frac{1}{16} x(2)(x+2) = \frac{x(x+2)}{8}$$

$$F_1(x) = \begin{cases} 0 & \text{for } x<0 \\ \dfrac{x(x+2)}{8} & \text{for } 0\le x\le 2 \\ 1 & \text{for } x>2 \end{cases}$$

$$F_2(y) = \lim_{x\to\infty} F(x,y) = \lim_{x>2} F(x,y) = F(2,y)$$
$$= \frac{1}{16}(2)\,y(2+y) = \frac{y(2+y)}{8}$$

$$F_2(y) = \begin{cases} 0 & \text{for } y<0 \\ \dfrac{y(2+y)}{8} & \text{for } 0\le y\le 2 \\ 1 & \text{for } y>2 \end{cases}$$

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial F(x,y)}{\partial y}\right]$$

$$F(x,y) = \frac{1}{16}(x^2y + xy^2)$$

$$f(x,y) = \frac{\partial}{\partial x}\left[\frac{1}{16}(x^2+2xy)\right] = \left[\frac{1}{16}(2x+2y)\right]$$
$$= \frac{1}{8}(x+y)$$

$$f(x,y) = \begin{cases} \dfrac{x+y}{8} & \text{for } 0<y<2,\ 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

H.W. (1) Prove that : $E(x+y) = E(x)+E(y)$

(2) " " : $E(xy) = E(x)E(y)$

Proof (1) If x & y are independent d.r.v.s ;

$$E(x_1+x_2) = \sum_{x_1}\sum_{x_2}(x_1+x_2)\,f(x_1,x_2)$$
$$= \sum_{x_1}\sum_{x_2}\big(x_1 f(x_1,x_2) + x_2 f(x_1,x_2)\big)$$
$$= \sum_{x_1}\sum_{x_2} x_1 f(x_1,x_2) + \sum_{x_1}\sum_{x_2} x_2 f(x_1,x_2)$$
$$= \Big(\sum_{x_1} x_1 f_1(x_1)\Big)\Big(\sum_{x_2} f_2(x_2)\Big) + \Big(\sum_{x_2} x_2 f_2(x_2)\Big)\Big(\sum_{x_1} f_1(x_1)\Big)$$
$$= E(x_1)\cdot(1) + E(x_2)(1)$$

$$\rho_{x,y} = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{\sigma_x^2}\ \sqrt{\sigma_y^2}}$$

$$\sigma_x^2 = E(X^2) - [E(X)]^2$$

$$\sigma_y^2 = E(Y^2) - [E(Y)]^2$$

$$E(X^2) = \left.\frac{\partial^2 M(t_1,0)}{\partial t_1^2}\right|_{t_1=0} = \left. -2(1-t_1)^{-3}(-1)\right|_{t_1=0} = \boxed{2}$$

$$E(Y^2) = \left.\frac{\partial^2 M(0,t_2)}{\partial t_2^2}\right|_{t_2=0} = \left. -2(1-t_2)^{-3}(-1)\right|_{t_2=0} = \boxed{2}$$

$$\sigma_x^2 = 2-(1)^2 = \boxed{1} \Rightarrow \boxed{\sigma_x = 1}$$

$$\sigma_y^2 = 2-(1)^2 = \boxed{1} \Rightarrow \boxed{\sigma_y = 1}$$

$$\rho_{x,y} = \frac{1-(1)(1)}{(1)\ (1)} = 0$$

$\therefore$ X & Y are independent (since $Cov(x,y)=0$).

H.W. Given a J.p.f of X & Y:

$$f(x,y) = \begin{cases} e^{-y} & \text{for } 0<x<y<\infty \\ 0 & \text{o.w.} \end{cases}$$

Find: ① the M.g.f. of X, Y $[M_{x,y}(t_1,t_2)]$.

② $E(XY)$

③ $f_1(x)$ & $f_2(y)$

④ $\mu_x = E(x)$ & $\mu_y = E(y)$ (by two methods).

⑤ $Cov(x,y)$.

Note: $R = \left\{ \begin{matrix} 0<x<y & , x<y<\infty \\ 0<x<\infty & , 0<y<\infty \end{matrix} \right\}$

$$\text{M.g.f of } X = M_{X,Y}(t_1,0) = M_X(t_1) = \frac{1}{1-t_1}$$

$$\mu_x = E(X) = \left.\frac{\partial M(t_1,0)}{\partial t_1}\right|_{t_1=0} = \left.\frac{\partial M_X(t_1)}{\partial t_1}\right|_{t_1=0}$$

$$= \left.\frac{1}{(1-t_1)^2}\right|_{t_1=0} = \boxed{1}$$

$$\text{M.g.f of } Y = M_{X,Y}(0,t_2) = M_Y(t_2) = \frac{1}{1-t_2}$$

$$\mu_y = E(Y) = \left.\frac{\partial M(0,t_2)}{\partial t_2}\right|_{t_2=0} = \left.\frac{\partial M_Y(t_2)}{\partial t_2}\right|_{t_2=0}$$

$$= \left.\frac{1}{(1-t_2)^2}\right|_{t_2=0} = \boxed{1}$$

$$\therefore \text{M.g.f of } X,Y = M_{X,Y}(t_1,t_2) = \frac{1}{(1-t_1)(1-t_2)}$$

$$E(XY) = \left.\frac{\partial^2 M(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0}$$

$$= \left.\frac{\partial}{\partial t_1}\left(\frac{\partial M(t_1,t_2)}{\partial t_2}\right)\right|_{t_1=t_2=0}$$

$$= \left.\frac{\partial}{\partial t_1}\left(\frac{1}{(1-t_1)(1-t_2)^2}\right)\right|_{t_1=t_2=0}$$

$$= \left.\frac{1}{(1-t_1)^2(1-t_2)^2}\right|_{t_1=t_2=0}$$

$$= \boxed{1}$$

The variance of X or y by using $M_{x,y}(t_1,t_2)$:

$$Var(X) = \sigma_x^2 = \frac{\partial^2 M(0,0)}{\partial t_1^2} - \left(\frac{\partial M(0,0)}{\partial t_1}\right)^2$$
$$= E(X^2) - [E(X)]^2$$

$$Var(y) = \sigma_y^2 = \frac{\partial^2 M(0,0)}{\partial t_2^2} - \left(\frac{\partial M(0,0)}{\partial t_2}\right)^2$$
$$= E(y^2) - [E(y)]^2$$

Example: Let $f(x,y)$ be the J.p.d.f. of X, y:

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } o \quad x>0, \; y>0 \\ 0 & \text{o.w.} \end{cases}$$

Find the M.g.f of X, y; also, find $\mu_x, \mu_y, E(XY)$, and $\rho_{x,y}$.

Sol.
$$M_{x,y}(t_1,t_2) = E\left[e^{t_1x+t_2y}\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dy\,dx$$
$$= \int_{0}^{\infty}\int_{0}^{\infty} e^{t_1x+t_2y}\, e^{-(x+y)}\,dy\,dx$$
$$= \left(\int_0^{\infty} e^{-x(1-t_1)}\,dx\right)\left(\int_0^{\infty} e^{-y(1-t_2)}\,dy\right)$$
$$= \frac{1}{(1-t_1)} \cdot \frac{1}{(1-t_2)} = \frac{1}{(1-t_1)(1-t_2)}$$

(eq. 2)

Moment Generating Function of two R.V.'s X & Y

(2) Let $f(x,y)$ be the J.p.d.f of X, Y ; and consider that:

(1) $E[e^{t_1 x + t_2 y}]$ exists for $|t_1| < h_1$, $|t_2| < h_2$; h_1, h_2 are positive constants, then:

$E[e^{t_1 X + t_2 y}]$ is the m.g.f of X & Y and is denoted by $M_{x,y}(t_1, t_2)$ and is defined as follows:

$$M_{x,y}(t_1,t_2) = E[e^{t_1 x + t_2 y}] = \begin{cases} \sum_{\forall x} \sum_{\forall y} e^{t_1 x + t_2 y} f(x,y) & \text{if } x,y \text{ are d.r.v.s} \\ \int_x \int_y e^{t_1 x + t_2 y} f(x,y)\, dy\, dx & \text{if } x,y \text{ are c.r.v.s} \end{cases}$$

and

$$M_x(t_1) = M_x(t_1, 0) = E[e^{t_1 x}] = \begin{cases} \sum_{\forall x} e^{t_1 x} f_1(x) & \text{if } X \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{t_1 x} f_1(x)\, dx & \text{if } X \text{ is c.r.v.} \end{cases}$$

(2) $$M_y(t_2) = M_y(0, t_2) = E[e^{t_2 y}] = \begin{cases} \sum_{\forall y} e^{t_2 y} f_2(y) & \text{if } y \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{t_2 y} f_2(y)\, dy & \text{if } y \text{ is c.r.v.} \end{cases}$$

and

(3) $$\frac{\partial M(0,0)}{\partial t_1} = \mu_x = E(X) \; ; \quad \frac{\partial M(0,0)}{\partial t_2} = \mu_y = E(Y)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1^2} = E(X^2) \; ; \quad \frac{\partial^2 M(0,0)}{\partial t_2^2} = E(Y^2)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} = E(XY)$$

Exersices About Ch. 5

Q1: Given a J.p.f. :

$$f(x,y) = \begin{cases} c\,y^2 & \text{for } 0 \le x \le 2 \ \& \ 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$c \int_0^2 \int_0^1 y^2 \, dy \, dx = c \int_0^2 \frac{y^3}{3}\Big|_0^1 dx = \frac{c}{3} \, x \Big|_0^2 = \frac{2}{3} c$

Find :
1. The value of (c)
2. $P(X+Y > 2)$
3. $P(Y < \frac{1}{2})$, 4. $P(X \le 1)$

Q2: Let the J.p.m.f of X & Y be:

$$f(x,y) = \begin{cases} \frac{xy^2}{30} & \text{for } x = 1,2,3 \ \& \ y = 1,2 \\ 0 & \text{o.w.} \end{cases}$$

(i) Find the Marginal p.f. of X ?
(ii) Find the Marginal p.f. of Y ?
(iii) Are X & Y independent ?

Q3: Let X & Y have the J.p.m.f. be :

$$f(x,y) = \begin{cases} \frac{x+2y}{18} , & x = 1,2 , \ y = 1,2 \\ 0 & \text{o.w.} \end{cases}$$

Find: $\mu_x = E(X)$, $\mu_y = E(Y)$, & $Cov(X,Y)$.

Q4: Suppose that X, Y are c.r.v.s with J.p.d.f. :

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } x > 0, \ y > 0 \\ 0 & \text{o.w.} \end{cases}$$

Find : $M_{X,Y}(t_1,t_2)$, $M_X(t_1)$, $M_Y(t_2)$.

Q5: Suppose X and Y are two r.v.s having the following J.p.d.f :

$$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(i) Find the conditional p.d.f $f(x|y)$ and $f(y|x)$.

(ii) Are these Conditional function $f(x|y)$ and $f(y|x)$ p.d.f.? (prove that).

Q6: Consider the following J.p.M.f. of X and Y whose values are given by:

$y \backslash x$	2	3	4	5	6	$f_2(y)$
0	1/9	0	4/27	0	2/27	1/3
1	2/27	0	8/81	0	4/81	2/9
2	4/27	0	16/81	0	8/81	4/9
$f_1(x)$	1/3	0	4/9	0	2/9	1

Find: ① $f(x|y)$ & $f(y|x)$

② $P(x|y=1)$ & $P(y|x=2)$

Q7: If J.p.m.f :

$P(1,1)=\frac{1}{9}$, $P(2,1)=\frac{1}{3}$, $P(3,1)=\frac{1}{9}$

$P(1,2)=\frac{1}{9}$, $P(2,2)=0$, $P(3,2)=\frac{1}{18}$

$P(1,3)=0$, $P(2,3)=\frac{1}{6}$, $P(3,3)=\frac{1}{9}$

Are X & Y independent ?

Q8: If X and Y are independent with M.p.d.f :

$$f_1(x) = \begin{cases} 3x^2 & \text{for } 0\le x\le 1 \\ 0 & \text{o.w.} \end{cases}, \quad f_2(y) = \begin{cases} 2y & \text{for } 0\le y\le 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(XY)$.

Q9: Suppose that J.p.d.f of X & Y is:

$$f(x,y) = \begin{cases} c\sin x & \text{for } 0 \le x \le \pi,\ 0 \le y \le \text{[illegible]} \\ 0 & \text{o.w.} \end{cases}$$

Find the value of C. $\left(C = \frac{1}{2\pi}\right)$

Q10: Let X and Y have the joint p.m.f. be:

(x,y)	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$
$P(x,y)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

Find the Correlation Coefficient $\rho_{x,y}$.

Q11: If $f(x,y) = \begin{cases} e^{-x-y} & \text{for } 0 < x < \infty \text{ \& } 0 < y < \infty \\ 0 & \text{o.w.} \end{cases}$

is the J.p.d.f of two r.v.s X & Y.
Show that X & Y are stochastically independent.

B

Q12: Let X & Y be independent r.v.'s with the following distribution:

X	1	2
$f_1(x)$	.6	.4

y	10	15	5
$f_2(y)$	.5	.3	.2

① Find the Joint pr. f of X & Y

② also, find $E(XY)$.

B

Q13: Let $f(x|y) = \begin{cases} \frac{C_1 x}{y^2} & \text{for } 0 < x < y,\ 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$

& $f_2(y) = \begin{cases} C_2 y^4 & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$; then:

$\left(f(x,y) = \qquad 0 < x < y < 1\right)$

Find:

(1) Determine the Constants C_1 and C_2.

(2) the $f_1(X)$ p.d.f. of X.

(3) the J.p.d.f. of X & Y ; $f(X,y)$

(4) $P(-\frac{1}{2} < X < \frac{1}{2} \mid y = \frac{5}{8})$

(5) $P(\frac{1}{4} < X < \frac{1}{2})$.

Q14: Let $f(X,y) = \begin{cases} \frac{1}{16} & \text{for } X = 1,2,3,4 \\ & \quad y = 1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

be the J.p.f of X & Y.

Show that X & Y are stochastically independent.

Q15: Given a J.p.f of X & Y:

$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1 \text{ \& } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$

Find the Correlation Coefficient of X & Y ($\rho_{x,y}$).

Solution of Q_1:

① by Cond. (2) $\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

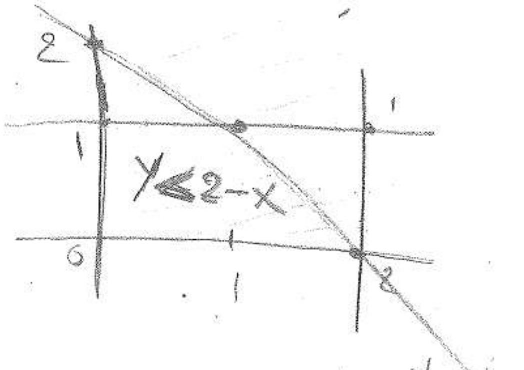
$C\int_0^2\int_0^1 y^2\,dy\,dx = 1 \Rightarrow \boxed{C = \frac{3}{2}}$

$P(X+Y>2)$

② $P(X+Y \leq 2) = 1 - P(X+Y > 2)$

Let $x+y = 2 \Rightarrow y = 2-x$

$x+y \leq 2 \Rightarrow y \leq 2-x$



$R = \left\{ \begin{array}{l} 0 \leq y \leq 2-x \\ 0 \leq x \leq 2 \end{array} \right\}$

$P(X+Y \leq 2) = \int_0^2\int_0^{2-x} \frac{3}{2} y^2\,dy\,dx = \cdots = \frac{1}{2}$

$\int_0^1\int_0^{2-y} \frac{3}{2} y^2\,dx\,dy$

③ $P(Y < \frac{1}{2}) = ?$

To find $f_2(y)$ in the first.

$f_2(y) = \int_0^2 f(x,y)\,dx = \frac{3}{2}\int_0^2 y^2\,dx = \cdots = \begin{cases} 3y^2 & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$

$P(Y < \frac{1}{2}) = 3\int_0^{1/2} y^2\,dy = \cdots = \frac{1}{8}$

④ $P(X \leq 1) = ?$

To find $f_1(x)$ in the begining.

$f_1(x) = \int_0^1 f(x,y)\,dy = \frac{3}{2}\int_0^1 y^2\,dy = \cdots = \begin{cases} \frac{1}{2} & \text{for } 0 \leq x \leq 2 \\ 0 & \text{o.w.} \end{cases}$

$P(X \leq 1) = \int_0^1 f_1(x)\,dx$

$= \int_0^1 \frac{1}{2}\,dx = \cdots = \frac{1}{2}$.

⑤ $P(X = 3Y) = \int_0^2\int_0^{\frac{1}{3}x} f(x,y)\,dy\,dx = \int_0^2\int_0^{\frac{1}{3}x} \frac{2}{3} y^2\,dy\,dx$.

$\Rightarrow \boxed{y = \frac{1}{3}x}$

$= \cdots = \frac{2}{27}$

Q2: If $f_1(x)\cdot f_2(y) = f(x,y)$, then X & Y are indep.

$$f_1(x) = \sum_{y=0}^{3} f(x,y) = \sum_{y=0}^{3} \frac{1}{30}(x+y)$$

$$= \frac{1}{30}\left[x + (x+1) + (x+2) + (x+3)\right]$$

$$= \frac{1}{30}(4x+6) = \begin{cases} \frac{1}{15}(2x+3) & \text{for } x=0,1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \sum_{x=0}^{2} f(x,y) = \sum_{x=0}^{2} \frac{1}{30}(x+y)$$

$$= \frac{1}{30}\left[y + (1+y) + (2+y)\right]$$

$$= \frac{1}{30}(3+3y) = \frac{1}{10}(y+1)$$

$$f_2(y) = \begin{cases} \frac{1}{10}(y+1) & \text{for } y=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

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$$f(x,y) = \begin{cases} \frac{x+y}{30} & y=0,1,2,3 \\ & x=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$f_1(x) f_2(y) = \frac{1}{15}(2x+3)\cdot\frac{1}{10}(y+1)$$

$$\neq \frac{1}{30}(x+y) = f(x,y)$$

$$\therefore f_1(x)\cdot f_2(y) \neq f(x,y)$$

Q3: To find f_1 & f_2 in the first time.

$$f_1(x) = \sum_{x_2=1}^{2} \frac{1}{18}(x_1+2x_2)$$

$$= \frac{1}{18}\sum_{x=1}^{2} x_1 + \frac{1}{18}\sum_{x_2=1}^{2} 2x_2$$

$$= \cdots = \begin{cases} \frac{x_1+3}{9} & \text{for } x_1=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_{x_1} = E(x_1) = \sum_{x_1=1}^{2} x_1 f_1(x_1) = \sum_{x_1=1}^{2} x_1\left(\frac{x_1+3}{9}\right) = \cdots = \frac{14}{19}$$

$$f_2(x_2) = \sum_{x_1=1}^{2} \frac{1}{18}(x_1+2x_2) = \begin{cases} \frac{1}{18}(3+4x_2) & \text{for } x_2=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_{x_2} = E(x_2) = \sum_{x_2=1}^{2} x_2 f_2(x_2) = \sum_{x_2=1}^{2} x_2\left(\frac{3+4x_2}{18}\right) = \cdots$$

$$= \frac{29}{18}$$

$$Cov(x,y) = E(X_1X_2) - E(X_1)E(X_2)$$
$$= \mu_{x_1x_2} - \mu_{x_1}\mu_{x_2}$$

$$\mu_{x_1x_2} = \sum_{x_1=1}^{2}\sum_{x_2=1}^{2} f(x_1,x_2) \cdot x_1 x_2$$
$$= \frac{1}{18}\sum_{x_1=1}^{2}\sum_{x_2=1}^{2} x_1x_2\,(x_1+2x_2)$$
$$= \frac{1}{18}\Big[(1)(1)(1+2(1)) + (1)(2)(1+2(2)) + (2)(1)(2+2(1))$$
$$+ (2)(2)(2+2(2))\Big] = \frac{9}{2}$$

$$\rho_{xy} = \frac{Cov(x,y)}{\sigma_x\ \sigma_y} = \frac{E(XY)-E(X)E(Y)}{\sqrt{V(X)}\ \sqrt{V(Y)}}$$

$$\sigma_{x_1} = \sqrt{\sigma_{x_1}^2} \quad ; \quad \sigma_{x_1}^2 = E(X_1^2) - [E(X_1)]^2$$
$$\sigma_{x_2} = \sqrt{\sigma_{x_2}^2} \quad ; \quad \sigma_{x_2}^2 = E(X_2^2) - [E(X_2)]^2$$

$$E(X_1^2) = \sum_{x_1=1}^{2} x_1^2 f_1(x_1) = \sum_{x_1=1}^{2} x_1^2\left(\frac{x_1+3}{9}\right) = \frac{24}{9}$$
$$E(X_2^2) = \sum_{x_2=1}^{2} x_2^2 f_2(x_2) = \sum_{x_2=1}^{2} x_2^2\left(\frac{1}{18}(3+4x_2)\right) = \frac{51}{18}$$

$$\mu_{x_1}^2 = [E(X_1)]^2 \ \& \ \mu_{x_2}^2 = [E(X_2)]^2$$

$$\sigma_{x_1}^2 = E(X_1^2) - [E(X_1)]^2$$
$$= \frac{24}{9} - \left(\frac{14}{9}\right)^2 = \sim$$
$$\sigma_{x_2}^2 = \frac{51}{18} - \left(\frac{29}{18}\right)^2 = \sim$$

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Q6: ① $f(x|y) = \dfrac{f(x,y)}{f_2(y)}$, $f(y|x) = \dfrac{f(x,y)}{f(x)}$

② $f(x|y=1) = \dfrac{f(x_i, y=1)}{f_2(y=1)} = \dfrac{f(x_i, y=1)}{2/9}$, $x_i = 2, 3, \dots, 6$.

$$= \begin{cases} \dfrac{2/27}{2/9} = \dfrac{1}{3} & \text{for } x=2 \\ \dfrac{0}{2/9} = 0 & \text{for } x=3 \\ \dfrac{8/81}{2/9} = \dfrac{4}{9} & \text{for } x=4 \\ \dfrac{0}{2/9} = 0 & \text{for } x=5 \\ \dfrac{4/81}{2/9} & \text{for } x=6 \end{cases}$$

$$f(y|x=2) = \frac{f(y|x=2)}{f(x=2)} = \frac{f(y_i \mid x=2)}{1/3}, \; y=0,1,2$$

$$= \begin{cases} \dfrac{1/9}{1/3} & \text{for } y=0 \\ \dfrac{2/27}{1/3} = \dfrac{2}{9} & \text{for } y=1 \\ \dfrac{4/27}{1/3} = \dfrac{4}{9} & \text{for } y=2 \end{cases}$$

Hence $f(y) = f(y|x=2)$; therefore X & Y are independent.

Q7:

$y \backslash x$	1	2	3	$f_2(y)$	
1	1/9	1/3	1/9	5/9	$= f_2(1)$
2	1/9	0	1/18	3/18	$= f_2(2)$
3	0	1/6	1/9	15/54	$= f_2(3)$
$f_1(x)$	2/9 $f_1(1)$	3/6 $f_1(2)$	3/18 $f_1(3)$	1	

$f(1,1) = \frac{1}{9} \neq f_1(1) f_2(1) = (\frac{2}{9})(\frac{5}{9})$

$f(2,1) = \frac{1}{3} \neq f_1(2) f_2(1) = (\frac{3}{9})(\frac{5}{9})$

$f(3,1) = \frac{1}{9} \neq f_1(3) f_2(1) = (\frac{5}{18})(\frac{5}{9})$

$f(1,2) = \frac{1}{9} \neq f_1(1) f_2(2) =$

$\therefore$ X & Y are dependent.

Q8: Since X & Y are independent; then:

$f_1(x) f_2(y) = f(x,y)$

$$f_1(x) \cdot f_2(y) = \begin{cases} 6yx^2 & \text{for } 0 \le x \le 1, \; 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(xy) = \int_x \int_y xy f(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 (6yx^2)\, xy\, dy\, dx = \cdots = \frac{1}{2}.$$

sol. Q10

$$f_1(x) = \sum_{y=1}^{3} f(x,y)$$

$$= \begin{cases} 9/15 & \text{for } x=1 \\ 6/15 & \text{for } x=2 \\ 0 & \text{o.w.} \end{cases}$$

J.p.f.

$f(x,y)$

y \ x	1	2	$f_2(y)$	
1	2/15	1/15	3/15	$f_2(1)$
2	4/15	1/15	5/15	$f_2(2)$
3	3/15	4/15	7/15	$f_2(3)$
$f_1(x)$	9/15	6/15	1	
	$f_1(1)$	$f_1(2)$		

$$f_2(y) = \sum_{x=1}^{2} f(x,y)$$

$$= \begin{cases} 3/15 & \text{for } y=1 \\ 5/15 & \text{for } y=2 \\ 7/15 & \text{for } y=3 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \sum_{x=1}^{2} x\, f_1(x)$$
$$= (1)(9/15) + (2)(\tfrac{6}{15})$$
$$= \frac{21}{15}$$

$$E(Y) = \sum_{y=1}^{3} y\, f_2(y)$$
$$= (1)(\tfrac{3}{15}) + (2)(\tfrac{5}{15}) + (3)(\tfrac{7}{15})$$
$$= \frac{34}{15}$$

$\rho_{xy} = Cov(x,y) / \sigma_x \sigma_y = ---$ اكمال الحل

Q12: ①

$f(1,5) = f_1(1) f_2(5) = (.6)(.2) = .12$

$f(1,10) = f_1(1) f_2(10) = (.6)(.5) = .30$

$f(1,15) = f_1(1) f_2(15) = (.6)(.3) = .18$

$f(2,10) = f_1(2) f_2(10) = (.4)(.5) = .20$

$f(2,5) = f_1(2) f_2(5) = (.4)(.2) = .08$

$f(2,15) = f_1(2) f_2(15) = (.4)(.3) = .12$

Since (X & Y are independent r.v.s)

$\Rightarrow f_1(x) f_2(y) = f(x,y)$ →

$$\therefore f(x,y) = \begin{cases} .12 & \text{for } x=1, y=5 \\ .08 & \text{for } x=2, y=5 \\ .30 & \text{for } x=1, y=10 \\ .20 & \text{for } x=2, y=10 \\ .18 & \text{for } x=1, y=15 \\ .12 & \text{for } x=2, y=15 \\ 0 & \text{o.w.} \end{cases}$$

$$② E(XY) = \sum_X \sum_Y XY f(x,y)$$

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Q13: Sol. $*_1$ ⓐ $\int_0^y f(x|y)\,dx = 1$ (By cond. ② of Pr.f)

$$C_1 \int_0^y \frac{x}{y^2}\,dx = \frac{C_1}{2y^2} x^2 \Big|_0^y = \dots = 1 \Longrightarrow C_1 = 2y^2$$

$$\frac{C_1}{2y^2}(y^2) = 1 \quad C_1 = 2$$

$$\therefore f(x|y) = \begin{cases} \frac{2x}{y^2} & \text{for } 0<x<y<1 \quad \cancel{0<x<1} \\ 0 & \end{cases} \quad 0<y<1$$

$$0<x<y<1 \qquad 0<x<1$$

$*_2$ ⓐ $\int_0^1 f_2(y)\,dy = 1$ (by Cond. (2) of Pr. f.)

$$C_2 \int_0^1 y^4\,dy = 1 \Longrightarrow C_2 = 5$$

$$\therefore f_2(y) = \begin{cases} 5y^4 & \text{for } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

ⓒ $f(x|y) = \frac{f(x,y)}{f_2(y)} \Longrightarrow f(x,y) = f(x|y)\, f_2(y)$

$$= (2x)(5y^4) = \begin{cases} 10xy^4 & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

ⓑ $f_1(x) = \int_y f(x,y)\,dy = 10\int_0^1 x y^4\,dy = \begin{cases} 2x & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

ⓓ $P(-\frac{1}{2} < x < \frac{1}{2} \mid y = \frac{5}{8}) = \int_0^y f(x|y=\frac{5}{8})\,dx$

$$= \int_0^{5/8} 10x\left(\frac{5}{8}\right)^4 dx = \dots$$

ⓔ $P(\frac{1}{4} < x < \frac{1}{2}) = \int_{1/4}^{1/2} f_1(x)\,dx = \int_{1/4}^{1/2} 2x\,dx = \dots = \frac{3}{16}$