وزارة التعليم العالي والبحث والعلمي
جامعة ديالى
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محاضرات مادة
الإحصاء والاحتمالية

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المرحلة الثالثة

Chapter Six

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Chapter (6)
Some Special Distributions

بعض التوزيعات الخاصة

(1) Binomial Distribution: توزيع ذي الحدين

Bernoulli experiment: تجربة برنولي
is an experiment that has only two possible outcomes, event A and event A^c

for example:-
Toss a coin once.
we get H or T
H, T are only two possible outcomes.
let A: to get H
A^c: to get T
suppose that $P(A) = p$, $0 < p < 1$
Then $P(A^c) = 1 - p$
If we repeat the Bernoulli exper. n - times. ($n \in I^+$) and if the r.v. X
$X \equiv$ number of times that A happens.
for Example:-
Toss a coine. 3 - times:-
A: to get H
$X \equiv$ number of H.
$X = 0, 1, 2, 3$.

Know:-

$\because X \equiv$ number of times that event A happens, then the function of x is

$$f(x)=\begin{cases}\binom{n}{x}p^{x}(1-p)^{n-x} & \text{for } x=0,1,2,\dots,n\\ 0 & \text{o.w}\end{cases}$$

Know:- To show that $f(x)$ is a p.m.f.

since (X is d.r.v.)

cond (1):- T.P $f(x) \geq 0$

since $n \in I^{+}$

$x = 0,1,2,3,\dots,n \;;\; x \leq n$

$n-x \geq 0$

then $\binom{n}{x} \geq 1$, $p^{x} > 0$ for $0<p<1$

$(1-p)^{n-x} > 0$

Then $f(x) > 0$ for $x = 0,1,\dots,n$
$ = 0$ o.w

* Notes:

- p احتمال النجاح
- $1-p$ احتمال الفشل
- n عدد مرات تكرار التجربة
- X عدد مرات حدوث النجاح

(25)

(IV) Gamma Distribution توزيع كاما

Def: " Gamma function Γ "

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} \quad ; \quad \alpha > 0$$

by use integration by parts we get

$$\boxed{\Gamma(\alpha) = (\alpha-1)!}$$

$$f(x, \alpha, \beta) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} ; & 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

Note: $f(x; \alpha, \beta)$ above is called p.d.f. for Gamma dist.

i.e

If $x \sim G(\alpha, \beta)$ then

$$f(x, \alpha, \beta) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} ; & 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

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Theorem (8) If $X \sim G(\alpha, \beta)$, then the m.g.f. of X is

$$M_x(t) = (1-\beta t)^{-\alpha}$$

proof:

$$M_x(t) = E(e^{tx})$$

$$M_x(t) = \int_0^\infty e^{tx}\left(\frac{x^{\alpha-1}\, e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\,\beta^{\alpha}}\right) dx$$

$$= \int_0^\infty \frac{x^{\alpha-1}\, e^{-x\left(\frac{1-\beta t}{\beta}\right)}}{\Gamma(\alpha)\,\beta^{\alpha}}\, dx$$

let $y = \frac{x(1-\beta t)}{\beta}$; $x = \frac{\beta}{1-\beta t}\, y$

$$dy = \frac{1-\beta t}{\beta}\, dx$$

$$dx = \frac{\beta}{1-\beta t}\, dy$$

$$\therefore M_x(t) = \int_0^\infty \frac{\left(\frac{(\beta y)}{1-\beta t}\right)^{\alpha-1} e^{-y}}{\Gamma(\alpha)\,\beta^{\alpha}} \left(\frac{\beta}{1-\beta t}\right) dy$$

$$= \underbrace{\int_0^\infty \frac{y^{\alpha-1}\, e^{-y}}{\Gamma(\alpha)}\, dy}_{=\Gamma(\alpha)} \;\frac{\cancel{\beta^{\alpha}}}{\cancel{\beta^{\alpha}}\,(1-\beta t)^{\alpha}}$$

$$\therefore M_x(t) = (1-\beta t)^{-\alpha}$$

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H.W Theorem (9): If $X \sim G(\alpha, \beta)$, then H.W

$\mu_x = E(X) = \alpha\beta$; $\sigma_x^2 = V(x) = \alpha\beta^2$

proof :-

$\because X \sim G(\alpha, \beta)$

$\therefore M_x(t) = (1-\beta t)^{-\alpha}$

$M'_x(t) = +\alpha\beta(1-\beta t)^{-\alpha-1}$

$M'_x(t) = \alpha\beta(1-\beta t)^{-(\alpha+1)}$

$M'_x(0) = \alpha\beta(1-0)^{-(\alpha+1)} = \alpha\beta$

$\therefore \mu_x = E(x) = M'_x(0) = \alpha\beta$

$M''_x = -\alpha\beta(\alpha+1)\cdot(1-\beta t)^{-(\alpha+2)}\cdot(-\beta)$

$= \alpha\beta^2(\alpha+1)(1-\beta t)^{-\alpha+2}$

$E(x^2) = M''_x(0) = \alpha\beta^2(\alpha+1)\cdot 1$

$= \alpha^2\beta^2 + \alpha\beta^2$

$\therefore V(x) = E(X^2) - [E(x)]^2$

$= \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2$

$= \cancel{\alpha^2\beta^2} - \cancel{\alpha^2\beta^2} + \alpha\beta^2$

$\therefore V(x) = \alpha\beta^2$

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EX(1): find the p.d.f. of a c.r.v. X whose M.g.f. is

$M_x(t) = (1-3t)^{-2}$

Sol.

$\therefore X \sim G(2, 3)$

$$\therefore f(x, 2, 3) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\, \beta^{\alpha}}$$

$$= \frac{x e^{-\frac{x}{3}}}{1 \cdot (3)^2} = \frac{x e^{-\frac{x}{3}}}{9}$$

EX) find the M.g.f. of X has a p.d.f.

$$f(x) = \begin{cases} \dfrac{x e^{-\frac{x}{2}}}{4} & ; \ 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

$\therefore X \sim G(2, 2)$

$\alpha - 1 = 1$

$\therefore \alpha = 2$

$\beta = 2$

$\therefore M_x(t) = (1-\beta t)^{-\alpha}$

$= (1-2t)^{-2}$

H.W EX)

$$f(x) = \begin{cases} \dfrac{x^2 e^{-\frac{x}{3}}}{\Gamma(3) \cdot (27)} & ; \ 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

Sol.

$M_x(t) = (1-\beta t)^{-\alpha}$; بما أن $X \sim G(3, 3)$

$M_x(t) = (1-3t)^{-3}$

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Lemma: If $X \sim \chi^2(\frac{r}{2})$, $\alpha = \frac{r}{2}$, $\beta > 0$, Then the r.v. $Y = \frac{2X}{\beta} \sim \chi^2(r)$

Ex: find $P(3.28 < X < 25.2)$

$\alpha = 3$, $\beta = 4$

let $y = \frac{2x}{\beta}$, $r = 2\alpha = 6$

$x = \frac{y\beta}{2} = \frac{y \cdot 4}{2} = 2y$

$P(3.28 < x < 25.2) = P(3.28 < 2y < 25.2)$

$= P(1.64 < y < 12.6)$

$= P(y < 12.6) - P(y < 1.64)$

$= 0.95 - 0.05$

$= 0.9$

(V) Chi - Square Dist.

Chi - Square dist. is aspecial Case of Gamma dist. where $\alpha = \frac{r}{2}$, r is positive integer and $\beta = 2$.
The new dist. is called chi - Square dist. (denoted by $\chi^2(r)$) where (r) is called degree of fredom (d. f.). درجات الحرية

There fore, if $X \sim \chi^2(r)$, then the p.d.f. of X is as followings:-

$$f(x;r) = \begin{cases} \dfrac{x^{\frac{r}{2}-1} e^{-\frac{x}{2}}}{\Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}} & \text{for } 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

Also; the M.g.f. of X is

$$M_X(t) = (1-2t)^{-\frac{r}{2}}$$

$$E(X) = \mu_X = \alpha\beta = (\tfrac{r}{2}) \cdot 2 = r$$
$$V(X) = \sigma^2 = \alpha\beta^2 = (\tfrac{r}{2})\, 4 = 2r$$

Ex: If a r.v. X has a p.d.f.

$$f(x) = \begin{cases} \dfrac{x\, e^{-\frac{x}{2}}}{4} & \text{for } 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

find M.g.f. of X ?

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sol. $f(x) = \begin{cases} \dfrac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\ \beta^{\alpha}} & 0 < x < \infty \\ 0 & o.w \end{cases}$

$\therefore X \sim G(2,2)$

$\therefore \frac{r}{2} = \alpha \Rightarrow 2 = \frac{r}{2} \Rightarrow r = 4$

$\therefore M_x(t) = (1-2t)^{-\frac{r}{2}} = (1-2t)^{-\frac{4}{2}}$

$= (1-2t)^{-2}$

$\therefore X \sim \chi^2(4)$

EX: If a r.v. X has a m.g.f. H.W

$M_X(t) = (1-2t)^{-7}, \ t < \frac{1}{2}$

then find the p.f. of X.

sol.

$\frac{r}{2} = 7 \Rightarrow r = 14, \ \beta = 2$

$\therefore X \sim G(7,2)$

$f(x) = \begin{cases} \\ \end{cases}$

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Table of ch. Square dist.

$$p(X \le x) = \int_0^x \frac{x^{\frac{r}{2}-1} e^{\frac{-x}{2}}}{\Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}}\, dx$$

$p(X \le x)$

dif (r)	(.001)	0.025	0.05	0.95	0.975	.99
1		.001	.004	.38		
2						
3	.115		.352		9.35	11.3
4	.554	.484	.711	.49	11.1	(13.3)
⋮						
10	2.56	3.25	3.44	18.3	20.3	23.2

Note, $p(a \le x \le b) = p(x \le b) - p(x \le a)$

Ex) If $X \sim \chi^2(10)$, then find $p(3.25 \le X \le 20.5)$

sol.

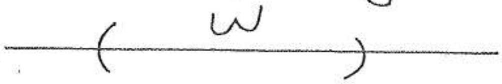
$p(3.25 \le X \le 20.5) = p(X \le 20.5) - p(X \le 3.25)$

$= 0.975 - 0.025$

$= 0.950.$

(3) Poisson Distribution توزيع بواسون

Given an interval of length W, $W>0$

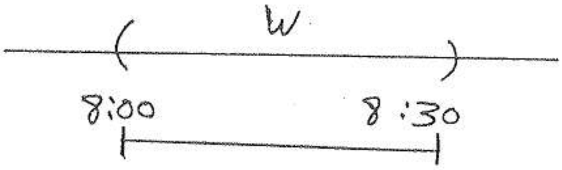


Let A be an event happen in W.

Note: W is interval of time or space or distance ... etc.

Example (1)

A : Car accident that happens in interval $(8:00, 8:30)$



W : 30 mintes = interval of times.

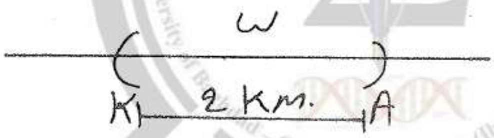
$X \equiv$ number of car accident in $W = 30$ minutes.

Example (2)

A : Car accident between Kadhmia and Adamia.

$X \equiv$ number of car accident in $W = 2$ Km.

W = interval of distance = 2 Km.



Example (3)

A : typing errors in one page.

$X \equiv$ number of typing errors in one page (W)

$W \equiv$ interval of space

Know;

If X = number of events A that happens in interval of length, then the function of X is:

$$f(x) = \begin{cases} \dfrac{e^{-\lambda W} (\lambda W)^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

where $\lambda \equiv$ number of event A, when $\boxed{w=1}$ & $w>0, \lambda>0$.

Let $\underline{m} = \lambda \cdot w$; $m>0$, $w>0$

A عدد الحوادث = λ

$$f(x;m) = \begin{cases} \dfrac{e^{-m}(m)^x}{x!} & \text{for } x=0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

(تكرار A في w=1)

where (m) is a parameter.

Know, to show that $f(x;m)$ is a p.m.f.

Cond. ① T.P $f(x;m) \geq 0 \quad \forall x \in R_X$

$m>0$, $X \geq 0$, $X! > 0$

$(m)^x > 0$, $e^{-m} = \dfrac{1}{e^m} > 0$

$$\therefore f(x;m) = \begin{cases} \dfrac{e^{-m}(m)^x}{x!} > 0 & \text{for } x=0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

Cond. ② T.P $\sum_{x=0}^{\infty} f(x;m) = 1$

$$\sum_{x=0}^{\infty} f(x;m) = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!}$$

$$\because e^x = \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\therefore \sum_{x=0}^{\infty} f(x;m) = e^{-m} \cdot e^{m} = 1$$

Cond. ① & Cond. ② are satisfied; then $f(x;m)$ is p.m.f.

Note: $f(x;m)$ is called Poisson p.m.f.

Theorem (6): If X has a poisson dist. with parameter (m), then the M.g.f. of X is:

$$M_X(t) = e^{m(e^t - 1)} \quad ; \quad -\infty < t < \infty$$

proof: $\because X \sim P(m)$

$$\therefore f(x;m) = \begin{cases} \dfrac{e^{-m} m^x}{x!} & \text{for } x=0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

Example ① The average number of homicides per day in (Los Angeles) Country is (2). Use the Poisson dist. to determine the pr. that on a given days:

القتلى

a. There will be three or less homicides in (Los Angeles) Country.

b. There will be exactly three homicides.

Sol.

$m = 2$; $w = 1$ day ; $X \equiv$ no. of homicides in a day. عدد القتلى في كل يوم

$\therefore X \sim P(2)$

$$f(x;2) = \begin{cases} \dfrac{e^{-2}\, 2^{x}}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

a. $P(X \leq 3) = \sum_{x=0}^{3} f(x) = f(0) + f(1) + f(2) + f(3)$

b. $P(X = 3) = f(3) = \dfrac{e^{-2}\, 2^{3}}{3!} = \dfrac{8\, e^{-2}}{6} = \dfrac{4}{3} e^{-2}$.

Example ② The average number of customers entering a store is (4) per minute. Assuming the number of customers has a poisson dist. Find the pr. that at least (2) customers enter the store in $(\frac{1}{2})$ minute.

معرض الداخلين الزبائن افرض

Sol. $X \equiv$ number of customers enter the store. عدد الزبائن الذين يدخلون المعرض

Given $m = 4$, $w = 1$ in one minute.

$m = \lambda \cdot w$
$4 = \lambda \cdot 1 \Rightarrow \lambda = 4$

if $w = \frac{1}{2}$; find $P(X \geq 2)$?

$m = \lambda \cdot w$
$= 4 \cdot (\frac{1}{2}) = 2 \Rightarrow m = 2$

$X \sim P(2)$

$$\therefore f(x;2) = \begin{cases} \dfrac{e^{-2}\, 2^{x}}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} P(X \geq 2) = 1 - P(X < 2) &= 1 - P[(X=0) \cup (X=1)] \\ &= 1 - [P(X=0) + P(X=1)] \\ &= 1 - [f(0) + f(1)] = 1 - [e^{-2} + 2e^{-2}] \\ &= 1 - 3e^{-2}. \end{aligned}$$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f(x;m)$$
$$= \sum_{x=0}^{\infty} e^{tx} \cdot e^{-m} \frac{m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!}$$

By $e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$M_X(t) = e^{-m} \cdot \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!} = e^{-m} \cdot e^{me^t} = e^{m(e^t-1)} \quad -\infty<t<\infty$$

Theorem (7): If X has a poisson dist. with parameter (m); then $\mu_X = E(X) = m$ & $\sigma_X^2 = Var(X) = m$.

Proof: $\because X \sim P(m)$

$\because M_X(t) = e^{m(e^t-1)} \quad ; -\infty<t<\infty$

$M'_X(t) = e^{m(e^t-1)} \cdot me^t$

$M'_X(0) = E(X) = 1 \cdot m \cdot 1 = m \Rightarrow \therefore E(X) = \mu_X = m$

$M''_X(t) = e^{m(e^t-1)} \cdot (me^t) + (me^t)\, e^{m(e^t-1)} \cdot (me^t)$

$M''_X(0) = E(X^2) = m + m^2$

$\sigma_X^2 = Var(X) = E(X^2) - (E(X))^2$
$= m + \cancel{m^2} - \cancel{m^2} = m \Rightarrow \sigma_X^2 = m$

H.W.

① If $\mu_X = \sigma_X^2 = 2$; Find $P(X \geq 1)$, $M_X(t)$, $P(X=0)$

② Given $M_X(t) = e^{4(e^t-1)}$ for $-\infty<t<\infty$

Show that:

$P(\mu_X - 2\sigma_X < X < \mu_X + 2\sigma_X) = 0.93$

Examples $\rightarrow$ about Poisson distribution

Example ③ The average number of defects on a tape is (4) per (1000) feet.

تسبب العيوب

Assuming that the number of defects has a poisson dist.

a. Find the pr. that a tape of long (2400) ft. has (2) defects.

b. Find the pr. that a tape of long (1200) ft. has no defects.

Sol. X = number of defects on a tape.

Given $m = 4$, $w = 1000$

$$m = \lambda \cdot w$$

$$4 = \lambda \cdot (1000) \Rightarrow \lambda = \frac{1}{250} = .004$$

a. if $w = 2400$; find $P(X=2)$

$$m = \lambda \cdot w = \frac{1}{250} \cdot (2400) = 9.6 \Rightarrow \therefore m = 9.6$$

$\therefore X \sim P(9.6)$

$$f(x; 9.6) = \begin{cases} \dfrac{e^{-9.6}(9.6)^x}{x!} & ; x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X=2) = f(2) = \frac{e^{-9.6}(9.6)^2}{2!}$$

b. if $w = 1200$; find $P(X=0)$

$$m = \lambda \cdot w = \frac{1}{250} \cdot (1200) = 4.8 \Rightarrow \therefore m = 4.8$$

$\therefore X \sim P(4.8)$

$$f(x; 4.8) = \begin{cases} \dfrac{e^{-4.8}(4.8)^x}{x!} & ; x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X=0) = f(0) = \frac{e^{-4.8}(4.8)^0}{0!} = e^{-4.8} \quad (\text{since } 0! = 1)$$

Example ④ The average number of phone calls is (5) per minute, find the pr. that there are (3) calls (15) per seconds.

مكالمات هاتفية — ثانية

Sol. X = number of phone calls in one minute.

عدد المكالمات الهاتفية في الدقيقة الواحدة

Given $m = 5$; $w = 1$ minute.

$$m = \lambda \cdot w$$

$$5 = \lambda \cdot 1 = 5 \Rightarrow \lambda = 5$$

if $w = 15$ seconds ; find $p(X=3)$?

$$w = \frac{15}{60} = \frac{1}{4} \text{ minute}$$

(تحول الثواني الى دقائق)

($W=1$ minute $\rightarrow W = 15$ sec.)

$$m = \lambda \cdot w$$

$$= 5 \cdot \left(\tfrac{1}{4}\right) = \frac{5}{4} = 1.25 \Rightarrow m = 1.25$$

$\therefore X \sim p(1.25)$

$$f(x;1.25) = \begin{cases} \dfrac{e^{-1.25}(1.25)^x}{x!} & \text{for } x = 0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

$$p(X=3) = f(3) = \frac{e^{-1.25}(1.25)^3}{3!}$$

Example (5) The average number of defects in a roll of a certain type of wall paper is 2.5. Use the poisson dist. to determine the pr. that a roll will have (4) or more defects.

sol. X = number of defects in a roll of a type of wall paper.

(العيوب ، رولة ، تيب ، ورق الحائط)

Given $m = 2.5$

$\therefore X \sim p(2.5)$

$$f(x;2.5) = \begin{cases} \dfrac{(2.5)^x e^{-2.5}}{x!} & \text{for } x = 0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

$$p(X \geq 4) = 1 - p(X<4)$$

$$= 1 - \sum_{x=0}^{3} f(x) = 1 - \sum_{x=0}^{3} \frac{(2.5)^x e^{-2.5}}{x!}$$

$$= 1 - [f(0) + f(1) + f(3)]$$

Example (6) The average number of traffic accidents that take place on the Baghdad high way on a week day between 7:00 A.M. and 8:00 A.m. is (.7) accident per hour. Use poisson dist. to determine the pr. that at least (2) traffic accidents would the Baghdad high way on Tuesday morning between 7:00 A.m. & 8:00 A.m.

(الحوادث المرورية ، تحدث ، الطريق السريع ، يوم بالاسبوع)

sol.

تعلم

$X \equiv$ number of traffic accidents that take place on the Baghdad high way on a week day between (7:00 – 8:00) A.m.

Given $m = .7$

$\therefore X \sim P(.7)$

$$f(x; .7) = \begin{cases} \dfrac{e^{-.7}(.7)^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$m = .7$, $w = 1$ (week)
$m = w \cdot \lambda$
$.7 = 1 \cdot \lambda \Rightarrow \lambda = .7$
w = Tuesday
$w = \frac{1}{7}$
$m = \frac{1}{7} \cdot \frac{7}{10} = .1$
$X \sim P(.1)$

$$P(X \leq 2) = \sum_{x=0}^{2} f(x) = \sum_{x=0}^{2} \frac{e^{-.7}(.7)^x}{x!} = f(0) + f(1) + f(2)$$

Example ⑦ A machine (ماكنة) produces (تنتج) a certain (معين) item with (0.05) defectives (عيوب). We select a sample of (100) item. Let the r.v. X be the number of defective. What is the pr. that no defective?

sol.

$X \equiv$ number of defective (عدد العيوب) in w.

$p = 0.05$; $n = 100$

$m = \lambda \cdot w$
$= (100) \cdot (0.05) = 5$

* يمكن حله بطريقة أخرى

$b(n, p)$
$= b(100, 0.05)$

$\therefore X \sim P(5)$

$$f(x; 5) = \begin{cases} \dfrac{e^{-5} 5^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X = 0) = f(0) = \frac{e^{-5} 5^0}{0!} = e^{-5}$$

Example ⑧ A car salesman (بائع سيارات) sells (بيع), on the average (2.5) cars per day. Use poisson dist. to determine the pr. that on a given day the salesman would sell:

a. at least (4) cars, b. exactly (4) cars.

sol. X = number of cars that salesman would sell.

Given $m = 2.5$

$\therefore X \sim P(2.5)$

$$f(x;2.5) = \begin{cases} \dfrac{e^{-2.5}(2.5)^x}{x!} & \text{for } x = 0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

a. $P(X \geq 4) = 1 - P(X < 4) = 1 - \sum_{x=0}^{3} f(x)$

$$= 1 - \sum_{x=0}^{3} \frac{e^{-2.5}(2.5)^x}{x!}$$

$$= 1 - (f(0) + f(1) + f(3))$$

b. $P(X=4) = f(4) = \dfrac{e^{-2.5}(2.5)^4}{4!}$

Example (9) The average number of major fires per month in a certain city is (1.5). Use the Poisson dist. to determine the pr. that there will be exactly one major fire in a period of two months.

حرائق كبيرة

Sol.

عدد الحرائق الكبيرة كل شهر

$X \equiv$ number of major fires per month.

Given $m = 1.5$, $w = 1$ month

$m = \lambda \cdot w$

$1.5 = \lambda \cdot 1 \Rightarrow \lambda = 1.5$

if $w = 2$; find $P(x=1)$?

التحويل من شهر الى شهرين

$w = 1 \longrightarrow w = 2$

$m = \lambda \cdot w$

$= (1.5)(2) = 3 \Rightarrow m = 3$

$\therefore X \sim P(3)$

$$f(x;3) = \begin{cases} \dfrac{e^{-3} 3^x}{x!} & \text{for } x = 0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

$P(x=1) = f(1) = 3e^{-3}$.

عباس نجم سلمان
استاذ مساعد

"Exponential Dist" التوزيع الاسي

يعتبر التوزيع الاسي حالة خاصة من توزيع كاما (γ) عندما $\alpha = 1$ و $\beta = \frac{1}{\lambda}$ حيث $(\lambda > 0)$

لذلك فان المتغير العشوائي X الذي يتبع التوزيع الاسي له دالة الكثافة الاحتمالية التالية

$$f(x|\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } 0 < x < \infty \\ 0 & \text{oth} \end{cases}$$

ويرمز له بالرمز $X \sim Exp.(\lambda)$

وايضا فان $E(x) = \mu = \frac{1}{\lambda}$

$$V(x) = \sigma^2 = \frac{1}{\lambda^2}$$

مثال: اذا كانت

$$f(x) = \begin{cases} 4 e^{-4x} & \text{for } 0 < x < \infty \\ 0 & \text{oth} \end{cases}$$

جد نوع التوزيع ثم جد بعد ذلك $E(x)$ ، $V(x)$

الحل: من خلال مقارنة هذه الدالة مع دالة التوزيع الاسي نلاحظ انها تتبع التوزيع الاسي مع المعلمة $(\lambda = 4)$

$$\therefore E(x) = \frac{1}{4}$$

$$V(x) = \frac{1}{4^2} = \frac{1}{16}.$$

t_distribution

let w be a c.r.v. s.t. $w \sim N(0,1)$

i.e $f(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$ for $-\infty < w < \infty$

let v be a c.r.v. s.t. $v \sim \chi^2(r)$

i.e $f(v) = \frac{1}{\Gamma(\frac{r}{2})\, 2^{r/2}} v^{\frac{r}{2}-1} e^{-\frac{v}{2}}$ $0 < v < \infty$

$\because$ w and v are s. Independent.

i.e $f(w,v) = f(w) \cdot f(v)$

$\therefore$ $f(w,v) = \frac{1}{\sqrt{2\pi}\, \Gamma(\frac{r}{2})\, 2^{r/2}} v^{\frac{r}{2}-1} e^{-\frac{(w^2+v)}{2}}$ $-\infty < w < \infty$, $0 < v < \infty$

Define $T = \frac{w}{\sqrt{\frac{v}{r}}}$

الاطلاع **To find $g(t)$** (use transfor——)

$t = \frac{w}{\sqrt{\frac{v}{r}}} = u_1(w,v)$

let $u = v = u_2(w,v)$

$w = t\sqrt{\frac{u}{r}} = u_1^{-1}(t,u)$

$v = u = u_2^{-1}(t,u)$

$$J = \begin{vmatrix} \frac{\partial w}{\partial t} & \frac{\partial w}{\partial u} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial u} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix} = \sqrt{u/r}$$

$g(t,u) = f[u_1^{-1}(t,u),\, u_2^{-1}(t,u)] \cdot |J|$

$$= \frac{1}{\sqrt{2\pi}\, \Gamma(\frac{r}{2})\, 2^{r/2}} u^{\frac{r}{2}-1} e^{-\left[\frac{\frac{t^2u}{r}+u}{2}\right]} \left(\sqrt{\frac{u}{r}}\right)$$

(2)

$$\therefore g(t,u) = \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})\,2^{r/2}}\; u^{\frac{r}{2}-1}\; e^{-\left[\frac{t^2\frac{u}{r}+u}{2}\right]}\left(\sqrt{\frac{u}{r}}\right)$$

for $-\infty < t < \infty$, $0 \le u < \infty$

$$g_1(t) = \frac{1}{\sqrt{2\pi r}\,\Gamma(\frac{r}{2})\,2^{r/2}} \int_0^\infty u^{\frac{r}{2}-\frac{1}{2}}\; e^{-u\left[\frac{\frac{t^2}{r}+1}{2}\right]}\, du$$

let $z = \frac{u[\frac{t^2}{r}+1]}{2}$, $2z = u(\frac{t^2}{r}+1)$

$u = \frac{2z}{\frac{t^2}{r}+1}$, $du = \frac{2}{\frac{t^2}{r}+1}\, dz$

$$\therefore g_1(t) = \frac{1}{\sqrt{2\pi r}\,\Gamma(\frac{r}{2})\,2^{r/2}} \int_0^\infty \left[\frac{2z}{(\frac{t^2}{r}+1)}\right]^{\frac{r-1}{2}} e^{-z}\, \frac{2}{(\frac{t^2}{r}+1)}\, dz$$

$2^{-1/2}\,2^{-1/2}\,2^{1}$

$$= \frac{2^{\cancel{\frac{r}{2}}-\frac{1}{2}}\cdot 2}{\sqrt{2\pi r}\,\Gamma(\frac{r}{2})\,(\frac{t^2}{r}+1)^{\frac{r+1}{2}}\cdot \cancel{2^{r/2}}} \int_0^\infty z^{\frac{r-1}{2}}\, e^{-z}\, dz$$

$$= \frac{1}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})\,(\frac{t^2}{r}+1)^{\frac{r+1}{2}}}\,\Gamma\left(\frac{r+1}{2}\right)$$

$\alpha - 1 = \frac{r-1}{2}$

$\alpha = \frac{r-1}{2}+1$

$\alpha = \frac{r+1}{2}$

$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$

احفظ

$\therefore$ if $X \sim t(r)$ [t-distribut with degree of freedom r]

then
$$f(t) = \begin{cases} \dfrac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})\,(\frac{t^2}{r}+1)^{\frac{r+1}{2}}} & \text{for } -\infty < t < \infty \\ 0 & \text{o.w} \end{cases}$$

$$P(X \le x) = \int_{-\infty}^{x} f(t)\, dt$$

Note (r) is called degree of freedom of t-dist.

(8) Sample of Table of t-dist. with (r) d.f.

$$P(T \le t) = \int_{-\infty}^{t} \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})} \left[1 + \frac{x^2}{r}\right]^{-\frac{(r+1)}{2}} dx$$

d.f (r)	$P(T \le t)$ 0.90	0.95	0.975	0.99	0.995
1	3.078	6.314	12.706		
⋮					
3	1.638	2.353	3.182	4.541	5.841
⋮					
7	1.415	1.895	2.365	2.998	3.499
14	1.345	1.761	2.145	2.629	2.977

Example (1) using the table of T-dist. Find

(i) $P(T \le 2.353)$, $r = 3$ ANS 0.95

(ii) $P(1.415 \le T \le 2.998)$, $r = 7$

$$P(1.415 \le T \le 2.998) = P(T \le 2.998) - P(1.415)$$
$$= 0.99 - 0.9$$
$$= 0.09 \quad \text{(ANS)}$$

(iii) $P(T > 2.145)$, $r = 14$

$$P(T > 2.145) = 1 - P(T \le 2.145) = 1 - 0.975$$
$$= 0.025 \quad \text{(ANS)}$$

(iii) $P(T \le -5.84)$, $r = 3$

$$P(T \le -5.84) = 1 - P(T \le 5.84) = 1 - 0.995 = 0.005 \quad \text{(ANS)}$$

(11)

Example ② $T \sim t(r=10)$

Find $P(|T| > 2.228)$

Sol

$$P(|T| > 2.228) = 1 - P(|T| \leq 2.228)$$
$$= 1 - P(-2.228 \leq T \leq 2.228)$$
$$= 1 - \left[P(T \leq 2.228) - P(T \leq -2.228)\right]$$
$$= 1 - \left[P(T \leq 2.228) - (1 - P(T \leq 2.228))\right]$$
$$= 2 - 2P(T \leq 2.228)$$
$$= 2 - 2[.975]$$

من جدول T

$r=10$

$$= 2 - [1.950]$$
$$= 0.05$$

Example ③ If $T \sim T(14)$, then find the value of b s.t. $P[-b < T < b] = 0.90$

Sol.

$$P[-b < T < b] = P[T < b] - P(T < -b)$$
$$= P(T < b) - [1 - P(T < b)]$$
$$0.90 = 2P(T < b) - 1$$
$$2P(T < b) = 1.90$$
$$P(T < b) = \frac{1.90}{2} = 0.95$$
$$\therefore \; b = 1.761$$

من جدول

$T, r=14$

(5) F- distribution

let u be c.r.v. s.t $u \sim \chi^2(r_1)$.
let v be c.r.v. s.t $v \sim \chi^2(r_2)$.

Suppose u and v are S. Indep.

$$\therefore \quad f(u,v) = \frac{1}{\Gamma(\frac{r_1}{2})\,\Gamma(\frac{r_2}{2})\,2^{\frac{r_1+r_2}{2}}}\, u^{\frac{r_1}{2}-1}\, v^{\frac{r_2}{2}-1}\, e^{-\frac{(u+v)}{2}}$$

$0 < u < \infty$
$0 < v < \infty$

Define a r.v F s.t

$$F = \frac{u/r_1}{v/r_2}$$

$g(F)$ نجد وبنفس طريقة التحويلات الخطية

بعد ان نفرض ما يأتي

$$f = \frac{u/r_1}{v/r_2} = u_1(u,v)$$

$$Z = v = u_2(u,v)$$

$$u = \frac{r_1}{r_2} f\, Z = u_1^{-1}(f,z)$$

$$v = Z = u_2^{-1}(f,z)$$

$$g(f,z) = F\left[u_1^{-1}, u_2^{-1}\right] \cdot |J|$$ ← Jacobian deter.

وبنفس الاسلوب السابقة نستنتج

$$g_1(f) = \begin{cases} \dfrac{\left(\frac{r_1}{r_2}\right)^{r_1/2} f^{\frac{r_1}{2}-1}\, \Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma(\frac{r_1}{2})\,\Gamma(\frac{r_2}{2})\left(1+\frac{r_1}{r_2} f\right)^{\frac{r_1+r_2}{2}}} & 0 < f < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$f \sim F(r_1, r_2)$$

[illegible]

(6) Sample of F-dist. table

$$P(F \le f) = \int_0^f f(w)\,dw$$

$P(F\le f)$	r_2	r_1 = 1	2	3	4	5	10	12	15
0.95	1	161		216		230	242		246
0.975		648		864		922	964		985
0.99		4052		5625		5764	6056		6157
	2								
0.95	3	10.1	9.55		9.12			8.74	
0.975		17.4	16.0		15.1			14.3	
0.99		34.1	30.8		28.7			27.1	

Example Find (1) $P(F \le 9.12)$, $r_1 = 4$, $r_2 = 3$

(2) $P(F \le -14.3)$, $r_1 = 12$, $r_2 = 3$

(3) $P(242 \le F \le 6056)$, $r_1 = 10$, $r_2 = 1$

Sol.

(1) $P(F \le 9.12) = 0.95$

(2) $P(F \le -14.3) = 1 - P(F \le 14.3)$, $r_1 = 12$, $r_2 = 3$

$= 1 - 0.975$

$= 0.25$

(3) $= P(242 \le F \le 6056) = P(F \le 6056) - P(F \le 242)$

$= 0.99 - 0.95$

$= 0.04$

(7) Exersies

(1) Show that t-dist. with $r=1$ (d.f) and the Cauchy dist. are the same

Sol.

$$g(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\ \Gamma\left(\frac{1}{2}\right)\left(\frac{t^2}{r}+1\right)^{\frac{r+1}{2}}} \qquad -\infty < t < \infty$$

$\boxed{\text{when } r=1}$

$$g(t) = \frac{\Gamma(1)}{\sqrt{\pi}\ \sqrt{\pi}\ (t^2+1)} = \frac{1}{\pi(t^2+1)} \qquad -\infty < t < \infty$$

Cauchy dist.

(2) let $T = \frac{W}{\sqrt{V/r}}$, where $W \sim N(0,1)$, $V \sim \chi^2(r)$

Show that $T^2 \sim F(1, r)$

Sol. $T = \frac{W}{\sqrt{\frac{V}{r}}}$

$$T^2 = \frac{W^2/1}{V/r}, \quad \text{where } W^2 \sim \chi^2(1),\ V \sim \chi^2(r)$$

$\therefore T^2 \sim F(1, r)$

(8) The Distribution of $\bar{X}$ and $\frac{nS^2}{\sigma^2}$

let $x_1, x_2, -, x_n$ be Random Sample of size $n \geq 2$ from a dist. have a p.d.f $f(x)$ with mean (μ) and var (σ^2), then

$E(\bar{X}) = \mu$, when $\bar{X}$ is the Sample mean

s.t $\bar{X} = \frac{\sum x_i}{n}$

$var(\bar{X}) = \frac{\sigma^2}{n}$

Thm. 1 let $x_1, x_2, -, x_n$ be a R.S. of size $n \geq 2$ from $N(\mu, \sigma^2)$, then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

proof

$$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E[e^{t(\frac{\sum x_i}{n})}]$$

$$= E[e^{\frac{t}{n}[x_1 + \cdots + x_n]}]$$

$$= E[e^{\frac{t}{n}x_1} \cdot e^{\frac{t x_2}{n}} \cdots e^{\frac{t}{n}x_n}]$$

$$= e^{n(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2})}$$

$$= e^{\mu t + \frac{\sigma^2}{n}\frac{t^2}{2}}$$

$\therefore \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

توضيح

$x_i \sim N(\mu, \sigma^2)$

$M_{x_i}(t) = E[e^{t x_i}] = e^{\mu t + \sigma^2 \frac{t^2}{2}}$

$M_{x_i}(\frac{t}{n}) = E[e^{\frac{t}{n} x_i}] = e^{\frac{\mu t}{n} + \frac{\sigma^2}{n^2}\frac{t^2}{2}}$

n^2

Note If $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

then $Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$

(a)

ex- let $x_1, \ldots, x_{25}$ be a r.s from $N(\overset{\mu}{71}, \overset{\sigma^2}{100})$, then Find $P(70 < \bar{x} < 72)$

Sol.

$$P(70 < \bar{x} < 72) = P\left(\frac{70-71}{10/\sqrt{25}} < Z < \frac{72-71}{10/\sqrt{25}}\right)$$

$$= P(-0.5 < Z < 0.5)$$

$$= P(Z \le 0.5) - P(Z \le -0.5)$$

$$= 2N(0.5) - 1$$

$$= 2(0.6911) - 1$$

$$= 0.382$$

Thm-2 let $x_1, x_2, \ldots, x_n$ be a r.s of Size $n \geq 2$ from $N(\mu, \sigma^2)$, then

$$\frac{nS^2}{\sigma^2} \sim \chi^2_{(n-1)}, \text{ where } S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

proof

$$\sum_{i=1}^{n}(x_i - \mu)^2 = \sum\left[(x_i - \bar{x}) + (\bar{x} - \mu)\right]^2$$

$$= \sum_{i=1}^{n}\left[(x_i - \bar{x})^2 + 2(\bar{x}-\mu)(x_i - \bar{x}) + (\bar{x}-\mu)^2\right]$$

$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + 2(\bar{x}-\mu)\sum_{i=1}^{n}(x_i - \bar{x}) + n(\bar{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x}-\mu)^2 \quad \div(n)$$

$$\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n} = S^2 + (\bar{x}-\mu)^2 \quad \times\left(\frac{n}{\sigma^2}\right)$$

$$\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma^2} = \frac{nS^2}{\sigma^2} + \frac{(\bar{x}-\mu)^2}{\sigma^2/n}$$

Since $\bar{x}$ and S^2 are S.Indep.

and $\sum\frac{(x_i-\mu)^2}{\sigma^2} \sim \chi^2_{(n)}$; $\frac{(\bar{x}-\mu)^2}{\sigma^2/n} \sim \chi^2_{(1)}$

then

(10)

$$E\left[e^{\frac{t\sum(x_i-\mu)^2}{\sigma^2}}\right] = E\left[e^{t\left[\frac{(\bar{x}-\mu)^2}{\sigma^2/n}+\frac{ns^2}{\sigma^2}\right]}\right]$$

$$\underbrace{E\left[e^{\frac{t\sum(x_i-\mu)^2}{\sigma^2}}\right]}_{m.g.f.f\ \frac{\sum(x_i-\mu)^2}{\sigma^2}} = \underbrace{E\left[e^{\frac{t(\bar{x}-\mu)^2}{\sigma^2/n}}\right]}_{m.g.f.f\ \frac{(\bar{x}-\mu)^2}{\sigma^2/n}} \cdot \underbrace{E\left[e^{t\left(\frac{ns^2}{\sigma^2}\right)}\right]}_{m.g.f.f\left(\frac{ns^2}{\sigma^2}\right)}$$

$$\frac{\sum(x_i-\mu)^2}{\sigma^2}: \quad // \quad (1-2t)^{-\frac{n}{2}} \sim \chi^2(n)$$

$$\frac{(\bar{x}-\mu)^2}{\sigma^2/n}: \quad // \quad (1-2t)^{-\frac{1}{2}} \sim \chi^2(1)$$

$$\therefore m.g.f.\left(\frac{ns^2}{\sigma^2}\right) = \frac{(1-2t)^{-\frac{n}{2}}}{(1-2t)^{-\frac{1}{2}}} = (1-2t)^{-\frac{(n-1)}{2}}$$

or $M_{\frac{ns^2}{\sigma^2}}(t)$

$$\therefore \frac{ns^2}{\sigma^2} \sim \chi^2(n-1)$$

<u>ex.</u> Let s^2 be the variance of the r.s. of size (6) from $N(\mu, 12)$, Find $P(2.30 < s^2 < 22.2)$

<u>Sol.</u>

$$\frac{ns^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\frac{6s^2}{12} \sim \chi^2(5)$$

<u>i.e</u> $\frac{s^2}{2} \sim \chi^2(5)$

$$P\left(\frac{(2.30)}{2} < \frac{s^2}{2} < \frac{22.2}{2}\right) = P(1.15 < W < 11.1)$$

$\boxed{W \sim \chi^2(5)}$

$$= P(W < 11.1) - P(W < 1.15)$$
$$= 0.95 - 0.05$$
$$= 0.90$$

من جدول
chi-squar
with r = 5

Thm, $\because \frac{ns^2}{\sigma^2} \sim \chi^2_{(n-1)}$

then

① $E\left(\frac{ns^2}{\sigma^2}\right) = (n-1)$

$$\therefore \frac{n}{\sigma^2} E(s^2) = (n-1)$$

$$\therefore E(s^2) = \frac{(n-1)}{n} \sigma^2$$

② $Var\left(\frac{ns^2}{\sigma^2}\right) = 2(n-1)$

$$\therefore \frac{n^2}{\sigma^4} Var(s^2) = 2(n-1)$$

$$Var(s^2) = \frac{2(n-1)}{n^2} \sigma^4$$

احصاء – [illegible] /٣

Random Sampling
(Ch: 7)

Remark:

(1) If $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$ then $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

(2) If $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$ then $\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$

(3) If $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$ then $\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(n)$

(4) $W = \frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$

(5) If $W \sim N(0,1)$, $V \sim \chi^2(r)$

$$T = \frac{W}{\sqrt{\frac{V}{r}}} \sim t(r)$$

(6) $U \sim \chi^2(r_1)$, $V \sim \chi^2(r_2)$

$$F = \frac{U/r_1}{V/r_2} \sim f(r_1, r_2)$$

(7) $T^2 = \frac{W^2}{V/r} \sim F(1, r)$

(8) $t(r=1) \sim$ Cauchy dist.

Example (1) Let $\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n}$ be the mean of a R.S. of size $n = 16$ from $N(\mu, \sigma^2)$ and $\sigma^2 = 25$, $\mu = 0$; find:

Sol. $P(\overline{X} < 0)$; $\overline{X} \sim N(0, \frac{25}{16})$.

$$P(\overline{X} < 0) = P\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < \frac{0-\mu}{5/\sqrt{16}}\right) \quad ; \mu = 0$$

$$= P(Z \le 0) = N(0) = 0.5$$

Example (2): Let $\overline{X}$ be the mean of a R.S. of size 5 from a normal distribution with $\mu=0$ and $\sigma^2=125$. Determine C so that
$P(\overline{X}<C)=0.90$.

Sol.

$$\overline{X}\sim N\left(0,\frac{125}{5}\right) \quad ; n=5,\ \mu=0,\ \sigma^2=125$$

$$Z=\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$$

$$Z=\frac{\overline{X}-0}{\sqrt{125}/\sqrt{5}}=\frac{\overline{X}-0}{5\sqrt{5}/\sqrt{5}}$$

$$P(\overline{X}<C)=0.90$$

$$P\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}<\frac{C-\mu}{\sigma/\sqrt{n}}\right)=0.90$$

$$P\left(\frac{\overline{X}-0}{5\sqrt{5}/\sqrt{5}}<\frac{C-0}{5\sqrt{5}/\sqrt{5}}\right)=0.90$$

$$P\left(Z<\frac{C}{5}\right)=0.90$$

$$N\left(\frac{C}{5}\right)=0.90$$

$$\frac{C}{5}=1.282 \Rightarrow C=(1.282)(5)=6.41$$

Ch.7

Example (3):

If $\overline{X}$ is the mean of a R.S of size n from $N(\mu,100)$ find n s.t. $P(\mu-5<\overline{X}<\mu+5)=0.954$.

Sol. $P(\mu-5<\overline{X}<\mu+5)=0.954$

$$P\left(\frac{-5}{10/\sqrt{n}}<\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}<\frac{5}{10/\sqrt{n}}\right)=0.954 \quad ; Z\sim N(0,1)$$

$$N\left(\frac{5}{10/\sqrt{n}}\right)-\left[1-N\left(\frac{5}{10/\sqrt{n}}\right)\right]=0.954$$

$$2N\left(\frac{5}{10/\sqrt{n}}\right)-1=0.954$$

$$N\left(\frac{5}{10/\sqrt{n}}\right)=\frac{0.954+1}{2}=\frac{1.954}{2}=0.977$$

$\therefore n = 16$; since :

$$N\left(\frac{5}{10/\sqrt{n}}\right) = 0.977 \Rightarrow \frac{5}{10/\sqrt{n}} = 2.00 \quad \left(\text{by Table}\right)$$

$$\Rightarrow \frac{\sqrt{n}}{2} = 2 \Rightarrow \sqrt{n} = 4 \Rightarrow \boxed{n = 16}$$

Examp(4): Let S^2 be the variance of a R.S of size $n=6$ from the normal $N(\mu, 12)$. Find $P(2.30 < S^2 < 22.2)$.

Sol.

$$P(2.30 < S^2 < 22.2) = P\left(\frac{2.30n}{\sigma^2} < \frac{nS^2}{\sigma^2} < \frac{22.2n}{\sigma^2}\right)$$

$n = 6$
$\sigma^2 = 12$

$$= P\left(\frac{2.30(6)}{12} < W < \frac{22.2(6)}{12}\right)$$

$$= P(1.15 < W < 11.1)$$

$$\overset{\text{by table}}{=} P(W < 11.1) - P(W < 1.15)$$

$$= 0.950 - 0.050$$

$$= 0.9$$

$\boxed{\begin{array}{l} W \sim \chi^2(n-1) \\ W \sim \chi^2(5) \end{array}}$

Example(5): Let $\bar{X}$ and S^2 be the mean and the variance of a R.S of size (25) from $N(3, 100)$; find :

① $P(0 < \bar{X} < 6)$ ② $P(55.2 < S^2 < 145.6)$

$\mu = 3$
$\sigma^2 = 100$
$n = 25 \therefore \sqrt{n} = 5$
$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

Sol.

① $$P(0 < \bar{X} < 6) = P\left(\frac{0-3}{10/5} < \frac{\bar{X}-3}{10/5} < \frac{6-3}{10/5}\right)$$

$$= P(-0.6 < Z < 0.6)$$

$$= P(Z < 0.6) - P(Z < -0.6)$$

$$= P(Z < 0.6) - [1 - P(Z < 0.6)]$$

$$= 2P(Z < 0.6) - 1$$

$$= 2N(0.6) - 1$$

$$= 2(0.726) - 1 = 0.452$$

by table of $N(0,1)$ ← 0.726

(2) $P(55.2 < S^2 < 145.6)$

$= P\left(\frac{55.2\,n}{\sigma^2} < \frac{nS^2}{\sigma^2} < \frac{145.6\,n}{\sigma^2}\right)$

$= P\left(\frac{55.2(25)}{100} < W < \frac{145.6(25)}{100}\right)$

$= P(13.8 < W < 36.4)$

$= P(W < 36.4) - P(W < 13.8)$

$= 0.95 - 0.05$

$= 0.9$

Table of Chi-square

$\boxed{\text{by } W \sim \chi^2(n-1) \Rightarrow W \sim \chi^2(24)}$

(by Table)

Note

Tables of Distributions

$Z = \frac{X-\mu}{\sigma} \sim N(0,1) \Rightarrow$ table of S.N. $N(0,1)$

$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \Rightarrow$ table of S.N. $N(0,1)$

$W = \frac{nS^2}{\sigma^2} \sim \chi^2(n-1) \Rightarrow$ table of Chi-square with d.f. $r = n-1$

$T = \frac{W}{\sqrt{\frac{V}{r}}} \sim t(r) \Rightarrow$ table of T-dist. with d.f. r

$F = \frac{U/r_1}{V/r_2} \sim f(r_1, r_2) \Rightarrow$ table of F-dist. with d.f. r_1, r_2

$Z^2 = \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2_{(1)}$

$\Rightarrow$ table of Chi-square with d.f. $r = 1$

$\sum_{i=1}^{n}\left(\frac{\bar{X}_i-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(n) \Rightarrow$ table of Chi-square with d.f. $r = n$

TABLE IV

*The t Distribution**

$$\Pr(T \le t) = \int_{-\infty}^{t} \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\,\Gamma(r/2)(1 + w^2/r)^{(r+1)/2}}\,dw$$

$$[\Pr(T \le -t) = 1 - \Pr(T \le t)]$$

	$\Pr(T \le t)$				
r	0.90	0.95	0.975	0.99	0.995
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750

* This table is abridged from Table III of Fisher and Yates; *Statistical Tables for Biological, Agricultural, and Medical Research*, published by Oliver and Boyd, Ltd., Edinburgh, by permission of the authors and publishers.

TABLE V

*The F Distribution**

$$\Pr(F \leq f) = \int_0^f \frac{\Gamma[(r_1 + r_2)/2](r_1/r_2)^{r_1/2} w^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)(1 + r_1 w/r_2)^{(r_1+r_2)/2}}\, dw$$

$Pr(F \leq f)$	r_2	r_1 = 1	2	3	4	5	6	7	8	9	10	12	15
0.95	1	161	200	216	225	230	234	237	239	241	242	244	246
0.975		648	800	864	900	922	937	948	957	963	969	977	985
0.99		4052	4999	5403	5625	5764	5859	5928	5982	6023	6056	6106	6157
0.95	2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4
0.975		38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4
0.99		98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4
0.95	3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70
0.975		17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.3	14.3
0.99		34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9
0.95	4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86
0.975		12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66
0.99		21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2
0.95	5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62
0.975		10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43
0.99		16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72
0.95	6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94
0.975		8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27
0.99		13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56
0.95	7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51
0.975		8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57
0.99		12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31
0.95	8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22
0.975		7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10
0.99		11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52
0.95	9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01
0.975		7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77
0.99		10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96
0.95	10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85
0.975		6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52
0.99		10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56
0.95	12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62
0.975		6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18
0.99		9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01
0.95	15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40
0.975		6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86
0.99		8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52

* This table is abridged and adapted from "Tables of Percentage Points of the Inverted Beta Distribution," *Biometrika*, 33 (1943). It is published here with the kind permission of Professor E. S. Pearson on behalf of the authors, Maxine Merrington and Catherine M. Thompson, and of the *Biometrika* Trustees.

TABLE III

The Normal Distribution

$$\Pr(X \le x) = N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$[N(-x) = 1 - N(x)]$$

x	$N(x)$	x	$N(x)$	x	$N(x)$
0.00	0.500	1.10	0.864	2.05	0.980
0.05	0.520	1.15	0.875	2.10	0.982
0.10	0.540	1.20	0.885	2.15	0.984
0.15	0.560	1.25	0.894	2.20	0.986
0.20	0.579	1.282	0.900	2.25	0.988
0.25	0.599	1.30	0.903	2.30	0.989
0.30	0.618	1.35	0.911	2.326	0.990
0.35	0.637	1.40	0.919	2.35	0.991
0.40	0.655	1.45	0.926	2.40	0.992
0.45	0.674	1.50	0.933	2.45	0.993
0.50	0.691	1.55	0.939	2.50	0.994
0.55	0.709	1.60	0.945	2.55	0.995
0.60	0.726	1.645	0.950	2.576	0.995
0.65	0.742	1.65	0.951	2.60	0.995
0.70	0.758	1.70	0.955	2.65	0.996
0.75	0.773	1.75	0.960	2.70	0.997
0.80	0.788	1.80	0.964	2.75	0.997
0.85	0.802	1.85	0.968	2.80	0.997
0.90	0.816	1.90	0.971	2.85	0.998
0.95	0.829	1.95	0.974	2.90	0.998
1.00	0.841	1.960	0.975	2.95	0.998
1.05	0.853	2.00	0.977	3.00	0.999

TABLE II

*The Chi-Square Distribution**

$$\Pr(X \le x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2}\, dw$$

	$\Pr(X \le x)$					
r	0.01	0.025	0.050	0.95	0.975	0.99
1	0.000	0.001	0.004	3.84	5.02	6.63
2	0.020	0.051	0.103	5.99	7.38	9.21
3	0.115	0.216	0.352	7.81	9.35	11.3
4	0.297	0.484	0.711	9.49	11.1	13.3
5	0.554	0.831	1.15	11.1	12.8	15.1
6	0.872	1.24	1.64	12.6	14.4	16.8
7	1.24	1.69	2.17	14.1	16.0	18.5
8	1.65	2.18	2.73	15.5	17.5	20.1
9	2.09	2.70	3.33	16.9	19.0	21.7
10	2.56	3.25	3.94	18.3	20.5	23.2
11	3.05	3.82	4.57	19.7	21.9	24.7
12	3.57	4.40	5.23	21.0	23.3	26.2
13	4.11	5.01	5.89	22.4	24.7	27.7
14	4.66	5.63	6.57	23.7	26.1	29.1
15	5.23	6.26	7.26	25.0	27.5	30.6
16	5.81	6.91	7.96	26.3	28.8	32.0
17	6.41	7.56	8.67	27.6	30.2	33.4
18	7.01	8.23	9.39	28.9	31.5	34.8
19	7.63	8.91	10.1	30.1	32.9	36.2
20	8.26	9.59	10.9	31.4	34.2	37.6
21	8.90	10.3	11.6	32.7	35.5	38.9
22	9.54	11.0	12.3	33.9	36.8	40.3
23	10.2	11.7	13.1	35.2	38.1	41.6
24	10.9	12.4	13.8	36.4	39.4	43.0
25	11.5	13.1	14.6	37.7	40.6	44.3
26	12.2	13.8	15.4	38.9	41.9	45.6
27	12.9	14.6	16.2	40.1	43.2	47.0
28	13.6	15.3	16.9	41.3	44.5	48.3
29	14.3	16.0	17.7	42.6	45.7	49.6
30	15.0	16.8	18.5	43.8	47.0	50.9

* This table is abridged and adapted from "Tables of Percentage Points of the Incomplete Beta Function and of the Chi-Square Distribution," *Biometrika*, **32** (1941). It is published here with the kind permission of Professor E. S. Pearson on behalf of the author, Catherine M. Thompson, and of the Biometrika Trustees.

④ Beta Distribution (توزيع بيتا)

A c.r.v. X have a Beta dist. it's p.d.f. has the following form:

$$f(x;a,b)=\begin{cases}\frac{1}{B(a,b)}X^{a-1}(1-X)^{b-1} & \text{for } 0<x<1\\ 0 & \text{o.w.}\end{cases}$$

where $a>0$ & $b>0$; denoted by $X\sim B(a,b)$

And $B(a,b)=\int_0^1 X^{a-1}(1-X)^{b-1}dX=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

The M.g.f. of $X\sim B(a,b)$ is:

$$M_X(t)=\sum_{x=0}^{\infty}\frac{t^n}{n!}\cdot\frac{B(a+n,b)}{B(a,b)}$$

$E(X)=\frac{a}{a+b}$, $Var(X)=\frac{ab}{(a+b)^2(a+b+1)}$

In particular:

1. If $a=b=1$, then Beta dist. reduces to uniform dist. on $(0,1)$:

$$f(x)=\begin{cases}1 & \text{for } 0<x<1\\ 0 & \text{o.w.}\end{cases}$$

Since $f(x,a=1,b=1)=\begin{cases}\frac{1}{B(1,1)}X^{1-1}(1-X)^{1-1} & 0<X<1\\ 0 & \text{o.w.}\end{cases}$

$$=\begin{cases}1 & 0<X<1\\ 0 & \text{o.w.}\end{cases}$$

& $B(1,1)=\frac{\Gamma(1)\Gamma(1)}{\Gamma(2)}=1 \Rightarrow \therefore X\sim B(1,1)$
$\Rightarrow X\sim \text{unif.}(0,1)$

2. If $a=1$ & $b=2$ then Beta dist. reduces to dist. which is known as triangular dist.:

$$f(x)=\begin{cases}2(1-X) & 0<X<1\\ 0 & \text{o.w.}\end{cases}$$

Theorem: If $X \sim P(m)$ & if $x_0 =$ mode of X, then

$$x_0 = \begin{cases} m,\ m-1 & \text{if } m \in I^+ \\ [m] & \text{if } m \notin I^+ \end{cases}$$

Ex. $X \sim P(5)$ then mode $= X = 5, 4$.

$X \sim P(2.6)$ then mode $= X = 2$.

Example: The (نسبة) percentage of betters who make money (نقود) of the race on a given day has Beta distribution with parameters $a=1$ and $b=5$. Find the pr. that fewer (اقل) than ten percent (نسبة مئوية) comment winners (الفائزين) on given day?

Sol. $P(X < .1)$?

$$P(X < .1) = \int_0^{.1} \frac{\Gamma(1+5)}{\Gamma(1)\Gamma(5)} \cdot X^0 (1-X)^4 \, dx$$

$$= \frac{5!}{0!\ 4!} \int_0^{.1} (1-X)^4 \, dx$$

$$= -5 \frac{(1-X)^5}{5} \Big|_0^{.1} = (.9)^5$$

عباس نجم سلمان
استاذ مساعد

(7) Uniform Distribution

when X is d.r.v.

$X \sim unif(K)$

$x = 1, 2, \ldots, K$

$$f(x) = \begin{cases} \frac{1}{K} & \text{for } x = 1, 2, \ldots, K \\ 0 & \text{o.w.} \end{cases}$$

$$E(x) = \sum_{\forall x} x f(x)$$

$$= \sum_{x=1}^{K} x \left(\frac{1}{K}\right)$$

$$= \frac{1}{\cancel{K}} \left[\frac{\cancel{K}(K+1)}{2}\right] \quad \text{when} \left(\sum_{x=1}^{K} x = \frac{K(K+1)}{2}\right)$$

$$= \left(\frac{K+1}{2}\right)$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{x=1}^{K} x^2 f(x)$$

$$= \left(\sum_{x=1}^{K} x^2\right)\left(\frac{1}{K}\right)$$

$$= \frac{1}{\cancel{K}}\left[\frac{\cancel{K}(K+1)(2K+1)}{6}\right] \;; \left[\sum_{x=1}^{K} x^2 = \frac{K(K+1)(2K+1)}{6}\right]$$

$$= \frac{(K+1)(2K+1)}{6},$$

$$Var(X) = E(X^2) - (E(X))^2 \geq 0$$

$$Var(X) = \frac{(K+1)(2K+1)}{6} - \frac{(K+1)}{2}$$

$$= \frac{(K+1)(K-1)}{3}$$

Ch.6 ملحق

تحويل التوزيع الطبيعي القياسي الى $\chi^2(1)$

Th. $X \sim N(0,1)$

$y = X^2 \sim \chi^2(1)$ (Using Moment Teqnique)

Proof. $X \sim N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$M_y(t) = E(e^{ty})$$

$$= E(e^{tx^2})$$

$$= \int_{-\infty}^{\infty} e^{tx^2} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2(1-2t)}{2}}\,dx$$

Let $W = \sqrt{1-2t}\; X$

$dW = \sqrt{1-2t}\; dX$

$$\therefore\ M_y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} \frac{dw}{\sqrt{1-2t}}$$

$$= \frac{1}{\sqrt{1-2t}} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} dw}_{=1}$$

since $(w \sim N(0,1))$

$$M_y(t) = (1-2t)^{-1}$$

$$\therefore\ Y \sim \chi^2(1)$$

Note $X \sim N(\mu, \sigma^2)$

Then $\Rightarrow$ ① $Z \sim N(0,1)$

② $Z^2 \sim \chi^2(1)$

③ $\sum_{i=1}^{n} Z^2 \sim \chi^2(n)$