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المرحلة الثالثة

Chapter Four

مدرس المادة: م.م هند إبراهيم محمد

CHAPTER FOUR

Expectation and Variance:

Def: Let x be a. r. v either d.r.v. or c.r.v."E(x)" is called the "Expectation of x" or "expected Value of x" or "mean of x" And denoted by μ.

Defined as followes;

1- $E(x)=\sum_{\forall x} x\, f(x)$, when x is a d.r.v.

2- $E(x)=\int_{-\infty}^{\infty} x f(x)\, dx$ when x is a c.r.v.

Note: 1- the value of E (x) is constant.

2- E (x) exists if $\sum_{\forall x} |x|\, f(x)<\infty$, when x is d.r.v.

3- E (x) exists if $\int_{-\infty}^{\infty} |x|\, f(x)\,dx<\infty$, when x is c.r.v.

Ex "1": Given p.m.f $f(x)=\begin{cases}\dfrac{\binom{3}{x}}{8} & for\ x=0,2,3,1\\ 0 & O.W\end{cases}$

Find E (X)?

Sol: $E(x)=\sum_{x=0}^{3} x\, f(x)$

X	$f(x)=\dfrac{\binom{3}{x}}{8}$	x.f(x)
0	$\binom{3}{0}/8=\frac{1}{8}$	0
1	$\binom{3}{1}/8=\frac{3}{8}$	$\frac{3}{8}$

2	$\binom{3}{2}/8=\frac{3}{8}$	$\frac{6}{8}$
3	$\binom{3}{3}/8=\frac{1}{8}$	$\frac{3}{8}$
	$\sum_{x=0}^{3} f(x)=1$	$\sum_{x=0}^{3} x f(x)=\frac{12}{8}$

$$\therefore E(x)=\sum_{x=0}^{3} x f(x)=\frac{3}{2}$$

H.W: given ap. M.f $f(x)=\begin{cases}\frac{x}{15} & \text{for } x=1,2,3,4,5\\ 0 & 0.w.\end{cases}$

Find the expected Value of x.

Ex "2": Given a p.d.f $f(x)=\begin{cases}\frac{x}{8} & \text{for } 0<x<4\\ 0 & 0.w.\end{cases}$ Find E(x) ?

Sol: $E(x)=\int_{-\infty}^{\infty} x f(x)dx=\int_{0}^{4} x\frac{x}{8}dx=\int_{0}^{4}\frac{x^2}{8}dx$

$$=\frac{1}{8}\frac{x^3}{3}\Big|_0^4=\frac{1}{24}[64-0]=\frac{64}{24}=\frac{8}{3}$$

H.W: Given ap. d.f $f(x)=\begin{cases}3x^2 & \text{for } 0<x<1\\ 0 & 0.w.\end{cases}$

Find E(x)?

تطابع ←

$+1 < X < e$

Ex: Given ap. d.f $f(x)=\begin{cases}\frac{1}{x} & for\ \cancel{0<x<1} \\ 0 & o.w\end{cases}$ $\quad -1 < X < 2.72 \simeq e^1$

Dose E(x) exist?

Sol: $E(x)=\int_1^e x\frac{1}{x}dx=\int_1^e dx=1-e$

$\therefore E(x)$ is exist.

Ex: Given ap .d.f $f(x)=\begin{cases}\frac{1}{x2} & \cancel{0<x<1}\ \ X>1 \\ 0 & o.w.\end{cases}$

Dose E(x) exist?

Sol: $E(x)=\int_1^\infty x\frac{1}{x^2}dx=\int_1^\infty\frac{1}{x}dx=\ln x\,|_1^\infty=[\infty-0]=+\infty \nless \infty$

$\therefore E(x)$ isnot exist.

Expectation of afunction of x:

Def: Let x be ar.v. and let g(x) be afunction of x then

$E[g(x)]=\sum_{\forall x} g(x)f(x)$, if x is d.r.v.

$=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx$, if x is c.r.v.

Ex "1": Given ap.m.f $f(x)=\begin{cases}\frac{x}{10} & for\ x=1,2,3,4 \\ 0 & o.w.\end{cases}$

find $E(x^2)$?

sol: $g(x)=x^2$

$E[g(X)]=\sum_{\forall x} g(X)f(x)$

$$E[x^2]=\sum_{x=1}^{4}(x)^2\frac{x}{10}=\sum_{x=1}^{4}\frac{x^3}{10}=\left[\frac{1}{10}+\frac{8}{10}+\frac{27}{10}+\frac{64}{10}\right]$$

$$=\frac{100}{10}=10$$

Ex"2": Given ap.d.f $f(x)=\begin{cases}\frac{x}{8} & \text{for } 0<x<4\\ 0 & \text{o.w.}\end{cases}$

find $E(\sqrt{x})$?

sol: $g(x)=\sqrt{x}=x^{\frac{1}{2}}$

$$E[g(x)]=E(x^{\frac{1}{2}})=\int_0^4 x^{\frac{1}{2}}\frac{x}{8}dx=\frac{1}{8}\int_0^4 x^{\frac{3}{2}}dx=\frac{1}{8}.\frac{2}{5}x^{\frac{5}{2}}\Big|_0^4$$

$$\frac{1}{20}\left[(\sqrt{4})^5-0\right]=\frac{1}{20}[32]=\frac{32}{20}$$

Note: if f(x) be a p. d.f of a r.v. x. then E(b)=b, where b is constant.

Proof: case "1": If x is a d.r.v. with p.m.f. f(x)

$E(b)=\sum_{\forall x} b\,f(x)=b\sum f(x)=b$.

Case "2": If x is a c.r.v. with p.d.f. f(x).

$$E(b)=\int_{-\infty}^{\infty} b f(x)\,dx=b\int_{-\infty}^{\infty} f(x)\,dx=b\,.$$

Properties of Expectation :

Theorem "1" If x is ar.v. have ap.f. f(x), and E(x) exists.

Let y=ax+b, a $b\in R$, then E(y)=aE(x)+b.

Sol:case :"1" If x is a c. r.v. With P.d.f f(x)

Y=g(x)=ax+b

$$E(y)=E[g(x)]=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx$$

$$E[ax+b]=\int_{-\infty}^{\infty}(ax+b)\ f(x)dx=\int_{-\infty}^{\infty} ax\ f(x)\ dx+\int_{-\infty}^{\infty} b\ f(x)\ dx$$

$$=a\int_{-\infty}^{\infty} x f(x)\ dx+b\int_{-\infty}^{\infty} f(x)\ dx=a\,E(x)+b$$

case"2": If x is ad.r.v. with p.m.f f(x).

y=g(x)=ax+b

$$E(y)=E[g(x)]=\sum_{\forall x} g(x)\ f(x)$$

$$E[ax+b]=\sum_{\forall x}(ax+b)\ f(x)=\sum\ ax\ f(x)+\sum b\ f(x)$$

$$=a\sum x f(x)+b\sum f(x)=aE(x)+b$$

theorem "2" : let x be a r.v. if u(x) and v(x) are two functions of x. then:

$$E[u(x)\mp v(x)]=E[u(x)]\mp E[v(x)]$$

proof: " case "1": If x is ad.r.v. with p.m.f f(x) let $g(x)=u(x)\mp v(x)$

$$E[u(x)\mp v(x)]=E[g(x)]=\sum_{\forall x} g(x)\ f(x)$$

$$=\sum_{\forall x}[u(x)\mp v(x)]f(x)=\sum_{\forall x} u(x)f(x)\mp\sum_{\forall x} v(x)\ f(x)$$

$$=E[u(x)]\mp E[v(x)]$$

case "2": If x is ac.r.v. with p.d.f f(x).

let $g(x)=u(x)\mp v(x)$

$$E[u(x)\mp v(x)]=E[g(x)]=\int_{-\infty}^{\infty} g(x)\ f(x)dx$$

$$=\int_{-\infty}^{\infty}[u(x)\mp v(x)]f(x)dx=\int_{-\infty}^{\infty} u(x)f(x)dx\mp\int_{-\infty}^{\infty} v(x)\ f(x)\ dx$$

$$=E[u(x)]\mp E[v(x)]$$

– **Ex "1"** Given ap.d.f $f(x)=\begin{cases}\frac{x+2}{18} & \text{for } -2<x<4\\ 0 & 0.w\end{cases}$

find $E\left[2x^3-1\right]$ & $E\left[(x+2)^2\right]$

Sol: $E\left[2x^3-1\right] \Rightarrow 1.g(x)=2x^3-1 \Rightarrow E[g(x)]=\int_{-2}^{4} g(x)\,f(x)\,dx$

Or 2. $E\left[2x^3-1\right]=2E(x^3)-1$

$$E(x^3)=\int_{-2}^{4} x^3\left(\frac{x+2}{18}\right)dx=\frac{1}{18}\int_{-2}^{4}(x^4+2x^3)dx$$

$$=\frac{1}{18}\left[\frac{x^5}{5}+\frac{x^4}{2}\right]_{-2}^{4}=\frac{1}{18}\left[\left(\frac{1024}{5}+\frac{256}{2}\right)-\left(\frac{-32}{5}+\frac{16}{2}\right)\right]$$

$$=\frac{1}{18}\left(\frac{1056}{5}+\frac{240}{2}\right)=-\frac{1}{18}\left(\frac{1056}{5}+120\right)$$

$E\left[(x+2)^2\right] \Rightarrow 1.g(x)=(x+2)^2 \Rightarrow E[g(x)]=\int_{-2}^{4} g(x)\,f(x)\,dx$

Or 2. $E\left[x^2+4x+4\right]=E(x^2)+4E(x)+4$

$$=\int_{-2}^{4} x^2 f(x)\,dx+4\int_{-2}^{4} x\,f(x)\,dx+4$$

$= \ldots$

H.W: 1 Given ap.d.f., $f(x)=\begin{cases}\frac{x}{2} & 0<x<4\\ 0 & o.w.\end{cases}$

Find $E\left[(x+2)^2\right]$?

2. Given ap. m.f, $f(x)=\begin{bmatrix}\frac{x}{10} & \text{for } x=1,2,3,4\\ 0 & \text{o.w.}\end{bmatrix}$ find $E\left[2x^3-1\right], E\left[(x-1)^3\right]$?

$(x-1)^3=(x^3-3x^2+3x-1)$

theorem "3": let x be ar.v.

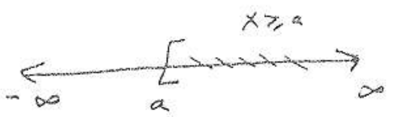
a. If $\exists(a)$ such that $p(x \geq a) = 1$, then $E(x) \geq a$.

b. If $\exists(b)$ such that $p(x \leq b) = 1$, then $E(x) \leq b$.

c. If $p(a \leq x \leq b) = 1$, then $a \leq E(x) \leq b$.

Proof: a. case "1": If x is a c.r.v. with p.d.f f(x)

$$P(x \geq a) = 1 \Rightarrow \int_a^{\infty} f(x)\,dx = 1$$



$$\because f(x) > 0 \quad \text{for } a \leq x < \infty$$
$$= 0 \quad \text{o.w.}$$

$$Rx = \{x : a \leq x < \infty\}$$

$$E(x) = \int_a^{\infty} x\, f(x)\,dx \geq \int_a^{\infty} a\, f(x)\,dx = a \underbrace{\int_a^{\infty} f(x)\,dx}_{=1}$$
$$= a$$

$$\therefore E(x) \geq a$$

Case (2) if x is d.r.v. with p.m.f f(x)

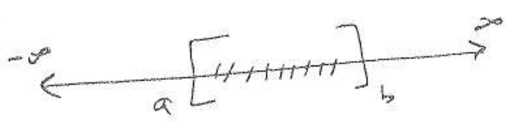
$$E(x) = \sum_{x=a}^{\infty} x\, f(x) \geq \sum a\, f(x) = a \underbrace{\sum f(x)}_{=1}$$
$$= a$$

$\therefore E(x) \geq a$.

b. Similary of (a).

c. Case "1": If x is a d.r.v. have a p.m.f f(x)

$$P(a \leq x \leq b) = 1 \Rightarrow \sum_{x=a}^{b} f(x) = 1$$



$$\therefore f(x) > 0 \quad \text{for } a \leq x \leq b$$
$$= 0 \quad \text{o.w}$$

$$E(x) = \sum_{x=a}^{b} x f(x) \geq \sum a f(x) = a \underbrace{\sum f(x)}_{=1}$$

$$\therefore E(x) \geq a \;\dots\dots\dots \quad (1)$$

$$E(x) = \sum_{x=a}^{b} x f(x) \leq \sum b f(x) = b \underbrace{\sum f(x)}_{=1}$$

$$\therefore E(x) \leq b \;\dots\dots\dots \quad (2)$$

$\therefore$ by "1" & "2" we get $a \leq E(x) \leq b$

theorem 4": If $p(x \geq a) = 1$ and $E(x) = a$ then $p(x = a) = 1$ and $p(x > a) = 0$.

proof: case "1": If x is ac.r.v. from ap.d.f f(x)

$$\therefore p(x \geq a) = 1 \Rightarrow \int_a^\infty f(x)dx = 1$$

$\therefore f(x) > 0$ for $a \leq x < \infty$

=0 ow

$E(x)= \int_a^\infty x \quad f(x)\ dx = a = a.1 = a\int_a^\infty f(x)dx = \int_a^\infty af(x)dx$

$\int_a^\infty x\, f(x)\ dx = a \int_a^\infty f(x)\ dx \Rightarrow \int_a^\infty x\, f(x)\ dx = \int_a^\infty a\, f(x)\ dx$

..this inquality hold only when x=a

i.e. (x>a)=ϕ $\Rightarrow P(X > a) = 0$

$p(x \geq a) = p[(x=a) \cup (x>a)] = p(x=a) + p(x>a) \Rightarrow 1 = p(x=a) + 0 \Rightarrow p(x=a) = 1$

case "2": If x is ad.r.v from ap.m.f f(x)

$\therefore p(x \geq a) = 1 \Rightarrow \sum_{x=a}^{\infty} f(x) = 1$

$\therefore f(x) > 0$ for $a \leq x < \infty$

=0 ow

$E(x)= \sum_{x=a}^{\infty} xf(x) = a = a.1 = a.\sum_{x=a}^{\infty} f(x) = \sum_{x=a}^{\infty} af(x)$

$\sum_{x=a}^{\infty} xf(x) = \sum_{x=a}^{\infty} af(x)$

This inequality hold only when x=a

i.e. (x>a)=ϕ$\Rightarrow p(x > a) = 0$

$p(x \geq a) = [(x=a) \cup (x>a)] = p(x=a) + p(x>a)$

$\Rightarrow 1 = p(x=a) + 0 \Rightarrow p(x=a) = 1$

variance of random variable:

Def: let x be ar.v. the variance of x denoted by v(x) or δ^2 is defined as

: $v(x)=E\{[x - E(x)]^2\}$

$\because E(x)=\mu$, then $v(x) = E[(x-\mu)^2]$

note: since $(x-\mu)^2 \geq 0$ *then* $E[(x-\mu)^2] \geq 0$

$\therefore v(x) \geq 0$ alwayes.

properties of variance:

theorem "5" let x be ar.v.,then $v(x)=E(x^2)-[E(x)]^2$

Proof: $v(x)=E\{[x-E(X)]^2\}$

$\because V(x)=E\{x^2-2E(x)x+[E(x)]^2\}, E(x)\ constant.$

$v(x)=E(x^2)-2E(x)E(x)+[E(x)]^2$

$v(x)=E(x^2)-2[E(x)]^2+[E(x)]^2 \Rightarrow v(x)=E(x^2)-[E(x)]^2$

note: 1. $v(x)=0 \Leftrightarrow E(x^2)=E[(x)]^2$

2. $\because v(x)\geq 0 \Rightarrow \therefore E(x^2)\geq [E(x)]^2$

3. $v(b)=0$, b is constant.

Theorem "6" let x be a r.v. and v(x) exist, If $y=ax+b$; $a,b\in R$ then $v(y)=a^2 v(x)$.

Proof: $y=ax+b \Rightarrow E(y)=aE(x)+b$

$V(y)=E\{[y-E(y)]^2\}=E\{[(ax+b)-(aE(x)+b)]^2\}$

$=E\{[ax-aE(x)]^2\}=E\{a^2[x-E(x)]^2\}$

$=a^2E\{[X-E(x)]^2\}=a^2V(x)$

Ex: Given a p.d.f $f(x)=\begin{cases}3x^2 & for\ 0<x<1\\ 0 & o.w.\end{cases}$

a. find E(x) & v(x). b. If y=1-2x, then find E(y), v(y).

SOL.

$$a.\ E(x)=\int_0^1 X(3X^2)dx=3\frac{X^4}{4}\Big|_0^1=\frac{3}{4}$$

$v(x)=E(x^2)-[E(x)]^2$

$$E(x^2)=\int_0^1 x^2(3x^2)dx=\frac{3}{5}x^5\Big|_0^1=\frac{3}{5}$$

$$\therefore v(x)=\frac{3}{5}-\frac{9}{16}=\frac{48-45}{80}=+\frac{3}{80}$$

b. $\because y=(-2)x+1 \Rightarrow \therefore E(y)=(-2)E(x)+1$

$$\therefore E(y)=(-2)(\frac{3}{4})+1=\frac{-3}{2}+1=-\frac{1}{2}$$

$$v(y)=(-2)^2 v(x)=4.(\frac{3}{80})=\frac{12}{80}$$

H.w.: Given a p.d.f. $f(x)=f(x)=\begin{cases}6x(1-x) & for\ 0<x<1\\ 0 & 0.w.\end{cases}$

If y=2-3x, then find E(y) & v(y)

Theorem "7" $v(x)=0$ iff $\exists K$ where k is constant such that $p(x=k)=1$

$\Leftarrow$ suppose that $p(x=k)=1$, t.p $V(X)=0$

Poof: $f(x)>0$ for $x=k$

$=0$ o.w.

$v(x)=v(x=k)=0$ by note.

$\Rightarrow$ suppose $V(X)=0$

,t.p $p(x=k)=1$

$\because v(x)=0 \Rightarrow E\{[x-E(x)]^2\}=0$

$Y=[x-E(x)]^2 \Rightarrow E(Y)=0$

by th. "4" $\Rightarrow$ $[E(x)=a \rightarrow p(x=a)=1]$

$\therefore p(y=0)=1 \Rightarrow p\{[x-E(x)]^2=0\}=1$

$\therefore p\{(x-E(x))=0\}=1 \Rightarrow \therefore p[x=E(x)]=1$

$\therefore x=k \Rightarrow E(x)=E(k)=k \Rightarrow \therefore p[x=k]=1$

* ***Existence of Mean and Variance:***

E(x) exists iff $E(|x|)<\infty$

$\because V(x)=E(x^2)-[E(x)]^2 \Rightarrow \therefore V(x)$ exists iff $E(x)$ & $E(x^2)$ exist

Ex. "1": Given acauchy p.d.f $f(x)=\dfrac{1}{\pi(1+x^2)}$ for $-\infty<x<\infty$

Show that E(x) dose not exist?

Sol.: $E(|x|)=\int_{-\infty}^{\infty}|x|\dfrac{1}{\pi(1+x^2)}dx=2\int_0^{\infty}x\dfrac{1}{\pi(1+x^2)}dx$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{2x}{1+x^2}dx=\frac{1}{\pi}\ln(1+x^2)\Big|_0^{\infty}=\frac{1}{\pi}[\ln\infty-\ln 1]\not<\infty$$

$E(|x|)\not<\infty \Rightarrow E(x)$ dose not exist

Exercises: 1. Let x be a c.r.v. have a p.d.f f(x) where

$f(x)>0$ for $0<x<b<\infty$, Show that $E(x)=\int_0^b[1-F(x)]dx$

$=0$ o.w.

Hint.: $f(x)=\dfrac{dF(x)}{dx} \leftrightarrow f(x)dx=dF(x)$

2. If x is d.r.v. have p.M.f $f(x)>0$ for x = -1, 0, 1

$=0$ o.w.

a. If $f(0)=\frac{1}{2}$, Find $E(x^2)$. $\sum f(x)=1 \Rightarrow f(-1)+f(0)+f(1)=1$

b. If $f(0)=\frac{1}{2}$, and $E(x)=\frac{1}{6}$, Find f(-1), f(1)

3. Given a p.d.f $f(x)=\begin{cases}1-|x| & \text{for } |x|<1 \\ 0 & \text{o.w.}\end{cases}$

a. Find E(x) & V(x), b. If y= 2-3 x, find E(y) & V(y).

Hint $f(x)=\begin{cases}1-|x| & -1<x<1 \\ 0 & \text{o.w.}\end{cases} = \begin{cases}1+x & -1<x<0 \\ 1-x & 0\le x<1 \\ 0 & \text{o.w.}\end{cases}$

Moments of Random Variables:

Def.: Let x be a r.v. either d.r.v. or c.r.v., let $K \in I^+$ then $E(x^k)$ is called "the k^{th} moment of x" or "the moment of order k of x".

when $k=1 \Rightarrow E(x^1) = 1^{st}$ moment of $x=\mu$

$E(x^2) = 2^{nd}$ moment of x

Note: $E(x^k)$ exists iff $E(|x|^k)<\infty$

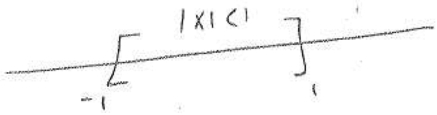
بدون برهان Theorem "8": If $E(x^k)$ exists then $E(x^j)$ exists, $j<k$ and $j, k \in I^+$

Proof: case "1" If x is c.r.v. with p.d.f f(x)

$\because E(x^k)$ exists $\Rightarrow \therefore E(|x|^k)<\infty$

T.p $E(x^j)$ exists, T.p $E(|x|^j)<\infty$

$E(|x|^j)=\int_{-\infty}^{\infty}|x|^j f(x)dx$



$E(|x|^j)=\int_{|x|\le 1}^{-\infty}|x|^j f(x)dx + \int_{|x|>1}|x|^j f(x)dx$

2^{nd} central moment of R.V.X is equal to V(x).

Ex.: Let x be a r.v.s.t $E(x) = 1$, $E(x^2) = 2$ and $E(x^3) = 5$. Find the 3^{rd} central moment of x.

Sol.: $E[(x-\mu)^3] = E\{x^3 - 3\mu x^2 + 3\mu^2 x - \mu^3\}$

$= E(x^3) - 3\mu E(x^2) + 3\mu^2 E(x) - \mu^3$

$= 5 - 3 . 1 . 2 + 3 . 1 . 1 - 1 = 1$

Exercises: 1. If x ~ uniform (a, b); a, b $\in$ R. Find the valued of 1^{st} central moment of x and also find the 2^{nd} central moment of x.

2. Let $\mu = E(x)$ and $\delta^2 = V(x)$ show that $E[(x-\mu)^4] \geq \delta^4$

i.e. 4^{th} central moment of x is greater than or equal the square of variance of x.

Moment Generating Function (M.g.f).

Def.: A moment generating function (M.g.f) of a r.v.x is a function that determines all moments of x, denoted by $M_x(t)$ suppose that $t \in (-h, h)$, $h > 0$. If $E[e^{tx}]$ exists $\forall t \in (-h, h)$ then $M_x(t) = E[e^{tx}]$, $-h < t < h$

There are two cases of $M_x(t)$.

Case "1": If x is a d.r.v. from a p.M.f f(x)

$$M_x(t) = E[e^{tx}] = \sum_{\forall x} e^{tx} f(x)$$

Case "2": If x is a c.r.v. have a p.d.f f(x)

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Ex.: Given a p.d.f $f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w} \end{cases}$

Find $M_x(t)$ and sketch it's graph.

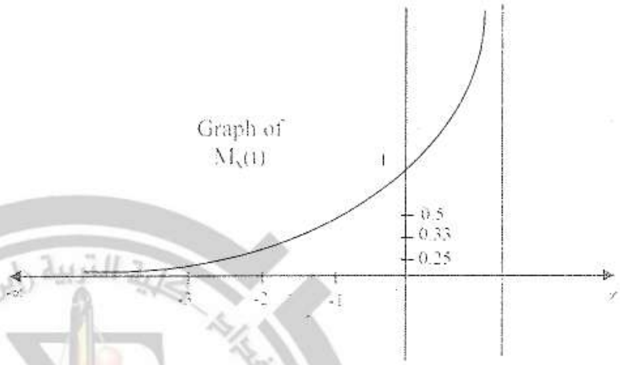
Sol.: $M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \cdot e^{-x} dx = \int_0^{\infty} e^{-(1-t)x} dx$

This integration exist only when (1-t) > 0, i.e. t < 1

$$M_x(t) = \frac{-1}{1-t} e^{-(1-t)x} \Big|_0^{\infty} = \frac{-1}{1-t}\left[e^{-\infty}_{=0} - e^{0}_{=1}\right] = \frac{1}{1-t}$$

$$M_x(t) = \frac{1}{1-t} \qquad for \quad t < 1$$

t	$M_x(t)$
1	∞
0	1
-1	0.5
-2	0.33
-3	0.25
.	
$-\infty$	0



Ex.: Given a p.M.f. $f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w} \end{cases}$

Find $M_x(t)$.

Sol.: $$M_x(t) = E(e^{tx}) = \sum_{x=1}^{5} e^{tx} f(x) = \sum_{x=1}^{5} \frac{x}{15} e^{tx}$$

$$= \frac{1}{15}\sum_{x=1}^{5} x e^{tx} = \frac{1}{15}[1\cdot e^{t} - 2\cdot e^{2t} + 3\cdot e^{3t} + 4\cdot e^{4t} + 5e^{5t}] \quad for \; -\infty < t < \infty$$

Theorem "9": Let x be a r.v. have a M.g.f. $M_x(t)$, then

$$M_x(0) = 1,\; M'_x(0) = E(x),\; M''_x(0) = E(x^2),\; M'''_x(0) = E(x^3), \ldots,$$
$$M_x^{(k)}(0) = E(x^k)$$

Proof: $* M_x(t) = E(e^{tx})$ by Macclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^k}{k!} + \ldots$$

$$\therefore e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots + \frac{t^kx^k}{k!} + \ldots$$

$$M_x(t) = E[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots + \frac{t^kx^k}{k!} + \ldots]$$

$$M_x(t) = 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \ldots + \frac{t^k}{k!}E(x^k) + \ldots$$

$M_x(t=0) = M_x(0) = 1$

$$* M'_x(t) = \frac{dM_x(t)}{dt} = E(x) + \frac{2t}{2!}E(x^2) + \frac{3t^2}{3!}E(x^3) + \ldots + \frac{kt^{k-1}}{k!}E(x^k) + \ldots$$

$M'_x(0) = E(x)$

$$* M''_x(t) = E(x^2) + \frac{6t}{3!}E(x^3) + \ldots + \frac{k(k-1)t^{k-2}}{k!}E(x^k) + \ldots$$

$M''_x(0) = E(x^2)$

Simillary we can find $E(x^3)$, $E(x^4)$, …. $E(x^k)$, …

Note: $M_x(t) = M_x(0) + tM'_x(0) + \frac{t^2}{2!}M''_x(0) + \ldots + \frac{t^k}{k!}M_x^{(k)}(0)$

This series is called the M. g.f. by Macclaurin series.

Ex.: Given $M_x(t) = \frac{1}{1-2t}$, $t < \frac{1}{2}$ Find $E(x)$ and $V(x)$

Sol.: $M_x(t) = (1-2t)^{-1}$, $t < \frac{1}{2}$

$M'_x(t) = -(1-2t)^{-2} \cdot (-2) = 2(1-2t)^{-2}$

$\therefore E(x) = M'_x(0) = 2(1-0)^{-2} = 2$

$V(x) = E(x^2) - [E(x)]^2$

$M''_x(t) = 8(1-2t)^{-3}$

$\therefore E(x^2) = M''_x(0) = 8(1-0)^{-3} = 8$

$\therefore V(x) = 8 - 2^2 = 4 \geq 0$

Theorem "10": Let x be a r.v. have a M.g.f $M_x(t)$. If $Y = ax + b$, a, b, $\in R$ then $M_y(t) = e^{bt} \cdot M_x(at)$

Proof: $Y = ax + b$

$$M_y(t) = E(e^{ty})\,(by\ def.) = E[e^{t(ax+b)}] = E[e^{(at)x+bt}]$$

$$M_y(t) = [e^{(at)x} \cdot e^{bt}] = e^{bt}E[e^{(at)x}] = e^{bt} \cdot M_x(at)$$

$$Since\ M_x(t) = E(e^{tx}) \Rightarrow M_x(at) = E[e^{(at)x}]$$

Ex: Given $M_x(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$, If y= 1-2x , find $M_y(t)$.

Sol: y= (-2)x+1 , a=-2 , b=1

By th. (10) $\Rightarrow M_y(t) = e^{bt} M_x(at) = e^t M_x(-2t)$

$\because M_x(t) = \dfrac{1}{1-3t}$ for $t < \dfrac{1}{3}$

$M_x(-2t) = \dfrac{1}{1-3(-2t)} = \dfrac{1}{1+6t}$ for $t > \dfrac{-1}{6}$

$M_y(t) = e^t \cdot \dfrac{1}{1+6t}$ for $t > \dfrac{-1}{6}$

The bounded of probability:

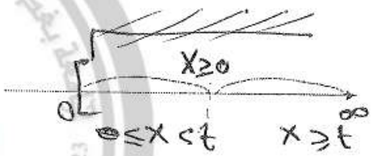
Theorem "11": ((Markov inquality))

If x is a r.v. and if $P(x \geq 0) = 1$ then $P(x \geq t) \leq \dfrac{E(x)}{t}$, for $t > 0$

Proof: case "1": if x is a c.r.v with a p.d.f f(x)

$$\because p(x \geq 0) = 1 \Rightarrow \int_0^\infty f(x)dx = 1$$

$$f(x) > 0 \quad \text{for } x \geq 0$$
$$= 0 \quad \text{O.W}$$



, (t>0)

$$E(x) = \int_0^\infty x f(x)dx = \int_0^t x f(x)dx + \int_t^\infty x f(x)dx \geq \int_t^\infty x f(x)dx \geq \int_t^\infty t f(x)dx \quad [x \geq t]$$

$$\therefore E(x) \geq \int_t^\infty t f(x)dx \Rightarrow E(x) \geq t \int_t^\infty f(x)dx$$

$$E(x) \geq t \cdot p(x \geq t)$$

$$\therefore p(x \geq t) \leq \frac{E(x)}{t}$$

Case "2": If x is a d.r.v. with a p.m.f. f(x) by the same method.

Theorem "12": "Chebyshev inqualities"

Let x be a r.v. where V(x) exists, then:

1. $p(|x-\mu| \geq t) \leq \dfrac{V(x)}{t^2}$, $t > 0$, 2. $p(|x-\mu| < t) \geq 1 - \dfrac{V(x)}{t^2}$, $t > 0$,

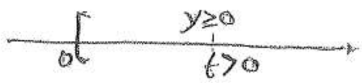
Note: 1. $\dfrac{V(x)}{t^2}$ *is called the upper buond of* $p(|x-\mu| \geq t)$

2. $1 - \dfrac{V(x)}{t^2}$ *is called the lower buond of* $p(|x-\mu| < t)$

Proof: 1. $V(x) = E[(x-\mu)^2] \geq 0$

$Let\ y=(x-\mu)^2 \geq 0 \Rightarrow E(y)=E[(x-\mu)^2 = V(x) \geq 0$

$\therefore$ by Markov inquality, we get



$$p(y \geq t^2) \leq \frac{E(y)}{t^2} \Rightarrow p[(x-\mu)^2 \geq t^2] \leq \frac{V(x)}{t^2}$$

$$p[|x-\mu| \geq t] \leq \frac{V(x)}{t^2}$$

2. $p(A^c) = 1 - p(A)$

$$p[|x-\mu| \geq t^2] = 1 - p[|x-\mu| < t^2] \leq \frac{V(x)}{t^2}$$

$$p[|x-\mu| < t^2] \geq 1 - \frac{V(x)}{t^2}$$

Ex. "1": If $x \sim$ unif. $(-\sqrt{3}, \sqrt{3})$ then:

a. Find the upper bound of $p(|x-\mu| \geq \frac{3}{2})$

b. Find the value of $p(|x-\mu_x| \geq \frac{3}{2})$

Sol.:

a. $f(x) = \begin{cases} \frac{1}{\sqrt{3}+(\sqrt{3})} = \frac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < x < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$

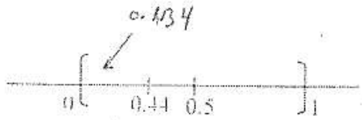
$u.b = \frac{V(x)}{t^2}, \qquad t = \frac{3}{2}$

$$E(x) = \int_{\sqrt{3}}^{\sqrt{3}} X \frac{1}{2\sqrt{3}} dx = \frac{1}{4\sqrt{3}} X^2 \Big|_{-\sqrt{3}}^{\sqrt{3}} = 0$$

$$E(x^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \frac{1}{2\sqrt{3}} dx = \frac{1}{6\sqrt{3}} x^3 \Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{6\sqrt{3}}[(\sqrt{3})^3 - (-\sqrt{3})^3] = \frac{1}{6\sqrt{3}}[3\sqrt{3}+3\sqrt{3}] = 1$$

$\therefore V(x) = E(x^2) - [E(x)]^2 = 1 - 0 = 1$

$\therefore u.b = \dfrac{V(x)}{t^2} = \dfrac{1}{(\frac{3}{2})^2} = \dfrac{4}{9} = 0.44$



b. $p(|x-\mu| \geq \frac{3}{2}) = p(|x-0| \geq \frac{3}{2}) = p(|x| \geq \frac{3}{2})$

$= 1 - p(|x| < \frac{3}{2}) = 1 - p(-\frac{3}{2} < x < \frac{3}{2}) = 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} dx$

$= 1 - \frac{1}{2\sqrt{3}} x \Big|_{-\frac{3}{2}}^{\frac{3}{2}} = 1 - \frac{1}{2\sqrt{3}}(\frac{3}{2} + \frac{3}{2}) = 1 - \frac{3}{2\sqrt{3}} = 1 - \frac{\sqrt{3}}{2}$

$= 1 - 0.866 = 0.134$

Ex. "2": Given a p.d.f $f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$

a. Find the lower bound of $p(\frac{5}{4} < x < \frac{11}{4})$

b. Find the value of $p(\frac{5}{4} < x < \frac{11}{4})$

Sol.: a. $L.b = 1 - \frac{V(x)}{t^2}$

$E(x) = \int_0^3 x \cdot \frac{2x}{9} dx = \frac{2}{27} x^3 \Big|_0^3 = \frac{2}{27}(27 - 0) = 2$

$E(x^2) = \int_0^3 x^2 \cdot (\frac{2x}{9}) dx = \frac{2}{36} x^4 \Big|_0^3 = \frac{1}{18}(81) = \frac{9}{2} = 4.5$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$= 4.5 - 4 = 0.5 = \frac{1}{2}$

$p(\frac{5}{4} < x < \frac{11}{4}) = p(\frac{5}{4} - 2 < x - 2 < \frac{11}{4} - 2) = p(-\frac{3}{4} < x - 2 < \frac{3}{4})$

$= p(|x - 2| < \frac{3}{4})$

$\therefore t = \frac{3}{4}$

$\therefore L.b = 1 - \dfrac{V(x)}{t^2} = 1 - \dfrac{\frac{1}{2}}{\frac{9}{16}} = 1 - \frac{1}{2} \times \frac{16}{9} = 1 - \frac{8}{9} = \frac{1}{9}$

b. $p(\frac{5}{4} < x < \frac{11}{4}) = \int_{\frac{5}{4}}^{\frac{11}{4}} \frac{2x}{9} dx$

$= \frac{1}{9} x^2 \Big|_{\frac{5}{4}}^{\frac{11}{4}} = \frac{1}{9}[\frac{121}{16} - \frac{25}{16}] = \frac{1}{9}[\frac{96}{16}] = \frac{6}{9}$

Median of Distribution of r.v.:

Def.: The median (m) is a value of x such that satisfing the two following inqualities:

$p(x < m) < \frac{1}{2}$ & $p(x \leq m) \geq \frac{1}{2}$

By properties of c.d.f F(x)

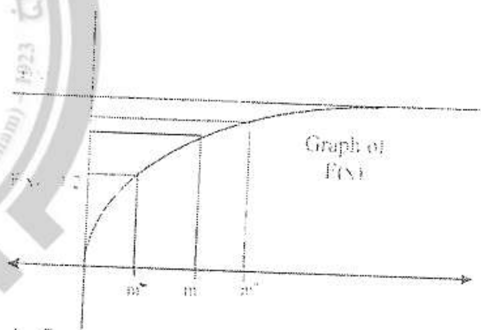
$p(x < m) = F(\overline{m})$, $p(x \leq m) = F(m) = F(m^{-})$

i.e.: $p(x < m) = F(\overline{m}) < \frac{1}{2}$, $p(x \leq m) = F(m^{-}) \geq \frac{1}{2}$

Note ① If $\overline{m} = m = m^{-}$ then

$p(x \leq m) = F(m) = \frac{1}{2}$

② The value of median unique



Ex.: Given a p.m.f $f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$

Find the median of x.

Sol.: $p(x < m) < \frac{1}{2}$ & $p(x \leq m) \geq \frac{1}{2}$

Suppose that m = 1

$p(x < 1) = 0 < \frac{1}{2}$, $p(x \leq 1) = f(1) = \frac{1}{15} \ngeq \frac{1}{2}$

$\therefore m \neq 1$

Suppose that m = 2

$p(x<2)=f(1)=\frac{1}{15}<\frac{1}{2}$

$p(x\le 2)=f(1)+f(2)=\frac{1}{15}+\frac{2}{15}=\frac{1}{15}\not\ge\frac{1}{2}$

$\therefore m\neq 2$

Suppose that m=3

$p(x<3)=f(1)+f(2)=\frac{1}{15}+\frac{2}{15}=\frac{1}{5}<\frac{1}{2}$

$p(x\le 3)=f(1)+f(2)+f(3)=\frac{1}{15}+\frac{2}{15}+\frac{3}{15}=\frac{6}{15}\not\ge\frac{1}{2}$

$\therefore m\neq 3$

Suppose that m = 4

$\frac{12}{30}<\frac{15}{30}$

$p(x<4)=f(1)+f(2)+f(3)=\frac{6}{15}<\frac{1}{2}$

$p(x\le 4)=f(1)+f(2)+f(3)+f(4)=\frac{10}{15}\ge\frac{1}{2}$

$\therefore m=4$ is median

Ex.: Given a p.d.f $f(x)=\begin{cases}\frac{1}{x^2} & \text{for } x>1\\ 0 & \text{o.w.}\end{cases}$

Find the median of x?

Sol.: $p(x<m)<\frac{1}{2}$ & $p(x\le m)\ge\frac{1}{2}$

$$p(x<m)=\int_1^m\frac{1}{x^2}\,dx=-\frac{1}{x}\Big|_1^m=-[\frac{1}{m}-1]<\frac{1}{2}$$

$$-\frac{1}{m}+1<\frac{1}{2}\Rightarrow-\frac{1}{m}<-\frac{1}{2}\Rightarrow m<2\quad .....(1)$$

$$p(x\le m)=\int_1^m\frac{1}{x^2}\,dx\ge\frac{1}{2}$$

$$-[\frac{1}{m}-1]\ge\frac{1}{2}\Rightarrow m\ge 2\quad .....(2)$$

From (1) & (2) $\Rightarrow$ median = 2

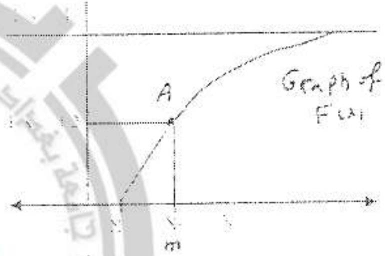
Note: We can also find the value of (Median) from the graph of F(x) such that. At the point $F(x)=\frac{1}{2}$, draw a line to cut the curve of F(x) at A, draw a line to cut the X-axis at m (median) from ex. "2" above

$$f(x)=\begin{cases}\frac{1}{x^2} & \text{for } x>1\\ 0 & \text{o.w.}\end{cases}$$

$$F(x)=p(X\le x)=\int_1^x \frac{1}{t^2}\,dt=1-\frac{1}{x}\Rightarrow F(x)=\begin{cases}1-\frac{1}{x} & \text{for } x\ge 1\\ 0 & \text{o.w.}\end{cases}$$

or : $F(x)=\frac{1}{2}$

$$1-\frac{1}{x}=\frac{1}{2}\Rightarrow -\frac{1}{x}=-\frac{1}{2}\Rightarrow x=2$$



To Find the Mode of Dist. Of R.V.X

Def.: Mode is a value of a r.v.x that maximize f(x)

i.e. If x_1 = mode. $\longrightarrow$ then $f(x_1)$ is a Max.

Note: $F(x_1)$ is Max. $\Leftrightarrow$ $f''(x_1)<0$

Ex.: Given a p.d.f. $f(x)=\begin{cases}12x^2(1-x) & 0<x<1\\ 0 & \text{o.w.}\end{cases}$

Find the mode of x?

Sol.: 0 < mode < 1

$f(x)=12x^2-12x^3\Rightarrow \quad f'(x)=24x-36x^2$

$\Rightarrow \quad 12x(2-3x)=0 \qquad$ either $x=0 \qquad$ or $x=\frac{2}{3}$

$f''(x)=24-72x\Rightarrow \ f''(0)=24>0 \qquad \Rightarrow f(0)$ is min.

$f''(\frac{2}{3})=24-72(\frac{2}{3})=-24<0\Rightarrow f(\frac{2}{3})$ *is* max

$x_1=\frac{2}{3}=\text{mode}$

Def.: Percentile

It is a value of x say (x_0) such that $p(x \le x_0) = \frac{t}{100}$ $\quad 0 < t < \infty$

denoted by P_t, $\quad 0 < t < 100$

i.e.: $P_t = x_0$

Ex.: Given a p.d.f $f(x) = \begin{cases} \frac{x}{2} & , 0 < x < 2 \\ 0 & , \text{o.w.} \end{cases}$

Find P_{40}, P_{65}

Sol.: let $p_{40} = x_0$, $t = 40$

$p(x \le x_0) = \frac{t}{100}$, $\quad p(x \le x_0) = \frac{40}{100} = 0.4$

$\int_0^{x_0} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^{x_0} = 0.4$

$[x^2 - 0^2] = 4(0.4) \Rightarrow x^2 = 1.6 \Rightarrow x_0 = \sqrt{1.6} = 1.26$

Exercises

① Given a p.f.

$$f(-1)=\frac{1}{8},\quad f(0)=\frac{6}{8},\quad f(1)=\frac{1}{8}$$

Find the u.b. of $P(|X| \geq 2\sigma)$.

<u>sol.</u>

$$E(x)=\sum_{x=-1}^{1} x f(x) = (-1)f(-1)+(0)f(0)+(1)f(1)$$
$$= -\frac{1}{8}+0+\frac{1}{8} = 0$$

$$\boxed{E(x)=\mu=0}$$

$$E(x^2)=\sum_{x=-1}^{1} x^2 f(x) = (-1)^2 f(-1)+(0)^2 f(0)+(1)^2 f(1)$$
$$=\frac{1}{8}+0+\frac{1}{8}=\frac{1}{4}$$

$$\sigma^2 = V(x) = E(x^2)-[E(x)]^2 = \frac{1}{4}-0=\frac{1}{4}$$

$$\boxed{\sigma=\frac{1}{2}}$$

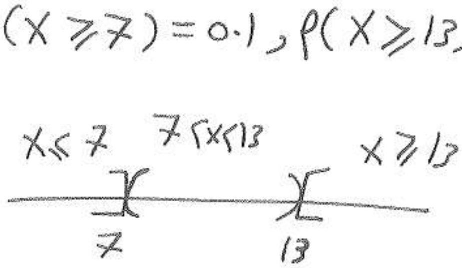
$$P(|X|\geq 2\sigma) = P(|X-0| \geq 2\cdot\frac{1}{2}) = P(|X-\mu|\geq 1)$$
$$= P(|X-\mu| \geq t) \leq \frac{V(x)}{t^2} \Rightarrow t=1$$

$$u.b = \frac{V(x)}{t^2} = \frac{\left(\frac{1}{4}\right)}{1} = \frac{1}{4}$$

<u>Note</u> If we have to find the exact value of above pr.:

$$P(|X|\geq 2\cdot\sigma) = P(|X|\geq 2\cdot\tfrac{1}{2}) = P(|X|\geq 1) = 1-P(|X|<1)$$
$$=1-P(-1<X<1) = 1-P(X=0) = 1-f(0)$$
$$=1-\frac{6}{8}=\frac{2}{8}$$

② If X is a r.v. with $E(x)=10$, $P(X\geq 7)=0.1$, $P(X\geq 13)=0.3$, then show that $V(x)\geq \frac{9}{2}$.



<u>sol.</u>

$$P(|X-\mu|<t) \geq 1-\frac{V(x)}{t^2}$$
$$P(|X-\mu|\geq t) \leq \frac{V(x)}{t^2}$$

$$P[(X\leq t)\cup(7<X<13)\cup(X\geq 13)] = P(S)$$
$$P(X\leq 7)+P(7<X<13)+P(X\geq 13)=1$$
$$0.2 + P(7<X<13) + 0.3 = 1$$
$$P(7<X<13) = 0.5 = \frac{1}{2}$$
$$\therefore E(x)=\mu=10$$

④ Let $X \sim b(2,p)$ & $Y \sim b(4,p)$. If $P(X \geq 1) = \frac{5}{9}$, Find $P(Y \geq 1)$.

Sol. $\because X \sim b(2,p)$

$$\therefore f(x,2,p) = \begin{cases} \binom{2}{x} p^{x}(1-p)^{n-x} & \text{for } x = 0,1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \geq 1) = \frac{5}{9} \Rightarrow 1 - P(X < 1) = \frac{5}{9} \Rightarrow 1 - P(X = 0) = \frac{5}{9}$$

$\therefore P(X=0) = \frac{4}{9} = f(0)$

$\binom{2}{0} p^{0}(1-p)^{2} = \frac{4}{9}$

$1 \cdot 1 \cdot (1-p)^{2} = \frac{4}{9}$

$(1-p)^{2} = \frac{4}{9}$

$(1-p) = \mp \frac{2}{3}$

If $1-p = \frac{2}{3}$

then $\boxed{p = \frac{1}{3}}$

If $1-p = -\frac{2}{3}$

since $p = 1 + \frac{2}{3} = \frac{5}{3} > 1$

$0 < p < 1$

$\therefore \boxed{p = \frac{1}{3}}$

$\therefore Y \sim b(4,p)$

$$\therefore f(y,4,p) = \begin{cases} \binom{4}{y} p^{y}(1-p)^{4-y} & ; y = 0,\dots,4 \\ 0 & \text{o.w.} \end{cases}$$

$\because p = \frac{1}{3} \Rightarrow 1-p = \frac{2}{3}$

$P(Y \geq 1) = 1 - P(Y<1) = 1 - P(Y=0)$

$= 1 - f(0)$

$= 1 - \binom{4}{0}\left(\frac{1}{3}\right)^{0}\left(\frac{2}{3}\right)^{4}$

$\therefore P(Y \geq 1) = 1 - \left(\frac{2}{3}\right)^{4} = 1 - \frac{16}{27} = \frac{11}{27} < 1$

⑤ If $X \sim G(\frac{1}{2})$, find Median of X

Sol. $X \sim G(p)$; $p = \frac{1}{2}$, $q = 1 - \frac{1}{2} = \frac{1}{2}$

$$f(x;p) = \begin{cases} \frac{1}{2}\left(\frac{1}{2}\right)^{x-1} & \text{for } x = 1,2,3,\dots \\ 0 & \text{o.w.} \end{cases}$$

$P(X<m) \leq \frac{1}{2}$ & $P(X \leq m) \geq \frac{1}{2}$, $m \equiv$ median

If $m=1$, $P(X<1) = P(X=0) = 0 < \frac{1}{2}$

$P(X \leq 1) = P(X=1) = \frac{1}{2}$

then $\boxed{m=1}$

Can $m = 2$?

$P(X<2) = P(X=1) = \frac{1}{2}$

$P(X \leq 2) = P(X=1,2) = P(X=1) + P(X=2) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)$

$= \frac{3}{4} > \frac{1}{2}$.

$\therefore \boxed{m=2}$

There are two values of Median 1, 2 .

(6) Given $M_x(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^9$, Show that :

$P(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} \binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x} = P(0 < X < 6)$

Sol.

$M_x(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^9 \Longrightarrow p = \frac{1}{3}, \; n = 9, \; 1-p = \frac{2}{3}$

$\therefore \boxed{X \sim b\left(9, \frac{1}{3}\right)}$

$M_x(t) = (1-p+pe^t)^n$

$$f\left(x, 9, \frac{1}{3}\right) = \begin{cases} \binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x} & \text{for } x = 0,1,2,\dots,9 \\ 0 & \text{o.w.} \end{cases}$$

$E(X) = \mu = np, \quad \sigma^2 = V(X) = np(1-p)$

$E(x) = 9\left(\frac{1}{3}\right) = 3$; $V(x) = 3\left(\frac{2}{3}\right) = 2$

$\mu = 3$; $\sigma^2 = 2$, $\sigma = \sqrt{2}$

$\mu - 2\sigma = 3 - 2\sqrt{2} = 3 - 2(1.4) = 3 - 2.8 = 0.2$ $\quad \boxed{\sqrt{2} = 1.4}$

$\mu + 2\sigma = 3 + 2.8 = 5.8$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(.2 < x < 5.8) = P(1 \leq x \leq 5)$

$= \sum_{x=1}^{5} f(x)$

$= \sum_{x=1}^{5} \binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}$

$= P(0 < X < 6)$

(7) If $X \sim b(n,p)$, and $Y = \frac{X}{n}$, Prove that :

$\forall \varepsilon > 0$, $\lim_{n \to \infty} P(|Y - p| \geq \varepsilon) = 0$

Sol.

$P(|X-\mu| \geq t) \leq \frac{V(X)}{t^2} = u.b.$

$P(|X-\mu| < t) \geq 1 - \frac{V(X)}{t^2} = L.b.$

$P(|\mu - p| \geq \varepsilon) = P(|\frac{X}{n} - p| \geq \varepsilon) = P(|\frac{X-np}{n}| \geq \varepsilon)$

$= P(|X - np| \geq n.\varepsilon)$

$\boxed{\mu = np}$

$P(|X-np| \geq n.\varepsilon) \geq \frac{V(X)}{t^2}$, $\boxed{V(X) = np(1-p)}$

$= \frac{np(1-p)}{n^2\varepsilon^2}$

$= \frac{p(1-p)}{n\varepsilon}$

a^2 $n \to \infty$; $P(|X-np| \geq n.\varepsilon) \to 0$

$\therefore \lim_{n\to\infty} P(|X-\mu| \geq t) = \lim_{n\to\infty} P(|X-np| \geq n.\varepsilon)$

$= 0$ By $(0 \leq P(A) \leq 1)$

$P(|Y-p| < \varepsilon) = P(|\frac{X}{n} - p| \geq \varepsilon) = P(|\frac{X-np}{p}| < \varepsilon)$

$= P(|X - np| < n.\varepsilon)$; $\mu = np$, $t = n.\varepsilon$

$< 1 - \frac{V(X)}{t^2}$; $V(X) = np(1-p)$

$= 1 - \frac{np(1-p)}{n^2\varepsilon^2} = 1 - \frac{p(1-p)}{n\varepsilon^2}$

$\therefore \lim_{n\to\infty} (|Y-p| < \varepsilon) = \lim_{n\to\infty} (1 - \frac{p(1-p)}{n\varepsilon^2}) = 1 - 0 = 1$

(8) Suppose that $X_1, X_2, \ldots, X_n$ are n- independent random variables, and for $i = 1, 2, \ldots, n$, let ψ_i be the M.g.f. of X_i. Let $Y = \sum_{i=1}^{n} x_i$ and let ψ be the M.g.f. of Y. Then for any t s.t. $\psi_i(t)$ exist,

$\psi(t) = \prod_{i=1}^{n} \psi_i(t)$.

Sol. X_i $(i = 1, 2, \ldots, n)$ is indep. r.v.s

ψ_i $(i = 1, 2, \ldots, n)$ be M.g.f. of X_i

6

If $y=\sum_{i=1}^{n} X_i$, then if ψ be the M.g.f. of ψ, then forevery t s.t. ($\psi_i(t)$ exists), then

$$\psi(t)=\prod_{i=1}^{n}\psi_i(t) \quad \text{s.t. } y=\sum_{i=1}^{n}X_i$$

$$\psi_y(t)=E(e^{ty})=E[e^{(\sum x_i)t}]=E[e^{tx_1+tx_2+\cdots+tx_n}]$$

$$=E(e^{tx_1})\cdot E(e^{tx_2})\cdots E(e^{tx_n})$$

$$=(\psi_1(t))(\psi_2(t))\cdots(\psi_n(t))$$

$$=\prod_{i=1}^{n}\psi_i(t)$$

(9) Let X be a r.v. with M.g.f. ψ_1, Let $Y=ax+b$; where $a,b\in R$ and let ψ_2 denote the M.g.f. of Y, then for any value of t such that $\psi_1(at)$ exists, $\psi_2(t)=\exp(bt)\cdot\psi_1(at)$.

sol.

$$\psi_2(t)=E(e^{ty})=E[e^{(ax+b)t}]=E[e^{axt}\cdot e^{bt}]=e^{bt}E(e^{axt})$$

$$=e^{bt}\cdot\psi_1(at)=\exp(bt)\,\psi_1(at)$$

since $\psi_1(t)=E(e^{tx})$

$\therefore \quad \psi_1(at)=E(e^{atx})$

نظرية 4 درجة

(10) If X is a r.v. with $M_X(t)$ exists for $t\in(-h,0)$, then

$$P(X\leq a)\leq \exp(-at)\cdot M_X(t)$$

sol. $M_X(t)=E(e^{tx})$

$$y-e^{tx}>0$$

$$e^{at}>0$$

$\therefore P(X\geq t)\leq \frac{E(x)}{t}$ by Markave

$$P(Y\geq e^{at})\leq\frac{E(Y)}{e^{at}}$$

$$P(e^{tx}\geq e^{at})\leq \exp(at)\cdot M_X(t)$$

$\therefore P(X\leq a)\leq \exp(at)\cdot M_X(t)$

X صفر

(11) Given a m.g.f. $M_X(t) = e^{3t^2+2t}$ for $-\infty < t < \infty$
Find the L.b. of $P(-\frac{1}{2} < X < \frac{9}{2})$.

Sol.

$$M'_X(t) = \left(e^{3t^2+2t}\right)(6t+2)$$

$$E(X) = M'_X(0) = e^0(0+2)$$

$$M''_X(t) = \left(e^{3t+2t}\right)(6) + (6t+2)^2\, e^{3t^2+2t}$$

$$E(X^2) = (e^0)(6) + (2)^2(e^0) = 6+4 = 10$$

$$V(X) = E(X^2) - [E(X)]^2 = 10 - (2)^2 = 6$$

$$\therefore P(-\tfrac{1}{2} < X < \tfrac{9}{2}) = P(-\tfrac{1}{2} - 2 < X-2 < \tfrac{9}{2} - 2)$$

$$= P(-2.5 < X-2 < 2.5)$$

$$= P(|X-2| < 2.5) \Rightarrow t = \tfrac{5}{2} = 2.5$$

$$L.b. = 1 - \frac{V(X)}{t^2} = 1 - \frac{6}{(\frac{5}{2})^2} = 1 - 6 \cdot \frac{4}{25} = 1 - \frac{24}{25}$$

$$= \frac{1}{25} \simeq 0.04$$

(12) Given a p.m.f. $f(x) = \begin{cases} \frac{x}{10} & \text{for } x = 0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$
Find the pr. percentile at the point $x = 3.5$.

Sol.

$$P(X \le x_0) = \frac{t}{100}$$

$$P(X \le 3.5) = \frac{t}{100} \Rightarrow \sum_{x=0}^{3} f(x) = \frac{t}{100}$$

$$f(0) + f(1) + f(2) + f(3) = \frac{t}{100}$$

$$\frac{0}{10} + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = \frac{t}{100} \Rightarrow \frac{6}{10} = \frac{t}{100} \Rightarrow 10t = 600$$

$$\boxed{t = 60\,\%}$$

(13) Find the mean, median and mode of Cauchy dist. if exists.

Sol. $f(x) = \dfrac{1}{\pi(1+x^2)}$ $\quad -\infty < x < \infty \quad$ (Cauchy dist.)

Mean: $E(X)$ exist only when $E(|X|) < \infty$

$$E(|X|) = \int_{-\infty}^{\infty} |x| \frac{1}{\pi(1+x^2)} dx = \int_0^{\infty} \frac{2x}{\pi(1+x^2)} dx = \frac{1}{\pi} \ln(1+x^2)\Big|_0^{\infty}$$

$$= \frac{1}{\pi}[\ln(\infty) - \ln(0)] = \infty$$

$E(X)$ not exist.

Median $P(X \leq m) \geq \frac{1}{2}$ & $P(X < m) \leq \frac{1}{2}$

$P(X < m) \leq \frac{1}{2}$

$$\int_{-\infty}^{m} \frac{1}{\pi(1+x^2)}\, dx \leq \frac{1}{2}$$

$$\frac{1}{\pi} \int_{-\infty}^{m} \frac{1}{1+x^2}\, dx \leq \frac{1}{2}$$

$$\frac{1}{\pi} \tan^{-1} x \Big|_{-\infty}^{m} \leq \frac{1}{2}$$

$$\frac{1}{\pi} [\tan^{-1}(m) - \underbrace{\tan^{-1}(\infty)}_{-\pi/2}] \leq \frac{1}{2}$$

$$\tan^{-1}(m) + \frac{\pi}{2} \leq \frac{\pi}{2}$$

$$\tan^{-1}(m) \leq 0$$

$$m \leq \tan(0)$$

$$\therefore m \leq 0$$

$P(X \leq m) \geq \frac{1}{2}$

$$\int_{-\infty}^{m} \frac{1}{\pi(1+x^2)}\, dx \geq \frac{1}{2}$$

$$\frac{1}{\pi} \int_{-\infty}^{m} \frac{1}{(1+x^2)}\, dx \geq \frac{1}{2}$$

$$\frac{1}{\pi} \tan^{-1} x \Big|_{-\infty}^{m} \geq \frac{1}{2}$$

$$\frac{1}{\pi} [\tan^{-1}(m) - \tan^{-1}(\infty)] \geq \frac{1}{2}$$

$$\tan^{-1}(m) \geq 0$$

$$m \geq \tan(0)$$

$$\therefore m \geq 0$$

$\therefore \boxed{m_e = 0}$ median.

Mode $f(x) = \frac{1}{\pi}(1+x^2)^{-1} \Longrightarrow f'(x) = -\frac{1}{\pi}(1+x^2)^{-2}(2x) = 0$

$$\frac{-2x}{\pi(1+x^2)^2} = 0 \Longrightarrow x = 0$$

$$f''(x) = \frac{2}{\pi}(1+x^2)^{-3}(2x)(2x) - \frac{1}{\pi}(1+x^2)^{-2}(2)$$

$$f''(0) = 0 - \frac{2}{\pi} < 0 \Rightarrow mode = 0$$

$\therefore \boxed{M_0 = 0}$ mode.

(14) Given a p.d.f. $f(x) = \begin{cases} e^x & \text{for } x < 0 \\ 0 & \text{o.w.} \end{cases}$

Find $M_X(t)$ and sketch it's graph. Also, find $E(x)$ by two methods.

Sol.

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{0} e^{tx} e^x\, dx = \int_{-\infty}^{0} e^{x(1+t)}\, dx$$

This integration exists only when $(1+t) > 0 \Longrightarrow \boxed{t > -1}$

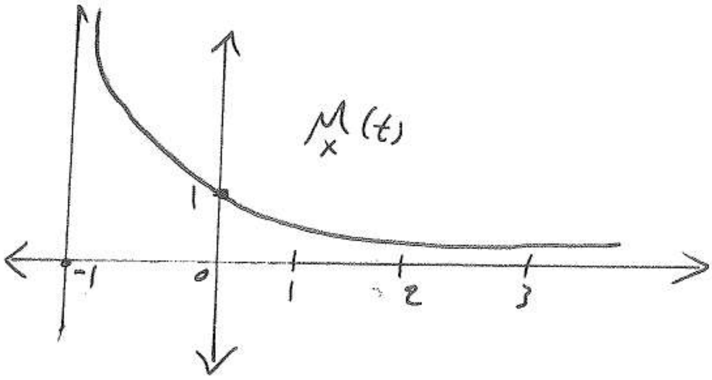
$$M_x(t) = \frac{1}{1+t} e^{x(1+t)} \Big|_{-\infty}^{0} = \frac{1}{1+t}\left[e^{0} - e^{-\infty}\right] = \frac{[1-0]}{1+t} = \frac{1}{1+t}$$

$$M_x(t) = \begin{cases} \frac{1}{1+t} & \text{for } t > -1 \\ 0 & \text{o.w.} \end{cases}$$

must $1+t>0$

$t>-1$

t	$M_x(t)$
-1	∞
0	1
1	$1/2$
2	$1/3$
3	$1/4$
$\vdots$	
∞	0



① $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cdot e^{x}\,dx$

$u = x$, $dv = e^{x}$

$du = 1$, $v = e^{x}$

$$= uv\Big|_{-\infty}^{0} - \int_{-\infty}^{0} v\,du$$

$$= xe^{x}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}\,dx$$

$$= \left[0\cdot e^{0} - (-\infty)e^{-\infty}\right] - e^{x}\Big]_{-\infty}^{0} = [0-0] - [e^{0} - e^{-\infty}] = -1$$

$$M_x(t) = \frac{1}{1+t} = (1+t)^{-1} \quad ; t > -1$$

$$M_x'(t) = -1(1+t)^{-2}(1)$$

$$E(X) = M_x'(0) = (-1)[1+0]^{-2} = -1$$

⑮ If $X \sim unif(0,2)$, Find the u.b. of $P(|X-\mu| \geq 2)$.

Sol: $u.b. = \frac{V(x)}{t^2}$, $t = 2$

$$f(x) = \begin{cases} \frac{1}{2-0} = \frac{1}{2} & 0 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu = E(X) = \int_0^2 x\, f(x)\,dx = \int_0^2 x \cdot \frac{1}{2}\,dx = \frac{1}{2}\cdot\frac{x^2}{2}\Big|_0^2 = \frac{1}{4}[4-0] = \frac{4}{4} = 1$$

$$\boxed{\mu = E(X) = 1}$$

$$E(X^2) = \int_0^2 x^2 \cdot \frac{1}{2}\,dx = \frac{1}{2}\frac{x^3}{3}\Big|_0^2 = \frac{1}{6}[8-0] = \frac{8}{6} = \frac{4}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$
$$= \frac{4}{3} - (1)^2$$
$$= \frac{4}{3} - \frac{3}{3} = \frac{1}{3}$$
$$U.b. = \frac{V(X)}{t^2} = \frac{(\frac{1}{3})}{(2)^2} = \frac{1/3}{4} = \frac{1}{12}$$

(16) Find the pr. percentite at the point $x=1$, if ~~[illegible]~~ Given a p.d.f.

$$f(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

$$P(X \leq 1) = \frac{t}{100} \Rightarrow \int_0^1 f(x)\,dx = \frac{t}{100} \Rightarrow \frac{3}{8}\int_0^1 x^2\,dx = \frac{t}{100}$$
$$\frac{3}{8} \cdot \frac{x^3}{3}\Big|_0^1 = \frac{1}{8} = \frac{t}{100} \Rightarrow t = \frac{100}{8} = 12.5\,\% = 0.125$$

(17) A Coin is tossed 4-times, X = number of heads. Find the mean, median and mode of X (if exists).

Sol.

$$f(x) = \begin{cases} \frac{\binom{4}{x}}{16} & x = 0, 1, 2, 3, 4 \\ 0 & \text{o.w.} \end{cases}$$

X	$f(x) = P(X = x)$
0	$f(0) = \frac{1}{16}$
1	$f(1) = \frac{4}{16}$
2	$f(2) = \frac{6}{16}$
3	$f(3) = \frac{4}{16}$
4	$f(4) = \frac{1}{16}$

$$\sum_{x=0}^{4} f(x) = \frac{16}{16} = 1$$

$f(2)$ is a Maximam

$\therefore$ mode $= 2$

Median

~~If $m=0$~~

$P(X<0) \leq \frac{1}{2}$; $P(X \leq 0) \geq \frac{1}{2}$

$0 \leq \frac{1}{2}$ $\qquad f(0) = \frac{1}{16} \not\geq \frac{1}{2}$

$\therefore \boxed{m \neq 0}$

If $m=1$

$P(X<1) \leq \frac{1}{2}$; $P(X \leq 1) \geq \frac{1}{2}$

$f(0) = \frac{1}{16} < \frac{1}{2}$; $f(0)+f(1) \geq \frac{1}{2}$

$\frac{1}{16} + \frac{4}{16} \geq \frac{1}{2}$

$\frac{15}{16} \not\geq \frac{1}{2}$

$\boxed{\therefore m \neq 1}$

if $m=2$

$P(X<2) \leq \frac{1}{2}$; $P(X \geq 2) \geq \frac{1}{2}$

$f(0)+f(1) \leq \frac{1}{2}$ $\quad$ $f(0)+f(1)+f(2) \geq \frac{1}{2}$

$\frac{1}{16}+\frac{4}{16} \leq \frac{1}{2}$ $\quad$ $\frac{1}{16}+\frac{4}{16}+\frac{6}{16} \geq \frac{1}{2}$

$\frac{5}{16} \leq \frac{1}{2}$ $\quad$ $\frac{11}{16} \geq \frac{1}{2}$

$\therefore$ median $= \boxed{m=2}$

Mean

$$E(x) = \sum_{x=0}^{4} x\, f(x) = 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3) + 4 \cdot f(4)$$

$$= 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16}$$

$$= \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16} = \frac{32}{16} = 2 = \mu$$

$\therefore \boxed{\mu = 2}$

(18) If $X \sim b(n,p)$, then find the value of (n) s.t.

$$P\left(\left|\frac{X}{n}-p\right| < 0.1\right) \geq 0.95$$

Sol. By theorem (chebyshive theorem)

$$P\left(\left|\frac{X}{n}-p\right|<0.1\right) \geq 0.95 \overset{\text{مقارنة}}{\Longleftrightarrow} P\left(\left|\frac{X}{n}-\mu\right|<t\right) \geq 1-\frac{V(x)}{t^2}$$

$$= P(|X-np|<(0.1)n) = P\left(|X-\mu|<\frac{n}{10}\right) \Rightarrow \boxed{t=\frac{n}{10}}, \ \mu = np$$

$$L.b. = 1-\frac{V(x)}{t^2}$$

$$V(x) = np(1-p)$$

$$L.b. = 1-\frac{np(1-p)}{\left(\frac{n}{10}\right)^2} = 1-\left[np(1-p)\frac{100}{n^2}\right]$$

$$= 1-\frac{100\,p(1-p)}{n} \geq 0.95$$

$$\frac{100\,p(1-p)}{n} \leq 0.05 = \frac{5}{100} = \frac{1}{20}$$

$$2000\,p(1-p) \leq n$$

$\boxed{n \geq 2000\,p(1-p)}$ $\quad \forall p \quad 0<p<1$

6

(19) If X has a M.g.f. as follows:

$$M_X(t) = \frac{2e^t}{5(1-\frac{3}{5}e^t)} \quad \text{for} \quad t < \ln(\tfrac{5}{3}), \text{ then find } P(X>7 \mid X>3).$$

memoryless

Sol.

$$M_X(t) = \frac{(\frac{2}{5})e^t}{(1-\frac{3}{5}e^t)} = \frac{pe^t}{1-qe^t}$$

$$\therefore X \sim G(p = \tfrac{2}{5})$$

$$\therefore P(X>7/X>3) = P(X>4) = (q)^4 = (\tfrac{3}{5})^4.$$

(20) If X has a p.m.f.

$$f(x) = \begin{cases} (\frac{2}{3})(\frac{1}{3})^{x-1} & \text{for } x = 1, 2, 3, \dots \\ 0 & \end{cases}$$

6

Find $P(X>5 \mid X>2)$

① which distribution that X have

② write the m.g.f. of X, $E(X)$ & $Var(X)$

Sol. $X \sim G(\frac{2}{3})$, $p = \frac{2}{3}$, $q = \frac{1}{3}$

$$\therefore P(X>5/X>2) = P(X>3) = (q)^3 = (\tfrac{1}{3})^3 = \frac{1}{27}$$

③ Compute $P(X>5 \mid X>2)$

(21) If $X \sim P(\lambda)$, then show that:

$E(X) = V(X) = \lambda$

derive the Expected value of X and the variance of X

$$\therefore M_X(t) = e^{\lambda(e^t-1)}$$

6

$$M'_X(t) = e^{\lambda(e^t-1)}(\lambda e^t)$$

$$E(x) = M'_X(0) = e^0(\lambda)e^0 = \lambda$$

$$M''_X(t) = e^{\lambda(e^t-1)}(\lambda e^t) + (\lambda e^t)e^{\lambda(e^t-1)}(\lambda e^t)$$

$$E(X^2) = M''_X(t=0) = 1.(\lambda) + (\lambda).(1).(\lambda)$$

$$E(X^2) = \lambda + \lambda^2$$

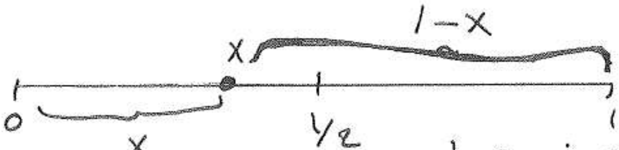
$$V(X) = E(X^2) - [E(X)]^2$$

$$= \lambda + \lambda^2 - \lambda^2 = \lambda$$

$$\boxed{E(X) = V(X) = \lambda}$$

(22) A point X is chosen at random from a stick of length one unite, and then the stick is broken at the chosen point into two unequal parts, find the expected value of longer part.

Sol.



مسطرة (stick)

تنقسم المسطرة الى قسمين غير متساويين ، فاحد الاقسام صغير X والآخر كبير (1−X)

$\because$ X is shorter part, (1−X) is longer part.

$\therefore \; X \in (0, 1/2)$

$\therefore \; X \sim unif.(0, \frac{1}{2})$

$$f(x) = \begin{cases} \frac{1}{1/2 - 0} = 2 & \text{for } 0 < x < \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

Find E[longer part]

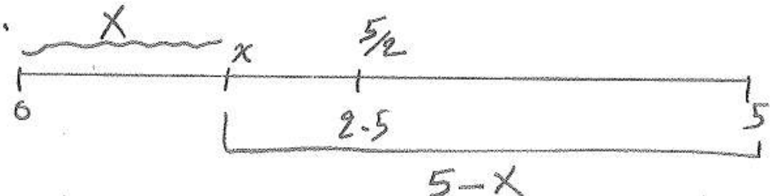
i.e Find E(1−X) ?

$$E(1-X) = 1 - E(X)$$
$$= 1 - \int_0^{1/2} x \cdot (2)\,dx = 1 - \frac{2}{2} x^2 \Big|_0^{1/2} = 1 - [(\tfrac{1}{2})^2 - 0]$$
$$= 1 - 1/4 = 3/4$$

$\therefore E(1-X) = \frac{3}{4}$ (القيمة المتوقعة للجزء الأكبر)

(23) A point X is chosen on a line of long 5 cm. This chosen point divides the line into two unequal parts. Find the expectation of shorter part.

Sol.



Let X divide the line into unequal two parts.

$\because$ X is the shorter part, (5−x) is the longer part.

$\therefore \; X \in (0, \frac{5}{2})$

$\therefore \; X \sim unif(0, \frac{5}{2})$

$$f(x) = \begin{cases} \frac{1}{\frac{5}{2} - 0} = \frac{2}{5} & \text{for } 0 < x < \frac{5}{2} \\ 0 & \text{o.w.} \end{cases}$$

Find E(shorter part) = E(x) ?

$$E(X) = \int x\, f(x)\, dx = \int_0^{5/2} x \left(\tfrac{2}{5}\right) dx = \tfrac{\cancel{2}}{5} \cdot \tfrac{x^2}{\cancel{2}} \Big|_0^{5/2}$$

$$= \tfrac{1}{5}\left[\left(\tfrac{5}{2}\right)^2 - 0\right] = \tfrac{1}{\cancel{5}} \cdot \tfrac{\cancel{25}}{4} = \tfrac{5}{4} = 1.25$$

(24) Given a p.d.f.

$$f(x) = \begin{cases} x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

ليست دالة احتمالية

Find mode of X.

Sol. $f(x) = 12x^2 - 2x^3$

$f'(x) = 24x - 36x^2 = 0$

$12x(2 - 3x) = 0$

$(12x = 0)$ or $(2 - 3x = 0)$

$x_1 = 0$ or $x_2 = \frac{2}{3}$

$$\int_0^1 12x^2 = \tfrac{12}{3} x^3 \Big|_0^1 = 4[1-$$

$$\int_0^1 ax^2 = 1$$

$$\tfrac{a}{3} x^3 \Big|_0^1 = 1$$

$$\tfrac{a}{3} = 1 \qquad a = 3$$

$f''(x) = 24 - 72x$

if $(x = 0) \Rightarrow f''(0) = 24 \Rightarrow f(0)$ is min. then mode $\neq 0$

if $(x = \frac{2}{3}) \Rightarrow f''(\frac{2}{3}) = 24 - 72(\frac{2}{3}) = -24 < 0$ then $f(\frac{2}{3})$ is Max.

$\therefore$ mode $= \frac{2}{3}$

(25) If X has a poisson dist. with parameter m, then $E(X) = V(X) = m$.

Theorem

Derive

Sol. $M_X(t) = e^{m(e^t - 1)}$ (by theorem)

$M'_X(t) = e^{m(e^t - 1)} \cdot me^t$

$\therefore E(X) = M'_X(0) = m\, e^{m(e^0 - 1)} = m$

$M''_X(t) = e^{m(e^t - 1)} \cdot me^t + me^t \cdot e^{m(e^t - 1)} \cdot (me^t)$

$E(X^2) = e^0 m e^0 + me^0 \cdot e^0 (me^0)$

$= m + m^2$

$V(X) = E(X^2) - [E(X)]^2$

$V(X) = (m + \cancel{m^2}) - \cancel{m^2}$

$V(X) = m = E(X)$

(26) Given a m.g.f. of X, $M_X(t) = \frac{1}{1-2t}$ for $t < \frac{1}{2}$
Find the mean and variance of X.

Sol. $M_X(t) = \frac{1}{1-2t} = (1-2t)^{-1}$

$M'_X(t) = -(1-2t)^{-2}(-2) = 2(1-2t)^{-2}$

$M''_X(t) = -4(1-2t)^{-3}(-2) = 8(1-2t)^{-3}$

$M'_X(0) = 2$, $M''_X(0) = 8$

$\therefore E(X) = 2$, $E(X^2) = 8$ (by theorem)

$V(X) = E(X^2) - [E(X)]^2$
$= 8 - (2)^2 = 4$

4

(27) If X has a uniform dist. on $(-\sqrt{3}, \sqrt{3})$, then find the upper bound of $P(|X-\mu| \geq \frac{3}{2})$.

Sol. $U.b. = \frac{V(x)}{t^2}$

$f(x) = \frac{1}{\sqrt{3}-(-\sqrt{3})} = \begin{cases} \frac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < x < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$

$E(x) = \int_{-\sqrt{3}}^{\sqrt{3}} x \left(\frac{1}{2\sqrt{3}}\right) dx = 0$

$E(x^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \left(\frac{1}{2\sqrt{3}}\right) dx = 1$

$\therefore V(x) = E(x^2) - (E(x))^2$
$= 1 - 0^2 = 1 \Rightarrow V(x) = 1$

$U.b. = \frac{V(x)}{t^2} = \frac{1}{(\frac{3}{2})^2} = \frac{4}{9} = 0.44$

(28) If X has a geometric dist. with $p = \frac{1}{4}$, then find $P(X > 8 | X > 3)$.

Sol. $p = \frac{1}{4} \Rightarrow q = \frac{3}{4}$

$P(X > 8 | X > 3) = P(X > 5) = [P(A^c)]^5 = q^5 = (\frac{3}{4})^5$

6

(29) If $X \sim b(n,p)$, then the M.g.f. of X is

$$M_X(t) = (1-p+pe^t)^n$$

Sol.

$\because X \sim b(n,p)$

$$\therefore f(x,n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x=0,1,2,\dots,n \\ 0 & \text{o.w.} \end{cases}$$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

since $(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}$

Let $b = pe^t$ and $a = 1-p$

then $(1-p+pe^t)^n = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = M_X(t)$

$$\therefore \boxed{M_X(t) = (1-p+pe^t)^n}$$

(30) Given a p.d.f. $f(x) = \begin{cases} e^{-x} & \text{for } x>0 \\ 0 & \text{o.w.} \end{cases}$

Find and sketch graph of $M_X(t)$.

① which distribution that X have
② derive $M_X(t)$
③ $E(X)$ $V(X)$

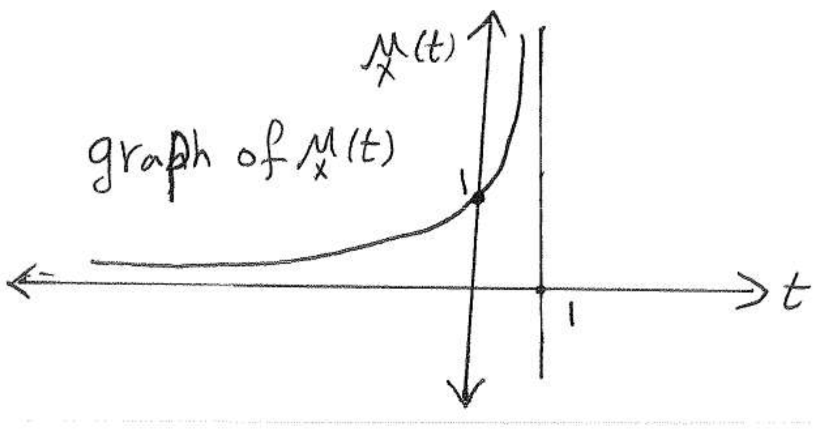
Sol.

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} (e^{-x})\, dx = \int_0^\infty e^{-(1-t)x}\, dx$$

$e^{-(1-t)x}$ exists when $t<1$

$$M_X(t) = \frac{-1}{1-t} \int_0^\infty e^{-(1-t)x} (-(1-t))\, dx = \frac{1}{1-t} \quad \text{for } t<1$$

t	$M_X(t) = \frac{1}{1-t}$
1	∞
0	1
-1	.5
$\vdots$	
$-\infty$	0



(31) Given $M_x(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$, if $y = 1-2X$, then find $M_y(t)$.

Sol. $M_y(t) = e^{bt} M_x(at)$, $y = ax+b$

$M_x(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$

$M_y(t) = e^{t} M_x(-2t)$, $y = 1-2X$

where $M_x(-2t) = \frac{1}{1+6t}$ for $t > \frac{-1}{6}$

(32) If a r.v. X with $M_x(t)$ for $-h<t<h$ show that $P(X \geq a) \leq e^{-at} M_x(t)$ for $0<t<h$

and $P(X \leq a) \leq e^{-at} M_x(t)$ for $-h<t<0$

Proof $M_x(t) = E(e^{tx})$, $y = e^{tx}$

$e^{at} > 0$; $a>0$ or $t>0$

$P(X \geq t) \leq \frac{E(X)}{t}$ (by theorem)

$P(Y \geq e^{at}) \leq \frac{E(Y)}{e^{at}}$

$P(e^{tx} \geq e^{at}) \leq \frac{E(e^{tx})}{e^{at}}$

$P(\ln e^{tx} \geq \ln e^{at}) \leq \frac{M_x(t)}{e^{at}} = e^{-at} M_x(t)$

$P(tx \geq at) \leq e^{-at} M_x(t)$ for $0<t<h$

$P(X \geq a) \leq e^{-at} M_x(t)$ —— ①

$P(tx \geq at) \leq e^{-at} M_x(t)$

$P(X \leq a) \leq e^{-at} M_x(t)$ for $-h<t<0$ —— ②

(33) If X has a geometric dist. with parameter p, then the m.g.f. of X is $M_x(t) = \frac{pe^t}{1-qe^t}$ for $t < \ln(\frac{1}{q})$.

Sol. $X \sim G(p,q) \Rightarrow f(x) = \begin{cases} p\, q^{x-1} & , x = 1,2,3,\ldots \\ 0 & \text{o.w.} \end{cases}$

derive

$$M_X(t) = E(e^{tx})$$
$$= \sum_{x=1}^{\infty} e^{tx} f(x)$$
$$= \sum_{x=1}^{\infty} e^{tx}(pq^{x-1}) = \frac{p}{q}\sum_{x=1}^{\infty}(qe^t)^x$$
$$= \frac{p}{q}\left[qe^t + (qe^t)^2 + \cdots + (qe^t)^n + \cdots\right]$$
$$= \frac{p}{q}\cdot qe^t\left[1 + qe^t + q^2e^{2t} + \cdots + q^{n-1}e^{(n-1)t} + \cdots\right]$$

$$\therefore\ r = \frac{qe^t}{1} = qe^t$$

$$|r| < 1 \Longrightarrow qe^t < 1 \Rightarrow e^t < \frac{1}{q} \Rightarrow \ln e^t < \ln\left(\frac{1}{q}\right)$$
$$\Rightarrow t < \ln\left(\frac{1}{q}\right)$$

$$S = \underbrace{1 - qe^t + (qe^t)^2 + \cdots}_{\text{Geometriq series}} \quad , \quad S = \frac{1}{1-r} = \frac{1}{1-qe^t}$$

$$M_X(t) = \frac{pe^t}{1-qe^t} \quad \text{for } t < \ln\left(\frac{1}{q}\right)$$

(34) $X \sim N(\mu, \sigma^2)$, $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$; find $E(Z)$, $V(Z)$.

Sol.

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right)$$
$$= E\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right)$$
$$= E\left(\frac{X}{\sigma}\right) - \frac{\mu}{\sigma}$$
$$= \frac{1}{\sigma}E(x) - \frac{\mu}{\sigma}$$
$$= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$$

$$Var(Z) = V\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right)$$
$$= \frac{V(X)}{\sigma^2} - V\left(\frac{\mu}{\sigma}\right)$$
$$= \frac{\sigma^2}{\sigma^2} - 0$$
$$= 1$$

$$M_Z(t) = e^{(tZ)} = E\left[e^{\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right)t}\right]$$
$$= e^{-\frac{\mu}{\sigma}t} E\left[e^{\frac{X}{\sigma}t}\right]$$
$$= e^{-\frac{\mu}{\sigma}t} M_{\frac{X}{\sigma}}(t)$$
$$= \cancel{e^{-\frac{\mu}{\sigma}t}}\ \cancel{e^{\frac{\mu}{\sigma}t}} + \frac{\sigma^2 t}{2\sigma}$$
$$= e^{\frac{\sigma^2 t^2}{2\sigma^2}} = e^{t^2/2} = e^{0 + \frac{t^2}{2}}$$

$$\therefore\ Z \sim N(0,1)$$

6

(35) If a r.v. X has Gamma dist. with parameters (a) & (b), then show that $V(x) = b \cdot E(x)$

Sol. $\because X \sim G(a,b)$

$\therefore E(x) = a \cdot b$ & $V(x) = a \cdot b^2$

$\therefore V(x) = (a \cdot b) \cdot b$

$= E(x) \cdot b$

$= b E(x)$

6

OR $\because X \sim G(a,b)$

$\therefore M_x(t) = (1-bt)^{-a}$

$M_x'(t) = (-a)(1-bt)^{-a-1} \cdot (-b) = (ab)(1-bt)^{-(a+1)}$

$M_x'(0) = (a \cdot b)(1-0)^{-(a+1)} = a \cdot b = E(x)$; (since $E(x) = M_x'(0)$)

$E(x^2) = ?$

$E(x^2) = M_x''(0)$

$M_x''(t) = (a \cdot b)(a+1)(1-bt)^{-(a+2)} \cdot (-b) = (ab^2)(a+1)(1-bt)^{-(a+2)}$

$E(x^2) = M_x''(0) = ab^2(a+1)(1-0) = a^2b^2 + ab^2$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$= (a^2b^2) + (ab^2) - (ab)^2 = ab^2$

$\therefore V(x) = (a \cdot b) \cdot b = E(x) \cdot b \Rightarrow V(x) = bE(x)$

(36) If a m.g.f. of X is as follows:

$M_x(t) = (1-2t)^{-7}$, then find the p.d.f. of x, E(x) & V(x).

Sol. $\beta = 2$, $\alpha = 7 \rightarrow r = 14$

$\therefore X \sim \chi^2(14)$ chi-squar with (14) d.f.

3

$\therefore f(x) = \begin{cases} \dfrac{x^6 e^{-\frac{x}{2}}}{\Gamma(7) 2^7} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$

6

$E(x) = r = 14$, $\sigma^2 = V(x) = 2r = 2(14) = 28$.

Find $P(X \leq a) = 0.95$
the value of a

(37) Given a p.d.f.

$$f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

Find the lower bounded of $P(\frac{5}{4} < X < \frac{11}{4})$.

Sol. $l.b = 1 - \frac{V(X)}{t^2}$

we must find $E(X)$, $E(X^2)$:

$$E(X) = \int_0^3 x f(x) dx = \int_0^3 x (\tfrac{2}{9} x) dx = \tfrac{2}{9} [\int_0^3 x^2 dx]$$

$$= \tfrac{2}{9} [\tfrac{x^3}{3}]_0^3 = \tfrac{2}{27} [27 - 0] = 2$$

$$E(X^2) = \int_0^3 x^2 f(x) dx = \int_0^3 x^2 (\tfrac{2}{9} x) dx = \tfrac{2}{9} [\int_0^3 x^3 dx]$$

$$= \tfrac{2}{36} x^4 \Big]_0^3 = \frac{1}{\cancel{18}_{2}} [\cancel{81}^{9}] = \tfrac{9}{2} = 4.5$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= 4.5 - (2)^2$$

$$= 4.5 - 4 = 0.5 = \tfrac{1}{2}$$

$$P(\tfrac{5}{4} < X < \tfrac{11}{4}) = P(\tfrac{5}{4} - 2 < X - 2 < \tfrac{11}{4} - 2)$$

$$= P(-\tfrac{3}{4} < X - 2 < \tfrac{3}{4})$$

$$= P(|X - 2| < \tfrac{3}{4})$$

$\therefore t = \frac{3}{4}$

$$l.b = 1 - \frac{V(X)}{t^2} = 1 - \frac{\frac{1}{2}}{(\frac{3}{4})^2} = 1 - \frac{1/2}{9/16} = 1 - \frac{8}{9} = \frac{1}{9}$$

$\therefore \boxed{l.b = \frac{1}{9}}$

(38) Given a p.d.f. of X

$$f(x) = \begin{cases} \frac{1}{4} e^{-\frac{x}{4}} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

① Find $M_x(t)$ ② If $y = 2 - x$, then find $M_y(t)$

Sol.

Sol.

① $M_X(t) = E(e^{tX}) = \frac{1}{4}\int_0^{\infty} e^{tx} \cdot e^{-\frac{1}{4}x}\, dx$

$$= \frac{1}{4}\int_0^{\infty} e^{-(\frac{1}{4}-t)x}\, dx$$

This integration exist only when $(\frac{1}{4}-t) > 0$

i.e $\frac{1}{4} > t \rightarrow t < \frac{1}{4}$

$\therefore M_X(t) = \frac{1}{4} \cdot \frac{-1}{(\frac{1}{4}-t)} e^{-(\frac{1}{4}-t)x} \Big|_0^{\infty} = \frac{-1}{4(\frac{1}{4}-t)}[e^{-\infty} - e^{0}]$

$$= \frac{-1}{4(\frac{1}{4}-t)}[0-1]$$

$$= \frac{1}{4(\frac{1}{4}-t)} \quad \text{for } t < \frac{1}{4}$$

② $Y = 2 - X$, $b = 2$, $a = -1$

$M_Y(t) = e^{bt} \cdot M_X(at)$

$= e^{2t} \cdot M_X(-t)$

$M_X(-t) = \frac{1}{4(\frac{1}{4}+t)}$ for $-t < \frac{1}{4}$

$M_X(-t) = \frac{1}{4(\frac{1}{4}+t)}$ for $t > \frac{-1}{4}$

$\therefore M_Y(t) = e^{2t} \cdot \frac{1}{4(\frac{1}{4}+t)}$ for $t > \frac{-1}{4}$

وزارة التعليم العالي والبحث والعلمي

جامعة ديالى

كلية تربية المقداد / قسم الرياضيات

محاضرات مادة

الإحصاء والاحتمالية

للعام الدراسي (2023-2024)

المرحلة الثالثة

Chapter Five

مدرس المادة: م.م هند إبراهيم محمد

Note: (Pr. dist. = Pr. fun. = f)

Probability Distribution of Two Random Variables

(I) Joint Probability Density Function (J.P.d.f.)

<u>Def.</u> Let X and Y be two c.r.v.'s. A function f defined over XY-plane is a joint p.d.f. of X and Y if for a subset $R \in$ XY-plane, then:

$$\boxed{P[(x,y)\in R] = \iint_R f(x,y)\,dx\,dy}$$

if $R = \{(x,y);\ a<x<b\ \&\ c<y<d\}$

Then
$$P[(x,y)\in R] = \int_c^d\int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b\int_c^d f(x,y)\,dy\,dx$$

Review

ch.5 (Pr. dist. of X & Y)

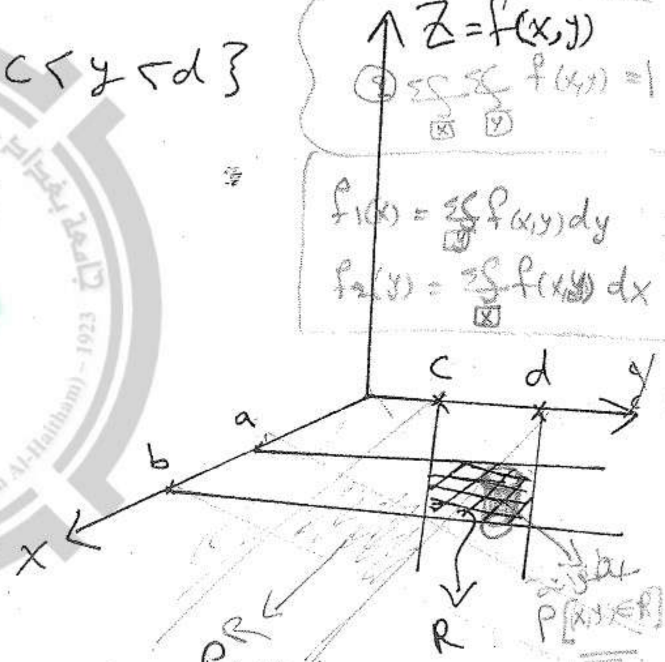
f(x,y) is J.P.d.f.

① $f(x,y) \geq 0\ \forall (x,y) \in R$

② $\sum\limits_{x}\sum\limits_{y} f(x,y) = 1$

$f_1(x) = \sum\int f(x,y)\,dy$

$f_2(y) = \sum\int f(x,y)\,dx$



Note: مقارنة بين J.P.f. و p.d.f.

① f(x,y) is a Solid over XY-plane. ⟶ f(x) is a curve

② R can be □, ▭, △, ○, ⬭, ... ⟶ (a,b) a—b

(= Domain of func. f(x,y))

③ P[(x,y)∈R] is the Volum inside Solid of f(x,y). ⟶ مساحة تحت المنحني Area under the curve

Properties of Joint p.d.f.

① A joint p.d.f. f(x,y) satisfies two conditions:

ⓐ $f(x,y) \geq 0 \quad \forall (x,y)\in R$

ⓑ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

② Also:

Region أنواع

ex.

① $R = \{(x,y);\ 0<x<1,\ 0<y<1\}$

② $R = \{(x,y);\ 0<x<y<1\}$

③ $R = \{(x,y);\ x^2+y^2 \leq 1\}$

∵ $f(x,y)$ is a function of x and y, from $f(x,y)$, we can find a function of X alone $f_1(x)$ and a function of Y alone $f_2(y)$.
s.t.

حدي

$f_1(x)$ is called Marginal p.d.f. of X

where $f_1(x) = \int_{\square}^{\square} f(x,y)\,dy$; ~~limits~~ of integral follows ~~limits~~ of y

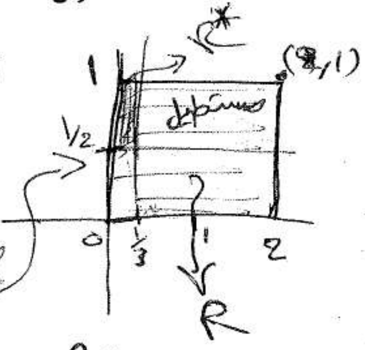
$f_2(y)$ is called Marginal p.d.f. of Y

where $f_2(y) = \int_{\square}^{\square} f(x,y)\,dx$; ~~limits~~ of integral follows ~~limits~~ of x

---

Example: Given a joint p.d.f. (J.p.d.f.) $f(x,y)$:

$$f(x,y) = \begin{cases} Kx^2y & \text{for } 0<x<2,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$



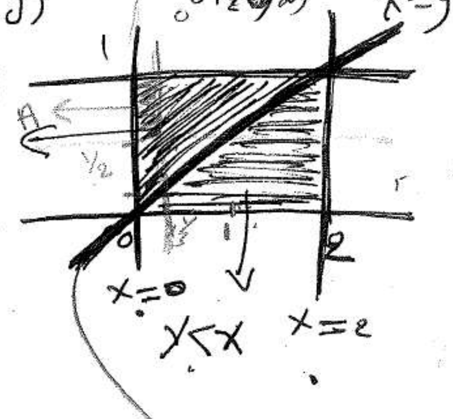
ملاحظة: في هذا المثال $\iint dxdy = \iint dydx$ لأن X و Y معزولة حدود

(a) Find the value of K, معزولة Isolation x, y

(b) Find $P(0<x<\frac{1}{3}, \frac{1}{2}<y<2)$, (f) $P(X>\frac{1}{2})$, $P(Y<\frac{1}{3})$

$\int_{1/2} f_1(x)\,dx$ $\qquad \int_0^{1/3} f_2(y)\,dy$

(c) Find $P(X+Y\geq 1)$, (d) Find $f_1(x)$ and $f_2(y)$



Sol. By cond. (2)

(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$R = \{(x,y): 0<x<2,\ 0<y<1\}$

$\int_0^1\int_0^2 K x^2 y\,dx\,dy = 1 \Rightarrow \int_0^1 Ky\left(\frac{x^3}{3}\Big|_0^2\right)dy = 1$

$\frac{1}{3}\int_0^1 Ky(8-0)\,dy = 1 \Rightarrow \frac{8K}{3}\int_0^1 y\,dy = 1$

الخط المساواة لا يعطيا المنطقة R لهذا يجب تحديد الأكبر والأصغر

$\frac{8K}{3}\,\frac{y^2}{2}\Big|_0^1 = 1 \Rightarrow \frac{8K}{6}[1^2-0] = 1 \Rightarrow \frac{4}{3}K = 1 \Rightarrow \boxed{K=\frac{3}{4}}$

ⓑ $f(x,y) = \begin{cases} \frac{3}{4}x^2 y & 0<x<2,\ 0<y<1 \\ 0 & \text{o.w} \end{cases}$

$$P\left(0<x<\tfrac{1}{3},\ \tfrac{1}{2}<y<2\right) = \int_{1/2}^{1}\int_{0}^{1/3} \frac{3}{4}x^2 y\, dx\, dy = \cdots = \frac{1}{32(9)} = \frac{1}{288}$$

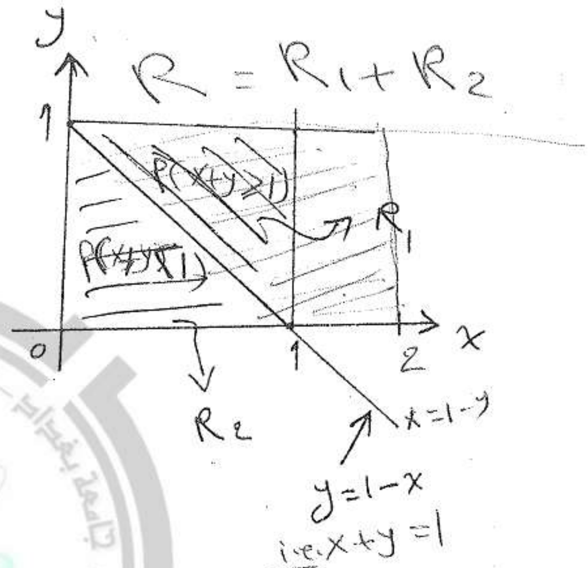
[illegible]

ⓒ $P(x+y \geq 1) = ?$ ⓓ $P(x+y<1) = ?$

Consider $(x+y)=1 \Rightarrow y = 1-x$

x	$y=1-x$
0	1
1	0

الرسم $\longrightarrow$



$P(x+y \geq 1) =$ Volum of $f(x,y)$ over R_1

$P(x+y<1) =$ Volum of $f(x,y)$ over R_2

$P(x+y \geq 1) = 1 - P(x+y<1)$

[illegible]

$$P(x+y<1) = \int_0^1\int_0^{1-x} f(x,y)\, dy\, dx$$

$$= \int_0^1 \left[\int_0^{1-x} \frac{3}{4}x^2 y\, dy\right] dx = \int_0^1 \frac{3}{4}x^2 \frac{y^2}{2}\Big|_0^{1-x} dx$$

$$= \int_0^1 \frac{3}{8}x^2\left[(1-x)^2 - 0\right] dx$$

$$= \frac{3}{8}\int_0^1 x^2\left[1-2x+x^2\right] dx$$

$$= \frac{3}{8}\left[\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5}\right]\Big|_0^1$$

$$= \frac{3}{8}\left[\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right) - (0-0+0)\right] = \frac{3}{8}\left[\frac{10-15+6}{30}\right]$$

$$= \frac{3}{8}\left(\frac{1}{30}\right) = \frac{1}{80}$$

$$P(x+y<1) = \frac{1}{80} \Rightarrow P(x+y \geq 1) = 1 - \frac{1}{80} = \frac{79}{80} \in [0,1]$$

ⓓ To find $f_1(x)$ & $f_2(y)$

$$f_1(x) = \int_0^1 f(x,y)\, dy$$

$$f_1(x)=\int_0^1 \frac{3}{4}x^2y\,dy=\frac{3}{4}x^2\frac{y^2}{2}\Big|_0^1=\frac{3}{8}x^2[1-0]=\frac{3}{8}x^2$$

$$f_1(x)=\begin{cases}\frac{3}{8}x^2 & \text{for } 0<x<2\\ 0 & \text{o.w.}\end{cases}$$

$$f_2(y)=\int_0^2 f(x,y)\,dx$$

$$=\int_0^2\frac{3}{4}x^2y\,dx=\frac{3}{4}y\frac{x^3}{3}\Big|_0^2=\frac{1}{4}y[8-0]=\frac{8}{4}y$$

$$=2y$$

$$f_2(y)=\begin{cases}2y & \text{for } 0<y<1\\ 0 & \text{o.w.}\end{cases},\quad P(y<\tfrac{1}{3})=\int_0^{1/3}2y\,dy=y^2\Big|_0^{1/3}=\frac{1}{9}$$

(لم يشرح)

* (يسمى اعتمادا على عدد المتغيرات)

univariat	احادي one
Bivariat	ثنائي two
multivariat	متعدد n

Example (10) Given a J.p.d.f.

$$f(x,y)=\begin{cases}2 & \text{for } 0<x<y<1\\ 0 & \text{o.w.}\end{cases}$$

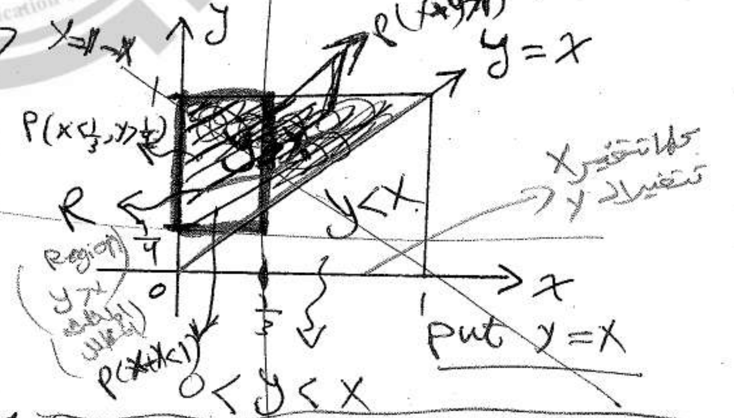
Find $f_1(x)$, $f_2(y)$, $P(X>\frac{1}{3})$, $P(Y<\frac{1}{2})$, $P(X<\frac{1}{3})$, Find (1) $P(X>\frac{1}{2})$ (2) $P(Y<\frac{1}{2})$ (3) $P(X>\frac{1}{2}, Y<\frac{1}{2})$

$P(X>1/2, Y<1/2)=\int_{1/2}\int_0^{1/2} f(x,y)\,dy\,dx=$

Sol.

$R=\{(x,y);\ 0<x<y<1\}$, Consider $y=x$, then:

$$R=\begin{cases}0<x<y & ; & x<y<1\\ 0<x<1 & ; & 0<y<1\end{cases}$$



$$f_1(x)=\int_x^1 2\,dy \quad \text{for } 0<x<1$$

$$=2y\Big|_x^1=2[1-x]$$

$$f_1(x)=\begin{cases}2(1-x) & \text{for } 0<x<1\\ 0 & \text{o.w.}\end{cases}$$

*(H.W. Prove that f_1 & f_2 are pr. fun.)

$$f_2(y)=\int_0^y 2\,dx \quad \text{for } 0<y<1$$

$$=2x\Big|_0^y=2y$$

$$f_2(y)=\begin{cases}2y & \text{for } 0<y<1\\ 0 & \text{o.w.}\end{cases}$$

2010/2/10

Find $P(X+Y<1)=?$ → $X+Y=1$, $Y=1-X$

تقاطع المستقيمات في:

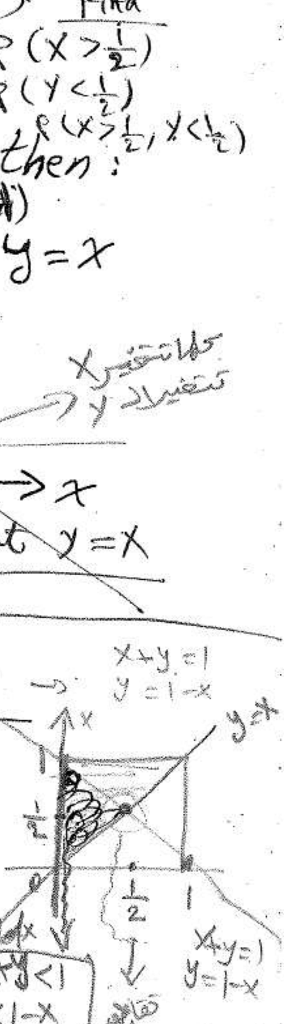
$Y=X$, $Y=1-X$

$X=1-X$ → $\boxed{X=\frac{1}{2}}$

$(X,Y)=(\frac{1}{2},\frac{1}{2})$

$\therefore P(X+Y<1)=\int_0^{1/2}\int_0^{1-x} f(x,y)\,dy\,dx$

$X+Y<1$
$Y<1-X$



$$P(X > \tfrac{1}{3}) = \int_{\frac{1}{3}}^{1} f_1(x)\,dx$$

$$= \int_{\frac{1}{3}}^{1} 2(1-x)\,dx = \frac{8}{18}$$

$$P(Y < \tfrac{1}{2}) = \int_{0}^{\frac{1}{2}} 2y\,dy = \cdots = \frac{1}{4}$$

$$P(X < \tfrac{1}{3}, Y > \tfrac{1}{4}) = \int_{\frac{1}{4}}^{1}\underbrace{\int_{0}^{\frac{1}{3}} 2\,dx}_{x}\,dy = \cdots = \frac{1}{2}$$

* لو لم يطلب f_1 f_2 ويطلب $\left(\begin{matrix} P(X<\frac{1}{4}) \\ P(Y>\frac{1}{2}) \end{matrix}\right)$ نحسب الـ f ضمنياً implicit

Find $P(X+Y \geq 1) = ?$

(II) Joint Probability Mass Function (J.P.m.f.)

Def. Let X and Y be two d.r.v.. A function $f(x,y)$ is a joint p.m.f. of X and Y iff

$$\boxed{f(x,y) = P(X=x, Y=y) \quad \forall (x,y) \in R}$$

X is d.r.v.
$f(x)$ is p.m.f.
$f(x) = P(X=x)$
$f(2) = P(X=2)$

and satisfies two conditions:

(a) $f(x,y) \geq 0 \quad \forall (x,y) \in R$

(b) $\underbrace{\sum_{\forall y}\sum_{\forall x}}_{\forall R} f(x,y) = 1$

Also,

$f_1(x) = \sum_{\forall y} f(x,y)$, limit of sumation followes limit of y.

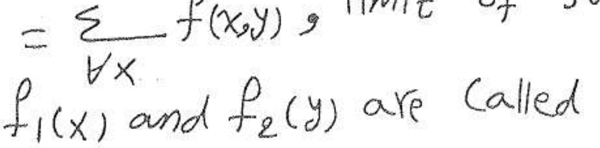
$f_2(x) = \sum_{\forall x} f(x,y)$, limit of Sumation followes limit of X.

where $f_1(x)$ and $f_2(y)$ are Called Marginal p.m.f. of X and Y respectively.

بالتعاقب

Example: A box has (3) balls (1), (2), (3). Choose two balls one by one without replac.; find $f(x,y)$.

Sol.



$n=3 \qquad k=2$

$$S = \left\{ \begin{matrix} \underset{(x_1,y_1)}{(1,2)} & \underset{(x_2,y_1)}{(2,1)} & \underset{(x_3,y_1)}{(3,1)} \\ \underset{(x_1,y_3)}{(1,3)} & \underset{(x_2,y_3)}{(2,3)} & \underset{(x_3,y_2)}{(3,2)} \end{matrix} \right\} ; \; S \text{ has "6" elts.}$$

OR A box has (3) balls which are numbered 1, 2, 3.
مرقمة

كم حالة لسحب 2 من 3 بدون ارجاع

سؤال مقترح 2

$P_2^3 = \frac{3!}{1!} = 6 = n(S)$

(نفس السؤال مع تغيير طريقة السحب (1-1) with rep.) $n(S) = (3)^2 = 9$

Graph of $f(x,y)$

$X = 1^{st}$ chosen ball ; $x = 1, 2, 3$ — تمثل الاختيار الاول

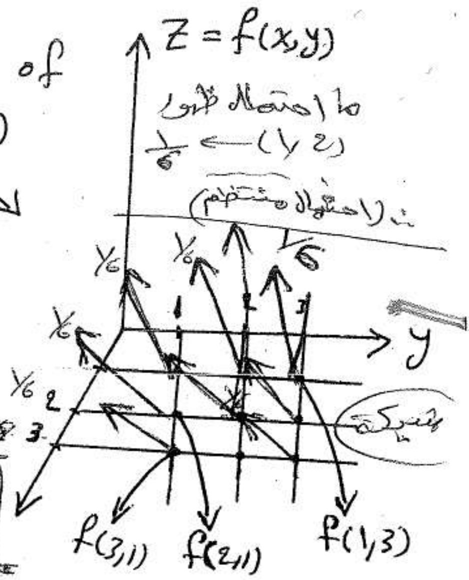
$Y = 2^{nd}$ chosen ball ; $y = 1, 2, 3$ $\Rightarrow X \neq Y$ — تمثل الاختيار الثاني

$P(X=2, Y=1) = \frac{1}{6} = f(2,1)$

$P(X=3, Y=2) = \frac{1}{6} = f(3,2)$ ex. لا توجد هذه الحالة $f(1,1) = 0$

$f(x,y) = P(X=x; Y=y) = \frac{1}{6}$

$\boxed{P(X \geq 2, Y \leq 2) = \sum_{x=2,3} \sum_{y=1,2} f(x,y)}$



قيمة الدالة ثابتة uniform(K)

$$f(x,y) = \begin{cases} \frac{1}{6} & \text{for } x = 1,2,3 \ ; \ y = 1,2,3 \ \& \ x \neq y \\ 0 & \text{o.w.} \end{cases}$$

الرسم الهندسي بالبعد (3) (3-dim.)

$\boxed{P(X=1) = f_1(1)}$

Find: $f_1(x) = \sum_{\forall y} f(x,y) = \begin{cases} \frac{2}{6} & x = 1,2,3 \\ 0 & \end{cases}$ & $f_2(y) = \sum_{\forall x} f(x,y) = \sum_{x=1}^{3} \frac{1}{6} = \frac{3}{6} \ldots \begin{cases} \frac{2}{6} & y = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$

$P(X \geq 2) = \sum_{x=2}^{3} f_1(x) = f_1(2) + f_1(3) = \frac{2}{6} + \frac{2}{6} = 1$

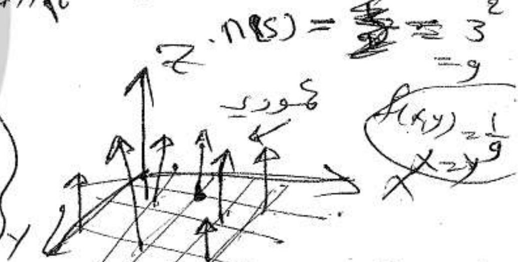
$P(Y \leq 2) = ?$

Example (4) Given a J.p.m.f. $f(x,y)$

$$f(x,y) = \begin{cases} \frac{1}{21}(x+y) & \text{for } x = 1,2,3 \\ & \quad\; y = 1, 2 \\ 0 & \text{o.w.} \end{cases}$$

Find $P(X \geq 2, Y \leq 2)$, $f_1(x)$, $f_2(y)$

نفس المثال اعلاه ولكن مع ارجاع $n(S) = 3^2 = 9$



sol.

$$P(X \geq 2, Y \leq 2) = \sum_{y=1}^{2} \sum_{x=2}^{3} \frac{1}{21}(x+y)$$
$$= \frac{1}{21} \sum_{y=1}^{2} [(2+y) + (3+y)]$$
$$= \frac{1}{21} \sum_{y=1}^{2} [2y+5] = \frac{1}{21}[(2(1)+5) + 2(2)+5]$$
$$= \frac{1}{21}[7+9] = \frac{16}{21}$$

$P(X \geq 2, Y \leq 2) = \sum_{x=2}^{3} \sum_{y=1}^{2} \frac{1}{6} = \sum \left(\frac{1}{6} + \frac{1}{6}\right) = \left(\frac{2}{6}\right)\left[\frac{1}{6} + \frac{1}{6}\right] = \frac{4}{6}$

$P(X=3, Y=1) = f(3,1) = \frac{1}{21}(3+1) = \frac{4}{21}$

$f_1(x) = \sum_{y} f(x,y)$

$$f_1(x) = \sum_{y=1}^{2} \frac{1}{21}(x+y) = \frac{1}{21}[(x+1) + (x+2)]$$
$$= \frac{1}{21}[3+2x]$$

$$f_1(x) = \begin{cases} \frac{1}{21}[3+2x] & \text{for } x = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \sum_{x} f(x,y)$$

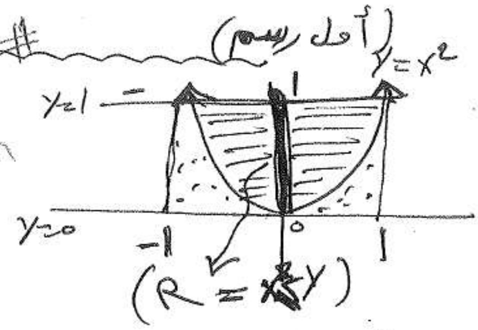
$$= \sum_{x=1}^{3} \frac{1}{21}(x+y)$$

$$= \frac{1}{21}\left[(1+y)+(2+y)+(3+y)\right]$$

$$= \frac{1}{21}(6+3\cancel{y}) = \frac{3}{21}(y+2) = \frac{1}{7}(y+2)$$

$$f_2(y) = \begin{cases} \frac{1}{7}(y+2) & \text{for } y=1,2 \\ 0 & \text{o.w.} \end{cases}$$

(أول رسم)



Example 5 Given a J.p.d.f.

$$f(x,y) = \begin{cases} cx^2y & \text{for } x^2 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Region

ⓐ Find the value of c ⓑ Find $P(X \leq y)$

Sol.

$$R = \{(x,y) ; x^2 \leq y \leq 1\} = \left\{ \begin{matrix} 0 \leq x^2 \leq y, & x^2 \leq y \leq 1 \\ 0 \leq x^2 \leq 1, & 0 \leq y \leq 1 \end{matrix} \right\}$$

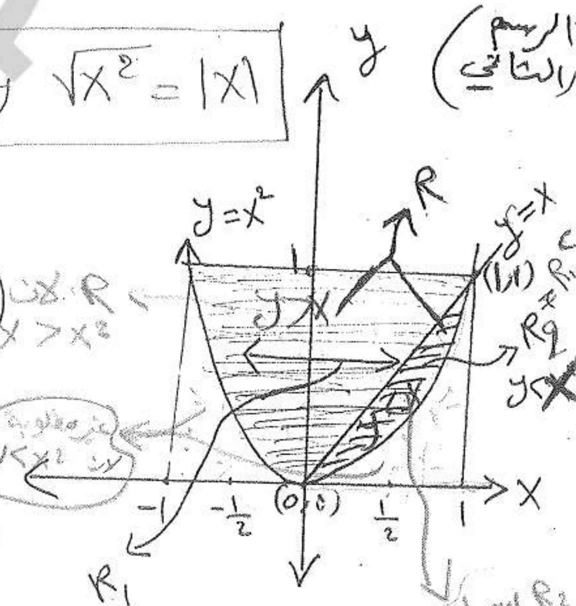
تفكيك R من النهاية لمعرفة الحدود والحكم

suppose $y = x^2 \Rightarrow x = \mp\sqrt{y}$

$0 \leq x^2 \leq y$	$0 \leq x^2 \leq y \leq 1$
$\lvert x \rvert \leq \sqrt{y}$	$0 \leq y \leq 1$
$-\sqrt{y} \leq x \leq \sqrt{y}$	$y = x^2$
$x^2 \leq 1$	$x = \mp\sqrt{y}$
$\lvert x \rvert \leq 1$	
$-1 \leq x \leq 1$	

$\boxed{\text{by } \sqrt{x^2} = \lvert x \rvert}$

(الرسم الثاني)



$$R = \left\{ \begin{matrix} -\sqrt{y} \leq x \leq \sqrt{y} & ; & x^2 \leq y \leq 1 \\ -1 \leq x \leq 1 & ; & 0 \leq y \leq 1 \end{matrix} \right\}$$

x	$y = x^2$
0	0
$\mp\frac{1}{2}$	$\frac{1}{4}$
∓ 1	1

Consider $y = x^2$

لإيجاد نقاط التقاطع : $y = x^2$, $y = x$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0,\ x = 1$

$\Rightarrow (0,0)$, $(1,1)$

R

$0 \leq x^2 \leq y \leq 1$

لذلك ... $P(Y \leq X) = 1 - P(X \leq Y)$

(a) By Cond. (2)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\int_{-1}^{1}\int_{x^2}^{1} c\,x^2 y\,dy\,dx = 1$$

$$1 = \frac{c}{2}\int_{-1}^{1} x^2 y^2 \Big|_{x^2}^{1} dx = \frac{c}{2}\int_{-1}^{1} x^2(1-x^4)\,dx$$

$$1 = \frac{c}{2}\int_{-1}^{1}[x^2 - x^6]\,dx = \frac{c}{2}\left[\frac{x^3}{3} - \frac{x^7}{7}\right]_{-1}^{1}$$

$$1 = \frac{c}{2}\left[\left(\frac{1}{3}-\frac{1}{7}\right) - \left(\frac{-1}{3} - \frac{-1}{7}\right)\right]$$

$$1 = \frac{c}{2}\left[\left(\frac{1}{3}-\frac{1}{7}\right) + \left(\frac{1}{3}-\frac{1}{7}\right)\right]$$

$$1 = \frac{c}{2}\left(\frac{2}{3} - \frac{2}{7}\right) = c\left(\frac{1}{3}-\frac{1}{7}\right) = c\left(\frac{7-3}{21}\right) = \frac{4}{21}c$$

$$\Rightarrow \boxed{c = \frac{21}{4}}$$

$$\therefore f(x,y) = \begin{cases} \frac{21}{4}x^2 y & \text{for } x^2 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

(الرسم الثاني)

(b) To find $P(X \le Y)$:

$$R = \begin{Bmatrix} -\sqrt{y} \le x \le \sqrt{y}, & x^2 \le y \le 1 \\ -1 \le x \le 1, & 0 \le y \le 1 \end{Bmatrix}$$

$P(X \le Y) =$ volum of $f(x,y)$ over R_1

but $P(Y \ge X) = 1 - P(Y < X)$

$P(Y < X) =$ volum of $f(x,y)$ over R_1^c

$$P(Y<X) = \int_0^1\int_{x^2}^{x} \frac{21}{4}x^2 y\,dy\,dx$$

($x=0$ & $x=1$) $\therefore$

$$= \frac{21}{8}\int_0^1 x^2 y^2 \Big|_{x^2}^{x} dx$$

$$= \frac{21}{8}\int_0^1 x^2[x^2 - x^4]\,dx = \frac{21}{8}\int_0^1 [x^4 - x^6]\,dx$$

$$= \frac{21}{8}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]\Big|_0^1$$

$$= \frac{21}{8}\left[\left(\frac{1}{5} - \frac{1}{7}\right) - (0-0)\right]$$

$$= \frac{21}{8}\left[\frac{7-5}{35}\right] = \frac{\cancel{21}^{\,3}}{\cancel{8}_{\,4}} \cdot \frac{\cancel{2}}{\cancel{35}_{\,5}} = \frac{3}{20}$$

$$\therefore P(X \le Y) = 1 - P(X > Y) \qquad \int_0^1 (x^2 - x)\,dx \quad \text{(المساحة بين منحنيين)}$$

$$= 1 - P(Y < X)$$

$$= 1 - \frac{3}{20} = \frac{17}{20}$$

(c) To find $f_1(x)$ and $f_2(y)$

$$R = \begin{cases} -\sqrt{y} \le x \le \sqrt{y}, & x^2 \le y \le 1 \\ -1 \le x \le 1, & 0 \le y \le 1 \end{cases}$$

$$f_1(x) = \int_{x^2}^{1} \frac{21}{4} x^2 y \, dy \quad \text{for } -1 \le x \le 1$$

$$= \frac{21}{4} x^2 \frac{y^2}{2}\Big|_{x^2}^{1} = \frac{21}{8} x^2 (1 - x^4)$$

$$f_1(x) = \begin{cases} \frac{21}{8} x^2 (1 - x^4) & \text{for } -1 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4} x^2 y \, dx \quad \text{for } 0 \le y \le 1$$

$$= \frac{21}{4} y \frac{x^3}{3}\Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{7}{4} y \left[y^{\frac{3}{2}} + y^{\frac{3}{2}}\right] = \frac{7}{2} y^{\frac{5}{2}}$$

$$f_2(y) = \begin{cases} \frac{7}{2} y^{\frac{5}{2}} & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$\boxed{P(X+Y<1) = ?}$ (H.W.)

Stochastic Independence

الاستقلال التصادفي

Def: X and y are stochastically independence iff:

$$\boxed{f_1(x)\, f_2(y) = f(x,y) \quad \forall (x,y)}$$

مقارنة مع
A & B are indep. events: $P(A)P(B)=P(AB)$

Denoted by (s. indep.)

شرح بـ(½ ساعة)
الوقت للتدريس

Example: Given a J.p.m.f

x \ y	1	2	3	4	$f_1(x)$ للاول
1	$f(1,1) = .1$	$f(1,2) = 0$	$f(1,3) = .1$	$f(1,4) = 0$	$f_1(1) = .1 + .1 = .2$
2	$f(2,1) = .3$	$f(2,2) = 0$	$f(2,3) = .1$	$f(2,4) = .2$	$f_1(2) = .3 + .1 + .2 = .6$
3	$f(3,1) = 0$	$f(3,2) = .2$	$f(3,3) = 0$	$f(3,4) = 0$	$f_1(3) = .2$
$f_2(y)$ بالثاني	$f_2(1) = .4$	$f_2(2) = .2$	$f_2(3) = .2$	$f_2(4) = .2$	$\sum_{x=1}^{3} f_1(x) = \sum_{y=1}^{4} f_2(y) = 1$

$$f(x,y) \equiv P(X=x, Y=y)$$

a. T.P. $f_1(X) \cdot f_2(Y) = f(X,Y) \quad \forall (x,y)$

Note $\sum_{x} f_1(x) = 1$, $f_1(x)$ is p.m.f. of X

$\sum_{y} f_2(y) = 1$, $f_2(y)$ is p.m.f. of y

Prove that

$f_1(1) \cdot f_2(1) = (.2)(.4) = .08$

$f(1,1) = .1$

$\therefore f_1(1) f_2(1) \neq f(1,1)$ & $f_1(2) f_2(2) \neq f(2,2)$

$\therefore$ X and y are not s. indep.

b. Find $P(X \geq 2, Y \leq 3)$

$$P(X \geq 2, Y \leq 3) = \sum_{y=1}^{3} \sum_{x=2}^{3} f(x,y) = \sum_{y=1}^{3} [f(2,y) + f(3,y)]$$

$$= [f(2,1) + f(2,2) + f(2,3)] + [f(3,1) + f(3,2) + f(3,3)]$$

$$= .03 + 0 + .1 + 0 + .2 + 0$$

$$= .6$$

c. Find $P(X \geq 2) = \sum_{x=2}^{3} f_1(x)$

$$= f_1(2) + f_1(3)$$

$$= 0.6 + 0.2 = 0.8$$

& $P(Y \leq 3) = \sum_{y=1}^{3} f_2(y)$

$$= f_2(1) + f_2(2) + f_2(3)$$

$$= .4 + .2 + .2 = .8$$

Uniform Distribution of two R.V.'s

الوقت شرح بـ ساعة 3 للتدريس

Def. If a c.r.v. X and y have a uniform distribution on a region R, then the joint p.d.f. of X and Y is:

$$\boxed{f(x,y) = \frac{1}{\text{Area}(R)} \quad \text{for } (x,y) \in R}$$

(مساحة المنطقة)

Example 6 X and Y have a uniform dist. on a triangle Δ with vertices $(0,0)$, $(0,1)$ and $(1,1)$. Find $f(x,y)$.

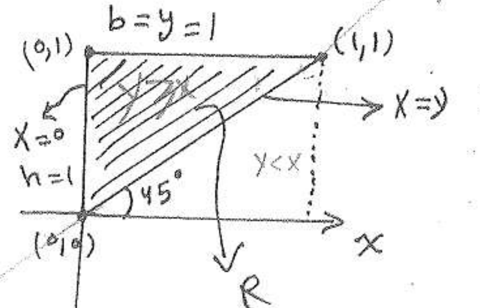
Sol.

R is a triangle Δ

$y \geq x \qquad x \geq 0$

$y \leq 1 \qquad \Rightarrow \quad 0 \leq x \leq y \leq 1$

dependent معتمدة



$R = \{(x,y);\ 0 \leq x \leq y \leq 1\}$

$\text{Area}(\Delta) = \frac{1}{2} \cdot h \cdot b$

$= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

X, Y are dep. $\Rightarrow f_1(x) f_2(y) \neq f(x,y)$

$$f(x,y) = \frac{1}{\text{Area}(R)} = \frac{1}{\frac{1}{2}} = \begin{cases} 2 & \text{for } 0 \le x \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Example 7 A point (x,y) is chosen from a region R which is bounded by a curve $y = x^2$ and a line $y = x$, find $f(x,y)$.

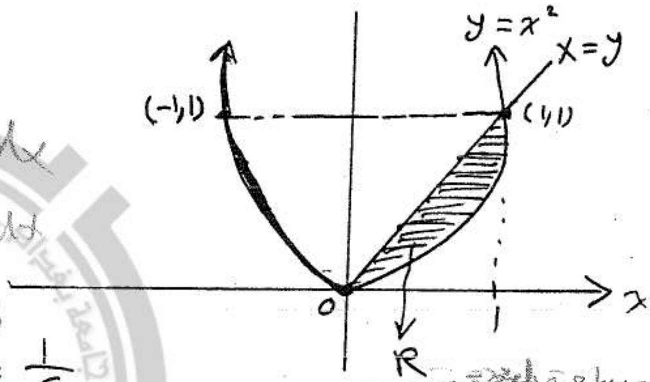
<u>Sol.</u> Sketch $y = x$ and $y = x^2$

A point $(x,y) \in R$ must be above the curve $y = x^2$ and under the line $y = x$, then $x^2 \le y \le x$. ($x^2 = x$ نقاط التقاطع $(0,0)$, $(1,1)$)
$\rightarrow x = 0, x = 1, y = 0, y = 1$

<u>Also</u> that point must be between $0 \le x \le 1$, $0 \le y \le 1$

$$0 \le x^2 \le y \le x \le 1$$

$$R = \{(x,y);\ 0 \le x^2 \le y \le x \le 1\}$$



$$\text{Area}(R) = \int_0^1\int_{x^2}^{x} dy\, dx = \int_0^1 |x - x^2|\, dx$$

$$= \int_0^1 |x^2 - x^2|\, dx$$

$$= \int_0^1 |(x - x^2)|\, dx$$

$x^2 \le y$ موجبة

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

مساحة المنطقة

$$\int_0^1 |x^2 - x|\, dx = \int_0^1 (x - x^2)\, dx$$

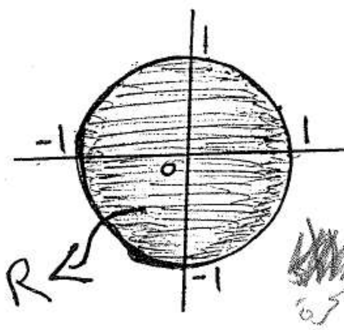
$$f(x,y) = \frac{1}{1/6} = \begin{cases} 6 & \text{for } 0 \le x^2 \le y \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Example 8 A point (x,y) is chosen from a region R where

$$R = \{(x,y) ;\ x^2 + y^2 \le 1\}$$

a. find $f(x,y)$

b. find $f_1(x)$ & $f_2(y)$



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<u>Sol.</u>

a. Let $x^2 + y^2 = 1 \rightarrow r = 1$ نصف القطر

$\text{Area}(R) = \pi r^2 = \pi$

radius $\leftarrow r = 1$ نصف القطر

then $f(x,y) = \begin{cases} \frac{1}{\pi} & \text{for } x^2 + y^2 \le 1 \\ 0 & \text{o.w.} \end{cases}$

b. We must find $f_1(x)$ & $f_2(y)$:

$$f_1(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy \quad \text{for } -1 \le x \le 1$$

$$= \frac{1}{\pi}\, y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \frac{1}{\pi}\left[\sqrt{1-x^2} + \sqrt{1-x^2}\right] = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{for } -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{1}{\pi}\, dx \quad \text{for } -1 \le y \le 1$$

$$= \frac{1}{\pi}\, x \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2} & \text{for } -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ 0 & \text{o.w.} \end{cases}$$

$$f_1(x) \cdot f_2(y) = \frac{2}{\pi} \cdot \frac{2}{\pi} \sqrt{(1-y^2)(1-x^2)} = \frac{4}{\pi^2}\sqrt{(1-y^2)(1-x^2)}$$

$$f(x,y) \ne f_1(x)\, f_2(y) \qquad \left(\text{since } f(x,y) = \frac{1}{\pi}\right)$$

---

* Conditional Function and Conditional probability

Def. Let X and Y be two r.v.'s, $f(y|x)$ denoted the conditional p.d.f or p.m.f. of y given $X = x$.
$f(x|y)$ denoted the conditional p.d.f. or p.m.f. of X given $Y = y$.

where

$$\boxed{f(y|x) = \frac{f(x,y)}{f_1(x)} \; ; \quad f_1(x) \ne 0}$$

$$\boxed{f(x|y) = \frac{f(x,y)}{f_2(y)} \; ; \quad f_2(y) \ne 0}$$

$$\left(P(A|B) = \frac{P(AB)}{P(B)} \right)$$

Note:

$$P(a < x < b \,/\, y = c) = \int_a^b f(x|y=c)\, dx$$
$$= \sum_{x=a}^{b} f(x|y=c)$$

$$P(c<y<d \mid x=a) = \int_c^d f(y \mid x=a)\,dy$$
$$= \sum_{y=c}^{d} f(y \mid x=a)$$

Example 8 Given a j.p.d.f.

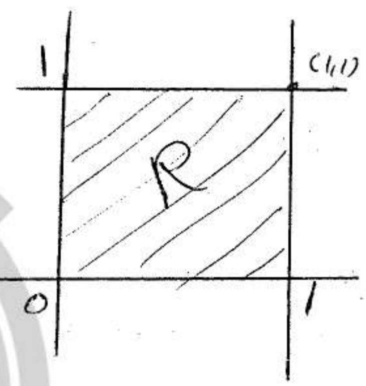
$$f(x,y) = \begin{cases} 8xy & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(1) find $f(x|y)$ & $f(y|x)$

(2) find $P(x>\frac{1}{4} \mid y=\frac{1}{2})$, $P(y<\frac{1}{4} \mid x=\frac{1}{2})$.

Sol. $R = \{(x,y) ; 0<x<y<1\}$

$$R = \begin{Bmatrix} 0<x<y & x<y<1 \\ 0<x<1 & 0<y<1 \end{Bmatrix}$$



(1) we must find $f_1(x)$ and $f_2(y)$

$$f_1(x) = \int_x^1 8xy\,dy = 8x \left.\frac{y^2}{2}\right|_x^1 = 4x(1-x^2)$$

$$= 4x(1-x^2) \quad \text{for } 0<x<1$$

$$f_2(y) = \int_0^y 8xy\,dx = \frac{8}{2}yx^2 = 4y^3 \quad \text{for } 0<y<1$$

$$f(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{8xy}{4y^3} = \begin{cases} \frac{2x}{y^2} & \text{for } \begin{cases} 0<x<y \\ 0<y<1 \end{cases} \\ 0 & \text{o.w.} \end{cases}$$

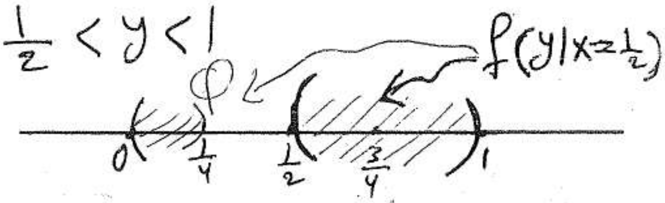
$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$$

$$\therefore f(y|x) = \begin{cases} \frac{2y}{1-x^2} & \text{for } x<y<1 \ \& \ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$x \neq 1$ لان مقام $f(y|x)$ = صفر

(2) $P(y<\frac{1}{4} \mid x=\frac{1}{2}) = \int_0^{\frac{1}{4}} f(y \mid x=\frac{1}{2})\,dy = \int_0^{\frac{1}{4}} 0\,dy = 0$

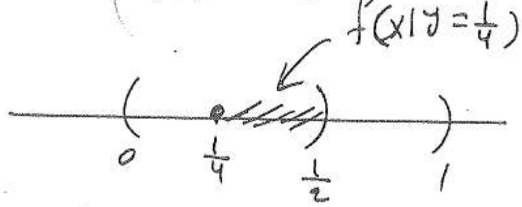
$$x<y<1, \quad x=\frac{1}{2} \implies \frac{1}{2}<y<1$$



المنطقة التي تعمل عليها الدالة $f(y|x=\frac{1}{2})$

$$P(X > \tfrac{1}{4} \mid y=\tfrac{1}{2}) = \int_0^{y} f(x \mid y=\tfrac{1}{2})\,dx$$ ← (نعوض y ب $f(x|y=\frac{1}{2})$ ثم نأخذ الاحتمالية)

$$\boxed{\begin{array}{c} P(X > \tfrac{1}{4} \mid y=\tfrac{3}{4}) \\ 0<x<y,\ y=\tfrac{3}{4} \\ \therefore 0<x<\tfrac{3}{4} \Rightarrow \tfrac{1}{4}<x<\tfrac{3}{4} \end{array}} = \int_{1/4}^{y} \frac{2x}{y}\,dx$$



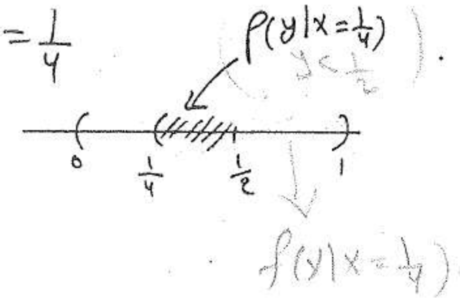
$$0<x<y,\quad y=\tfrac{1}{2} \Rightarrow 0<x<\tfrac{1}{2}$$

$$= \int_{1/4}^{1/2} \frac{2x}{(\frac{1}{2})^2}\,dx = 4\cdot 2\cdot \frac{x^2}{2}\Big|_{1/4}^{1/2}$$

$$= 4\left[\tfrac{1}{4} - \tfrac{1}{16}\right] = 4\cdot\left(\tfrac{3}{16}\right) = \tfrac{3}{4}$$

$$P(y<\tfrac{1}{2} \mid X=\tfrac{1}{4}) = \ ?$$

$$x<y<1,\quad x=\tfrac{1}{4}$$
$$\tfrac{1}{4}<y<1$$



$$P(y<\tfrac{1}{2} \mid x=\tfrac{1}{4}) = \int_x^{1/2} f(y \mid x=\tfrac{1}{4})\,dy$$ ← ملاحظة

نعوض ب $f(y|x=\frac{1}{4})$ ونعوض ب $x=\frac{1}{4}$ ثم نأخذ الاحتمالية

$$= \int_x^{1/2} \frac{2y}{1-x^2}\,dy = \int_{1/4}^{1/2} \frac{2y}{1-(\frac{1}{4})^2}\,dy$$

$$= \int_{1/4}^{1/2} \frac{2y}{15/16}\,dy = \frac{16}{15}\cdot\cancel{2}\cdot\frac{y^2}{\cancel{2}}\Big|_{1/4}^{1/2}$$

$$= \frac{16}{15}\left(\frac{1}{4}-\frac{1}{16}\right) = \frac{\cancel{16}}{\cancel{15}_{5}}\cdot\frac{\cancel{3}}{\cancel{16}} = \frac{1}{5}$$

\# Example 6: If $X \sim unif(0,1)$ and

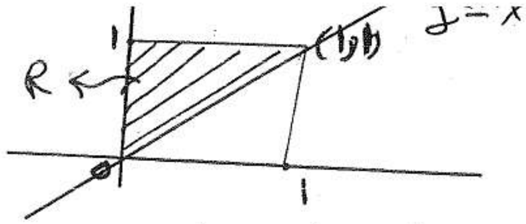
$$f(y|x) = \begin{cases} \frac{1}{1-x} & \text{for } x<y<1,\ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

(a) find $f(x,y)$?

$\because X \sim unif(0,1)$

$$f_1(x) = \frac{1}{1-0} = 1 \Longrightarrow f_1(x) = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$



$$f(x,y) = f_1(x) \cdot f(y|x)$$

$$= (1)\left(\frac{1}{1-x}\right) = \begin{cases} \frac{1}{1-x} & \text{for } x<y<1,\ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore \boxed{f(y|x) = f(x,y)}$$

b. Find $P(X>\frac{1}{2} \mid y=\frac{3}{4}) = \int_0^y f(x|y=\frac{3}{4})\,dx$

$$f(x|y) = \frac{f(x,y)}{f_2(y)}$$

$$f_2(y) = \int_0^y f(x,y)\,dx = \int_0^y \frac{1}{1-x}\,dx = -Ln(1-x)\Big|_0^y$$

$$= -[Ln(1-y) - \underbrace{\cancel{Ln}(1-0)}_{\text{Zero}}]$$

$$= -Ln(1-y) \quad \text{for } 0<y<1$$

$$\therefore f_2(y) = -Ln(1-y) \quad \text{for } 0<y<1$$

$$\therefore f(x|y) = \frac{\frac{1}{1-x}}{-Ln(1-y)}$$

$$f(x|y) = \begin{cases} \frac{1}{-Ln(1-y)} \cdot \frac{1}{1-x} & \text{for } 0<x<y \ \& \ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X>\tfrac{1}{2} \mid y=\tfrac{3}{4}) = \int_{1/2}^{y} f(x|y=\tfrac{3}{4})\,dx$$

$(0<X<y,\ y=\frac{3}{4})$

$\Rightarrow 0<X<\frac{3}{4}$

بشكل عام

$X>\frac{1}{2}$ ثم

$\Rightarrow \frac{1}{2}<X<\frac{3}{4}$

$$= \int_{1/2}^{y} \frac{-1}{Ln(1-y)} \cdot \frac{1}{1-x}\,dx = \int_{1/2}^{3/4} \frac{-1}{Ln(1-\frac{3}{4})} \cdot \left(\frac{1}{1-x}\right) dx$$

$$= \frac{-1}{Ln1 - Ln4} \cdot \Big[-Ln(1-x)\Big]_{1/2}^{3/4}$$

$$= \frac{+1}{-Ln4}\left[Ln(1-\tfrac{3}{4}) - Ln(1-\tfrac{1}{2})\right]$$

$$= \frac{-1}{Ln4}\left[Ln(\tfrac{1}{4}) - Ln(\tfrac{1}{2})\right]$$

$$= \frac{-1}{Ln2^2}\left[(\underbrace{\cancel{Ln1}}_{\text{Zero}} - Ln4) - (\underbrace{\cancel{Ln1}}_{\text{Zero}} - Ln(2))\right]$$

$$= \frac{-1}{2\ln 2}(-2\ln 2 + \ln 2) = \frac{-1}{2\ln 2}(-\ln 2) = \frac{1}{2}$$

H.W Find $P(Y < \frac{3}{4} \mid X = \frac{1}{2})$

الموضوع الثاني

Expectation of two Random Variables

Review

① Def. $E(XY)$ is the expectation or the mean of both X and Y such that:

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy \quad \text{if } X \text{ and } Y \text{ are c.r.v.s}$$

$$= \sum_{y}\sum_{x} xy f(x,y) \quad \text{if } X \text{ and } Y \text{ are d.r.v.s}$$

Note: ① $E(X) \equiv$ mean of $X = \mu_x$

s.t. $E(X) = \int_{-\infty}^{\infty} x f_1(x)\,dx$ if X is c.r.v.

$= \sum_{x} x f_1(x)$ if X is d.r.v.

② $E(Y) \equiv$ mean of $Y = \mu_y$

s.t. $E(y) = \int_{-\infty}^{\infty} y f_2(y)\,dy$ if Y is c.r.v.

$= \sum_{y} y f_2(y)$ if Y is d.r.v.

③ (a) $V(X) = E(X^2) - [E(X)]^2$

(b) $V(Y) = E(Y^2) - [E(Y)]^2$

④ Covariance of both X & Y

We use the symbole $Cov(X,Y)$ (it's mean Covariance), s.t

$$Cov(X,Y) = E\{[X - E(X)][Y - E(Y)]\}.$$

Theorem (1) If X and Y are random variables, then:

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Proof

$$Cov(X,Y) = E\{[X-E(X)][Y-E(Y)]\}$$

$$= E\{XY - YE(X) - XE(Y) + E(X)\cdot E(Y)\}$$

$$= E(XY) - E(X)E(Y) - E(X)E(Y) + E(Y)E(X)$$

$$= E(XY) - E(X)E(Y)$$

Theorem (2) If X and Y are independent, then

$$E(XY) = E(X)E(Y)$$

OR $Cov(X,Y) = 0$

Proof **Case ①** If X and Y are c.r.v.s

$\because$ X & Y are indep. $\Rightarrow f_1(x) f_2(y) = f(x,y)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy) f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_1(x) f_2(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x f_1(x)\, dx \cdot \int_{-\infty}^{\infty} y f_2(y)\, dy \quad \text{by Comm. Property}$$

$$= E(X) \cdot E(Y)$$

Case ② By the same method.

Note By Th. 2 above, if X & Y are indep., then

$$Cov(X,Y) = 0$$

Example: Given a J.p.d.f.

$$f(x,y) = \begin{cases} 2 & \text{for } 0<y<x<1 \\ 0 & \text{o.w.} \end{cases}$$

هناك ٣ طرق للاجابة

Are X & Y indep.? (If $Cov(X,Y) = 0$ Then X & Y are indep.)

① By using $Cov(X,Y)$
or ② By using $E(X,Y)$
③ $f_1(x) f_2(y) = f(x,y)$

$$E(XY) = \int_0^1 \int_0^x xy\,(2)\,dy\,dx$$

$$= \int_0^1 \cancel{2}x \frac{y^2}{\cancel{2}} \Big]_0^x dx$$

$$= \int_0^1 x \cdot x^2\,dx$$

$$= \frac{x^4}{4}\Big]_0^1$$

$$= \frac{1}{4}.$$

Rho

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{V(X) \cdot V(Y)}}$$

$$\rho = \frac{\sigma_{X,Y}}{\sigma_X \, \sigma_Y}$$

$\rho_{X,Y}$

Correlation coefficient

$$E(X^2) = \int_0^1 x^2 (2x)\,dx = \frac{2}{4} x^4 \Big]_0^1 = \frac{2}{4} = \frac{1}{2}$$

$$V(y) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

$$E(y^2) = \int_0^1 y^2 \left[2(1-y)\right] dy = 2\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$$

$$= 2\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= 2\left[\frac{4-3}{12}\right]$$

$$= \frac{2}{12} = \frac{1}{6}$$

$$V(y) = \frac{1}{6} - \left[\frac{2}{6}\right]^2$$

$$= \frac{1}{6} - \frac{4}{36}$$

$$= \frac{6}{36} - \frac{4}{36}$$

$$= \frac{2}{36} = \frac{1}{18}$$

$$R = \begin{Bmatrix} 0 < y < x & , & y < x < 1 \\ 0 < y < 1 & , & 0 < x < 1 \end{Bmatrix}$$

$$E(XY) = \int_0^1 \int_y^1 xy \cdot f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_y^1 xy(2)\, dx\, dy = \int_0^1 \int_0^X (XY) f(X,Y)\, dy\, dx$$

$$= \int_0^1 2y \frac{x^2}{2} \Big|_y^1 dy = \int_0^1 y(1-y^2)\, dy$$

$$= \int_0^1 [y - y^3] = \frac{y^2}{2} - \frac{y^4}{4} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E(X) = \int_0^1 x\, f_1(x)\, dx \quad , \quad E(y) = \int_0^1 y\, f_2(y)\, dy$$

$$f_1(x) = \int_0^x 2\, dy = \int_0^x 2\, dy = 2y \Big|_0^x = 2x \Rightarrow f_1(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_y^1 f(x,y)\, dx = \int_y^1 2\, dx = 2x \Big|_y^1 = 2[1-y] \Rightarrow f_2(y) = \begin{cases} 2(1-y) & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \int_0^1 x f_1(x)\, dx = \int_0^1 x(2x)\, dx = \int_0^1 2x^2\, dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$E(y) = \int_0^1 y f_2(y)\, dy = \int_0^1 2y(1-y)\, dy = \int_0^1 (2y - 2y^2)\, dy = y^2 - \frac{2}{3} y^3 \Big|_0^1$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \operatorname{Cov}(X,y) = E(Xy) - E(X)E(y)$$

$$= \frac{1}{4} - \left(\frac{2}{3}\right) \cdot \left(\frac{1}{3}\right) = \frac{9-8}{36} = \frac{1}{36}$$

$\operatorname{Cov}(X,y) \neq 0 \Longrightarrow$ X and y are dependent.

~~$\because \operatorname{Cov}(X,Y) > 0$ all ways~~

(تصحيح)

$$\left.\begin{array}{l} V(X) = 1/18 \\ V(y) = 1/18 \end{array}\right] \rho_{X,y} = \frac{\operatorname{Cov}(X,y)}{\sqrt{V(X) \cdot V(y)}} = \frac{\frac{1}{36}}{\sqrt{\frac{1}{18} \cdot \frac{1}{18}}} = \frac{\frac{1}{36}}{\frac{1}{18}} = \frac{1}{2}$$

معامل الارتباط Correlation Coefficient

Def. when X and y are dependent and we want to know how much X depends on y we use the correlation to measure it, denoted by "$\rho_{x,y}$" read (Rho) where:

$-1 \le \rho_{x,y} \le 1$

$$\boxed{\rho_{x,y} = \frac{Cov(X,y)}{\sqrt{V(X)\,V(y)}} = \frac{E(Xy) - E(X)E(y)}{\sqrt{V(X)\,V(y)}}} = \frac{Cov(X,y)}{\sigma_x\,\sigma_y}$$

$Cov(X,y) \neq 0$ if X & y are dependent

where $-1 \le \rho \le 1$

H.w. Find $\rho_{x,y}$ from above example. $\longrightarrow Cov(X,y) = \frac{1}{2} \neq 0$

② Conditional Mean and Conditional Variance

① $f(x|y) = \dfrac{f(x,y)}{f_2(y)}$

$f(y|x) = \dfrac{f(x,y)}{f_1(x)}$

$E(y|X)$ denote to the conditional mean of y given $X = x$.

$E(x|y)$ denote to the conditional mean of X given $Y = y$.

where $\underset{g(X) = \text{func. of } X}{E(y|X)} = \int_{\square}^{\square} y\, f(y|X)\, dy$, limit of integral follows, limit of y.

$\underset{g(y) = \text{func. of } y}{E(X|y)} = \int_{\square}^{\square} x\, f(X|y)\, dx$, limit of integral follows, limit of X.

Note Since $f(y|X)$ defined for interval of y interm of X, then $E(y|X)$ is a function of X. Also $E(X|y)$ is a function of y.

$V(y|X)$ denote to the conditional variance of y given $X = x$.

$V(X|y)$ denote to the Conditional Variance of X given $Y = y$.

where

$$V(y|X) = E\{[y-E(y|X)]^2 | X\}$$

$$= \int_{[Y]} [y-E(y|x)]^2 f(y|x)\,dy$$

$$\cancel{\sum_{[Y]} [y-E(y|x)]^2 f(y|x)}$$ } by def. of $E(Y|x)$

$$V(X|y) = E\{[X-E(X|y)]^2 | y\}$$

$$= \cancel{\sum_{[X]} [x-E(X|y)]^2 f(x|y)}$$

$$= \int_{[X]} [x-E(X|y)]^2 f(x|y)\,dx$$ } by def. of $E(X|y)$

Example: Given a J.p.d.f $f(x,y)$

$$f(x,y) = \begin{cases} 2 & \text{for } 0 \le x+y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(X|y), E(Y|x), V(X|y), V(Y|x)$

Sol.

$$R = \{(x,y);\ 0 \le x+y \le 1\} = \left\{ \begin{matrix} 0 \le x \le 1-y, & 0 \le y \le 1-x \\ 0 \le x \le 1, & 0 \le y \le 1 \end{matrix} \right\}$$

$$E(X|y) = \int_X x f(x|y)\,dx$$

$$f(x|y) = \frac{f(x,y)}{f_2(y)}$$

$$f_2(y) = \int_0^{1-y} f(x,y)\,dx = \int_0^{1-y} 2\,dx = 2x\Big|_0^{1-y} = 2(1-y)$$

$$f_2(y) = \begin{cases} 2(1-y) & \text{for } 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f(x|y) = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

$$f(x|y) = \begin{cases} \frac{1}{1-y} & \text{for } 0 \le x \le 1-y \\ 0 & \text{o.w.} \end{cases}$$ (هي دالة لـ X وقيمة لامعلومة)

$$E(X|y) = \int_0^{1-y} x \cdot \frac{1}{(1-y)}\,dx = \frac{1}{(1-y)} \cdot \frac{x^2}{2}\Big|_0^{1-y} = \frac{(1-y)^2}{2(1-y)} = \left(\frac{1-y}{2}\right) \quad 0 \le y \le 1$$ دالة لـ Y

$$V(X|y) = \int_0^{1-y} [x - E(X|y)]^2 \underbrace{\,}_{f(x|y)} dx = \int_0^{1-y} \left[x - \frac{(1-y)}{2}\right]^2 dx$$

$$= \int_0^{1-y} \left[x^2 - (1-y)x + \frac{(1-y)^2}{4}\right] dx = \frac{1}{12}(1-y)^2 \text{ for } 0 \le y \le 1$$

طريقة ثانية للحل

Find $E(Y|X)$ & $V(Y|X)$ H.W

or

$$E(X^2|y) = \int_0^{1-y} x^2 f(x|y)\,dx = \frac{x^3}{3(1-y)}\Big]_0^{1-y} = \frac{(1-y)^3}{3(1-y)} = \frac{(1-y)^2}{3}$$

$$V(X|Y) = E(X^2|y) - [E(X|y)]^2 = \frac{(1-y)^2}{3} - \left(\frac{1-y}{2}\right)^2$$

Theorem (3) If X and Y are two random variables, then:

a. $E[E(Y|X)] = E(Y)$, b. $E[E(X|Y)] = E(X)$

Proof Case (1): If X and Y are c.r.v. with p.d.f $f(x)$.

$E(Y|X)$ is a function of X.

Suppose that $E(Y|X) = g(X)$ = as a func. of X

$$E[E(Y|X)] = E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_1(x)\,dx = \int_{-\infty}^{\infty} [E(Y|X)] f_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y f(y|x)\,dy\right] f_1(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(y|x) f_1(x)\,dy\,dx$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} \Rightarrow f(y|x) \cdot f_1(x) = f(x,y)$$

$$\int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} y\, f_2(y)\,dy = E(Y)$$

Case (2): If X and Y are d.r.v. with p.m.f. $f(x)$.

$E(Y|X)$ is a function of X.

Suppose that $E(Y|X) = g(X)$

$$E(E(Y|X)) = E(g(X)) = \sum_x g(x) \cdot f_1(x) = \sum_x [E(Y|X)] f_1(x)$$

$$= \sum_x \left[\sum_y y f(y|x)\right] f_1(x) = \sum_x \sum_y y\, f(y|x) f_1(x)$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} \Rightarrow f(y|x) \cdot f_1(x) = f(x,y)$$

$$\sum_y y \sum_x f(x,y) = \sum_y y\, f_2(y) = E(Y)$$

b. By the same method.

Example: Given a J. p.d.f. $f(x,y)$

$$f(x,y) = \begin{cases} 2 & \text{for } 0 < x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E[E(Y|X)]$.

Sol.

$E(E(Y|X)) = E(Y)$ (by Theorem 3)

$$E(Y) = \int_{-\infty}^{\infty} y\, f_2(y)\,dy \qquad f_2(y) = \int_0^y f(x,y)\,dx = \int_0^y 2\,dx = 2x\Big|_0^y = 2y$$

$$f_2(y) = \begin{cases} 2y & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore E(Y) = \int_0^1 y(2y)\,dy = 2\int_0^1 y^2\,dy = \frac{2}{3} y^3 \Big|_0^1 = \frac{2}{3}$$

Joint Distribution function (J.d.f) of two aR.v.s X and y

<u>Def.</u> The J.d.f of X and y is defined to a function such that for all values of X and y $(-\infty < x < \infty, -\infty < y < \infty)$ then:

$$F(x,y) = P(X \le x, y \le y)$$

$F(x) = P(X \le x)$ (Rev.)

① $\int_{-\infty}^{x} f(t)\,dt$ if X is c.r.v.

② $\sum_{x_j \le x} f(x_j)$ if X is d.r.v.

Remarks

1. $P(a < X \le b, c < y \le d) = [F(b,d) - F(a,d)] - [F(b,c) - F(a,c)]$

نقارن اعلاه للمتغير العشوائي الواحد (X):

$$\begin{Bmatrix} 1.\ F(x) = P(X \le x) \\ 2.\ P(a < X \le b) = F(b) - F(a) \end{Bmatrix}$$

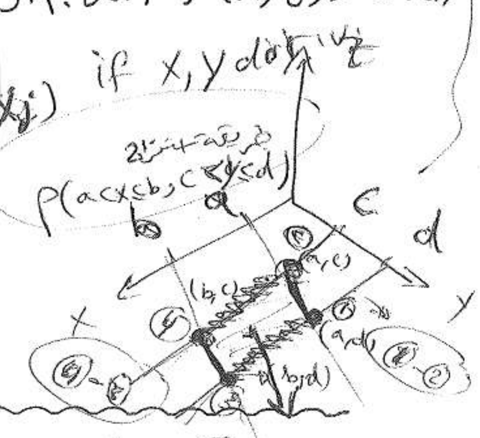
2. (i) $F_1(x) = P(X \le x) = \lim\limits_{y \to \infty} P(X \le x \text{ and } y \le y) = \lim\limits_{y \to \infty} F(x,y)$

(ii) $F_2(y) = \lim\limits_{x \to \infty} F(x,y)$

3. If X and y have Cont. J.d.f $F(x,y)$ with J.P.d.f $f(x,y)$, then for any values of X and y the J.d.f. is

① $F(x,y) = \sum_{x_i \le x} \sum_{y_j \le y} f(x_j, y_j)$ if x, y d.r.v.

② $F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(r,s)\,dr\,ds$ therefore



Note

$* \; f(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x \, \partial y}$, $f(x,y)$ diff.

<u>Example:</u> Given a J.d.f $F(x,y)$

$F(x,y) = P(X \le x, Y \le y)$

$$F(x,y) = \begin{cases} 0 & \text{for } x < 0, y < 0 \\ \frac{1}{16} xy(x+y) & \text{for } 0 \le x \le 2, 0 \le y \le 2 \\ 1 & \text{for } x > 2, y > 2 \end{cases}$$

Find $F_1(x)$, $F_2(y)$, $f(x,y)$.

<u>Sol.</u> $F_1(x) = \lim\limits_{y \to \infty} F(x,y) = \lim\limits_{y > 2} F(x,y) = F(x,2)$

$$= \frac{1}{16} x(2)(x+2) = \frac{x(x+2)}{8}$$

$$F_1(x) = \begin{cases} 0 & \text{for } x<0 \\ \frac{x(x+2)}{8} & \text{for } 0\le x\le 2 \\ 1 & \text{for } x>2 \end{cases}$$

$$F_2(y) = \lim_{x\to\infty} F(x,y) = \lim_{x>2} F(x,y) = F(2,y)$$
$$= \frac{1}{16}(2)\,y(2+y) = \frac{y(2+y)}{8}$$

$$F_2(y) = \begin{cases} 0 & \text{for } y<0 \\ \frac{y(2+y)}{8} & \text{for } 0\le y\le 2 \\ 1 & \text{for } y>2 \end{cases}$$

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial F(x,y)}{\partial y}\right]$$

$$F(x,y) = \frac{1}{16}(x^2y + xy^2)$$

$$f(x,y) = \frac{\partial}{\partial x}\left[\frac{1}{16}(x^2+2xy)\right] = \left[\frac{1}{16}(2x+2y)\right]$$
$$= \frac{1}{8}(x+y)$$

$$f(x,y) = \begin{cases} \frac{x+y}{8} & \text{for } 0<y<2,\ 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

H.W. (1) Prove that : $E(X+Y) = E(X)+E(Y)$

(2) " " : $E(XY) = E(X)E(Y)$

Proof (1) If X & Y are independent d.r.v.s ;

$$E(X_1+X_2) = \sum_{x_1}\sum_{x_2}(x_1+x_2)\,f(x_1,x_2)$$
$$= \sum_{x_1}\sum_{x_2}\left(x_1 f(x_1,x_2) + x_2 f(x_1,x_2)\right)$$
$$= \sum_{x_1}\sum_{x_2} x_1 f(x_1,x_2) + \sum_{x_1}\sum_{x_2} x_2 f(x_1,x_2)$$
$$= \left(\sum_{x_1} x_1 f_1(x_1)\right)\left(\sum_{x_2} f_2(x_2)\right) + \left(\sum_{x_2} x_2 f_2(x_2)\right)\left(\sum_{x_1} f_1(x_1)\right)$$
$$= E(X_1)\cdot(1) + E(X_2)(1)$$

$$\rho_{x,y} = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{\sigma_x^2}\ \sqrt{\sigma_y^2}}$$

$$\sigma_x^2 = E(X^2) - [E(X)]^2$$

$$\sigma_y^2 = E(Y^2) - [E(Y)]^2$$

$$E(X^2) = \left. \frac{\partial^2 M(t_1,0)}{\partial t_1^2} \right|_{t_1=0} = -2(1-t_1)^{-3}(-1) \Big|_{t_1=0} = \boxed{2}$$

$$E(Y^2) = \left. \frac{\partial^2 M(0,t_2)}{\partial t_2^2} \right|_{t_2=0} = -2(1-t_2)^{-3}(-1) \Big|_{t_2=0} = \boxed{2}$$

$$\sigma_x^2 = 2-(1)^2 = \boxed{1} \Rightarrow \boxed{\sigma_x = 1}$$

$$\sigma_y^2 = 2-(1)^2 = \boxed{1} \Rightarrow \boxed{\sigma_y = 1}$$

$$\rho_{x,y} = \frac{1-(1)(1)}{(1)(1)} = 0$$

$\therefore$ X & Y are independent (since $Cov(x,y) = 0$).

H.W. Given a J.p.f of X & Y:

$$f(x,y) = \begin{cases} e^{-y} & \text{for } 0<x<y<\infty \\ 0 & \text{o.w.} \end{cases}$$

Find: ① the M.g.f. of X, Y $[M_{x,y}(t_1,t_2)]$.

② $E(XY)$

③ $f_1(x)$ & $f_2(y)$

④ $\mu_x = E(x)$ & $\mu_y = E(y)$ (by two methods).

⑤ $Cov(X,Y)$.

Note: $R = \left\{ \begin{array}{ll} 0<x<y & , x<y<\infty \\ 0<x<\infty & , 0<y<\infty \end{array} \right\}$

$$M.g.f \text{ of } X = M_{X,Y}(t_1,0) = M_X(t_1) = \frac{1}{1-t_1}$$

$$\mu_x = E(X) = \left.\frac{\partial M(t_1,0)}{\partial t_1}\right|_{t_1=0} = \left.\frac{\partial M_X(t_1)}{\partial t_1}\right|_{t_1=0}$$

$$= \left.\frac{1}{(1-t_1)^2}\right|_{t_1=0} = \boxed{1}$$

$$M.g.f \text{ of } Y = M_{X,Y}(0,t_2) = M_Y(t_2) = \frac{1}{1-t_2}$$

$$\mu_y = E(Y) = \left.\frac{\partial M(0,t_2)}{\partial t_2}\right|_{t_2=0} = \left.\frac{\partial M_Y(t_2)}{\partial t_2}\right|_{t_2=0}$$

$$= \left.\frac{1}{(1-t_2)^2}\right|_{t_2=0} = \boxed{1}$$

$$\therefore M.g.f \text{ of } X,Y = M_{X,Y}(t_1,t_2) = \frac{1}{(1-t_1)(1-t_2)}$$

$$E(XY) = \left.\frac{\partial^2 M(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0}$$

$$= \left.\frac{\partial}{\partial t_1}\left(\frac{\partial M(t_1,t_2)}{\partial t_2}\right)\right|_{t_1=t_2=0}$$

$$= \left.\frac{\partial}{\partial t_1}\left(\frac{1}{(1-t_1)(1-t_2)^2}\right)\right|_{t_1=t_2=0}$$

$$= \left.\frac{1}{(1-t_1)^2(1-t_2)^2}\right|_{t_1=t_2=0}$$

$$= \boxed{1}$$

The variance of X or Y by using $M_{x,y}(t_1,t_2)$:

$$Var(X) = \sigma_x^2 = \frac{\partial^2 M(0,0)}{\partial t_1^2} - \left(\frac{\partial M(0,0)}{\partial t_1}\right)^2$$

$$= E(X^2) - [E(X)]^2$$

$$Var(Y) = \sigma_y^2 = \frac{\partial^2 M(0,0)}{\partial t_2^2} - \left(\frac{\partial M(0,0)}{\partial t_2}\right)^2$$

$$= E(Y^2) - [E(Y)]^2$$

Example: Let $f(x,y)$ be the J.p.d.f. of X, Y:

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } o \quad x>0,\ y>0 \\ 0 & \text{o.w.} \end{cases}$$

Find the M.g.f of X, Y ; also, find μ_x, μ_y, $E(XY)$, and $\rho_{x,y}$.

Sol.
$$M_{x,y}(t_1,t_2) = E[e^{t_1x+t_2y}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dy\,dx$$

$$= \int_0^{\infty}\int_0^{\infty} e^{t_1x+t_2y}\, e^{-(x+y)}\,dy\,dx$$

$$= \left(\int_0^{\infty} e^{-x(1-t_1)}\,dx\right)\left(\int_0^{\infty} e^{-y(1-t_2)}\,dy\right)$$

$$= \frac{1}{(1-t_1)} \cdot \frac{1}{(1-t_2)} = \frac{1}{(1-t_1)(1-t_2)}$$

(eq. 2)

Moment Generating Function of two R.V's X & Y

(2) Let $f(x,y)$ be the J.p.d.f of X, Y ; and consider that:

(1) $E\left[e^{t_1 X + t_2 Y}\right]$ exists for $|t_1| < h_1$, $|t_2| < h_2$; h_1, h_2 are positive constants, then:

$E\left[e^{t_1 X + t_2 Y}\right]$ is the m.g.f of X & Y and is denoted by $M_{X,Y}(t_1, t_2)$ and is defined as follows:

$$M_{X,Y}(t_1,t_2) = E\left[e^{t_1 X + t_2 Y}\right] = \begin{cases} \sum\limits_{\forall x} \sum\limits_{\forall y} e^{t_1 x + t_2 y} f(x,y) & \text{if } x, y \text{ are d.r.v.s} \\ \int\limits_x \int\limits_y e^{t_1 x + t_2 y} f(x,y)\, dy\, dx & \text{if } x, y \text{ are c.r.v.s} \end{cases}$$

and

$$M_X(t_1) = M_X(t_1, 0) = E\left[e^{t_1 X}\right] = \begin{cases} \sum\limits_{\forall x} e^{t_1 x} f_1(x) & \text{if } X \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{t_1 x} f_1(x)\, dx & \text{if } X \text{ is c.r.v.} \end{cases}$$

(2)

$$M_Y(t_2) = M_Y(0, t_2) = E\left[e^{t_2 Y}\right] = \begin{cases} \sum\limits_{\forall y} e^{t_2 y} f_2(y) & \text{if } Y \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{t_2 y} f_2(y)\, dy & \text{if } Y \text{ is c.r.v.} \end{cases}$$

and

(3)

$$\frac{\partial M(0,0)}{\partial t_1} = \mu_x = E(X) \quad ; \quad \frac{\partial M(0,0)}{\partial t_2} = \mu_y = E(Y)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1^2} = E(X^2) \quad ; \quad \frac{\partial^2 M(0,0)}{\partial t_2^2} = E(Y^2)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} = E(XY)$$

Exersicees About Ch.5

Q1: Given a J.p.f. :

$$f(x,y) = \begin{cases} cy^2 & \text{for } 0 \le x \le 2 \ \& \ 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$c\int_0^2\int_0^1 y^2\,dy\,dx = c\int_0^2 \left[\frac{y^3}{3}\right]_0^1 dx = \frac{c}{3}\,x\Big|_0^2 = \frac{2}{3}c$$

Find :

1. The value of (c)
2. $P(X+Y > 2)$
3. $P(Y < \frac{1}{2})$, 4. $P(X \le 1)$

Q2: Let the J.p.m.f of X & Y be:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{30} & \text{for } x = 1,2,3 \ \& \ y = 1,2 \\ 0 & \text{o.w.} \end{cases}$$

(i) Find the Marginal p.f. of X ?
(ii) Find the Marginal p.f. of Y ?
(iii) Are X & Y independent ?

Q3: Let X & Y have the J.p.m.f. be :

$$f(x,y) = \begin{cases} \dfrac{x+2y}{18} , & x = 1,2 ,\ y = 1,2 \\ 0 & \text{o.w.} \end{cases}$$

Find: $\mu_x = E(X)$, $\mu_y = E(Y)$, & $Cov(X,Y)$.

Q4: Suppose that X, Y are c.r.v.s with J.p.d.f. :

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } x > 0,\ y > 0 \\ 0 & \text{o.w.} \end{cases}$$

Find : $M_{x,y}(t_1,t_2)$, $M_x(t_1)$, $M_y(t_2)$.

* [illegible]

Q5: Suppose X and Y are two r.v.s having the following J.p.d.f:

$$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(i) Find the conditional p.d.f $f(x|y)$ and $f(y|x)$.
(ii) Are these conditional function $f(x|y)$ and $f(y|x)$ p.d.f.? (prove that).

Q6: Consider the following J.p.M.f. of X and Y whose values are given by:

$y \backslash x$	2	3	4	5	6	$f_2(y)$
0	1/9	0	4/27	0	2/27	1/3
1	2/27	0	8/81	0	4/81	2/9
2	4/27	0	16/81	0	8/81	4/9
$f_1(x)$	1/3	0	4/9	0	2/9	1

Find: ① $f(X|Y)$ & $f(Y|X)$
② $P(X|Y=1)$ & $P(Y|X=2)$

Q7: If J.p.m.f:

$P(1,1)=\frac{1}{9}$, $P(2,1)=\frac{1}{3}$, $P(3,1)=\frac{1}{9}$
$P(1,2)=\frac{1}{9}$, $P(2,2)=0$, $P(3,2)=\frac{1}{18}$
$P(1,3)=0$, $P(2,3)=\frac{1}{6}$, $P(3,3)=\frac{1}{9}$

Are X & Y independent?

Q8: If X and Y are independent with M.p.d.f:

$$f_1(x)=\begin{cases} 3x^2 & \text{for } 0\le x\le 1 \\ 0 & \text{o.w.} \end{cases}, \quad f_2(y)=\begin{cases} 2y & \text{for } 0\le y\le 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(XY)$.

Q_9: Suppose that J.p.d.f of X & Y is:

$$f(x,y) = \begin{cases} C\sin x & \text{for } 0 \le x \le \pi,\ 0 \le y \le \text{[redacted]} \\ 0 & \text{o.w.} \end{cases}$$

Find the value of C. $\left(C = \frac{1}{2\pi}\right)$

Q_{10}: Let X and Y have the joint p.m.f. be:

(x,y)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
P(x,y)	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

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Find the Correlation Coefficient $\rho_{x,y}$.

Q_{11}: If $f(x,y) = \begin{cases} e^{-x-y} & \text{for } 0 < x < \infty \ \& \ 0 < y < \infty \\ 0 & \text{o.w.} \end{cases}$

is the J.p.d.f of two r.v.s X & Y.
Show that X & Y are stochastically independent.

B

Q_{12}: Let X & Y be independent r.v.'s with the following distribution:

X	1	2
$f_1(x)$	.6	.4

y	10	15	5
$f_2(y)$	.5	.3	.2

① Find the Joint pr. f of X & Y
② also, find E(XY).

B

Q_{13}: Let $f(x|y) = \begin{cases} \frac{C_1 x}{y^2} & \text{for } 0 < x < y,\ 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$

& $f_2(y) = \begin{cases} C_2 y^4 & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$; then:

$(f(x,y) = \quad 0 < x < y < 1)$

Find:

(1) Determine the Constants C_1 and C_2.

(2) the $f_1(X)$ p.d.f. of X.

(3) the J.p.d.f. of X & Y; $f(x,y)$

(4) $P(-\frac{1}{2}<X<\frac{1}{2} \mid Y=\frac{5}{8})$

(5) $P(\frac{1}{4}<X<\frac{1}{2})$.

Q14: Let $f(x,y) = \begin{cases} \frac{1}{16} & \text{for } x=1,2,3,4 \\ & \quad\; y=1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

be the J.p.f of X & Y.

Show that X & Y are stochastically independent.

Q15: Given a J.p.f of X & Y:

$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1 \text{ \& } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$

Find the Correlation Coefficient of X & Y $(\rho_{x,y})$.

Solution of Q_1:

(1) by Cond. (2) $\Longrightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

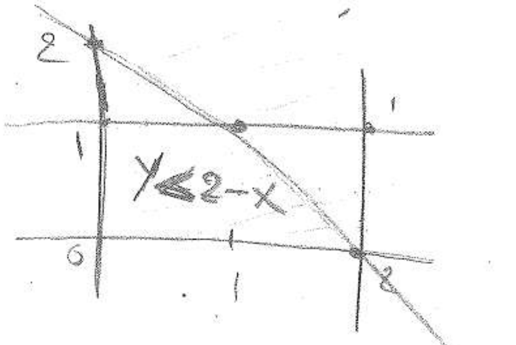
$$C\int_0^2\int_0^1 y^2\,dy\,dx = 1 \Longrightarrow \boxed{C=\frac{3}{2}}$$

$P(X+Y>2)$

(2) $P(X+Y\le 2) = 1-P(X+Y>2)$

Let $x+y=2 \Longrightarrow y=2-x$

$x+y\le 2 \Longrightarrow y\le 2-x$



$$R=\left\{\begin{array}{l} 0\le y\le 2-x \\ 0\le x\le 2 \end{array}\right\}$$

$$P(X+Y\le 2)=\int_0^2\int_0^{2-x}\frac{3}{2}y^2\,dy\,dx = \cdots = \frac{1}{2}$$

$\int_0^1\int_0^{2-y}\frac{3}{2}y^2\,dx\,dy$

(3) $P(Y<\frac{1}{2})=?$

To find $f_2(y)$ in the first.

$$f_2(y)=\int_0^2 f(x,y)\,dx=\frac{3}{2}\int_0^2 y^2\,dx=\cdots=\begin{cases}3y^2 & \text{for } 0\le y\le 1\\ 0 & \text{o.w.}\end{cases}$$

$$P(Y<\tfrac{1}{2})=3\int_0^{1/2}y^2\,dy=\cdots=\frac{1}{8}$$

(4) $P(X\le 1)=?$

To find $f_1(x)$ in the begining.

$$f_1(x)=\int_0^1 f(x,y)\,dy=\frac{3}{2}\int_0^1 y^2\,dy=\cdots=\begin{cases}\frac{1}{2} & \text{for } 0\le x\le 2\\ 0 & \text{o.w.}\end{cases}$$

$$P(X\le 1)=\int_0^1 f_1(x)\,dx$$

$$=\int_0^1\frac{1}{2}\,dx=\cdots=\frac{1}{2}$$

(5) $P(X=3Y)=\int_0^2\int_0^{\frac{1}{3}x} f(x,y)\,dy\,dx=\int_0^2\int_0^{\frac{1}{3}x}\frac{2}{3}y^2\,dy\,dx$

$\Longrightarrow \boxed{y=\frac{1}{3}x}$

$$=\cdots=\frac{2}{27}$$

Q2: If $f_1(x)\cdot f_2(y) = f(x,y)$, then X & Y are indep.

$$f_1(x) = \sum_{y=0}^{3} f(x,y) = \sum_{y=0}^{3} \frac{1}{30}(x+y)$$

$$= \frac{1}{30}\left[x+(x+1)+(x+2)+(x+3)\right]$$

$$= \frac{1}{30}(4x+6) = \begin{cases} \frac{1}{15}(2x+3) & \text{for } x=0,1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \sum_{x=0}^{2} f(x,y) = \sum_{x=0}^{2} \frac{1}{30}(x+y)$$

$$= \frac{1}{30}\left[y+(1+y)+(2+y)\right]$$

$$= \frac{1}{30}(3+3y) = \frac{1}{10}(y+1)$$

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$$f(x,y) = \begin{cases} \frac{x+y}{30} & y=0,1,2,3 \\ & x=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \begin{cases} \frac{1}{10}(y+1) & \text{for } y=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$f_1(x) f_2(y) = \frac{1}{15}(2x+3)\cdot\frac{1}{10}(y+1)$$

$$\neq \frac{1}{30}(x+y) = f(x,y)$$

$$\therefore f_1(x)\cdot f_2(y) \neq f(x,y)$$

Q3: To find f_1 & f_2 in the first time.

$$f_1(x) = \sum_{x_2=1}^{2} \frac{1}{18}(x_1+12x_2)$$

$$= \frac{1}{18}\sum_{x=1}^{2} x_1 + \frac{1}{18}\sum_{x_2=1}^{2} 2x_2$$

$$= \dots = \begin{cases} \frac{x_1+3}{9} & \text{for } x_1=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_{x_1} = E(x_1) = \sum_{x_1=1}^{2} x_1 f_1(x_1) = \sum_{x_1=1}^{2} x_1\left(\frac{x_1+3}{9}\right) = \dots = \frac{14}{19}$$

$$f_2(x_2) = \sum_{x_1=1}^{2} \frac{1}{18}(x_1+2x_2) = \begin{cases} \frac{1}{18}(3+4x_2) & \text{for } x_2=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_{x_2} = E(x_2) = \sum_{x_2=1}^{2} x_2 f_2(x_2) = \sum_{x_2=1}^{2} x_2\left(\frac{3+4x_2}{18}\right) = \dots$$

$$= \frac{29}{18}$$

$$Cov(X,Y) = E(X_1X_2) - E(X_1)E(X_2)$$
$$= \mu_{x_1x_2} - \mu_{x_1}\mu_{x_2}$$

$$\mu_{x_1x_2} = \sum_{x_1=1}^{2}\sum_{x_2=1}^{2} f(x_1,x_2)\cdot x_1x_2$$
$$= \frac{1}{18}\sum_{x_1=1}^{2}\sum_{x_2=1}^{2} x_1x_2(x_1+2x_2)$$
$$= \frac{1}{18}\Big[(1)(1)(1+2(1)) + (1)(2)(1+2(2)) + (2)(1)(2+2(1)) + (2)(2)(2+2(2))\Big] = \frac{9}{2}$$

$$\rho_{xy} = \frac{Cov(X,Y)}{\sigma_x\ \sigma_y} = \frac{E(XY)-E(X)E(Y)}{\sqrt{V(X)}\ \sqrt{V(Y)}}$$

$$\sigma_{x_1} = \sqrt{\sigma_{x_1}^2}\ ;\ \sigma_{x_1}^2 = E(X_1^2) - [E(X_1)]^2$$
$$\sigma_{x_2} = \sqrt{\sigma_{x_2}^2}\ ;\ \sigma_{x_2}^2 = E(X_2^2) - [E(X_2)]^2$$

$$E(X_1^2) = \sum_{x_1=1}^{2} X_1^2 f_1(x_1) = \sum_{x_1=1}^{2} X_1^2\left(\frac{x_1+3}{9}\right) = \frac{24}{9}$$
$$E(X_2^2) = \sum_{x_2=1}^{2} X_2^2 f_2(x_2) = \sum_{x_2=1}^{2} X_2^2\left(\frac{1}{18}(3+4x_2)\right) = \frac{51}{18}$$

$$\mu_{x_1}^2 = [E(X_1)]^2 \ \&\ \mu_{x_2}^2 = [E(X_2)]^2$$

$$\sigma_{x_1}^2 = E(X_1^2) - [E(X_1)]^2$$
$$= \frac{24}{9} - \left(\frac{14}{9}\right)^2 = \sim$$
$$\sigma_{x_2}^2 = \frac{51}{18} - \left(\frac{29}{18}\right)^2 = \sim$$

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Q6 : ① $f(X|Y) = \frac{f(x,y)}{f_2(y)}$, $f(Y|X) = \frac{f(x,y)}{f(x)}$

② $f(X|Y=1) = \frac{f(x,y=1)}{f_2(y=1)} = \frac{f(x_i,y=1)}{2/9}$, $x_i = 2,3,\dots,6$.

$$= \begin{cases} \frac{2/27}{2/9} = \frac{1}{3} & \text{for } x=2 \\ \frac{0}{2/9} = 0 & \text{for } x=3 \\ \frac{8/81}{2/9} = \frac{4}{9} & \text{for } x=4 \\ \frac{0}{2/9} = 0 & \text{for } x=5 \\ \frac{4/81}{2/9} & \text{for } x=6 \end{cases}$$

$$f(y|x=2) = \frac{f(y|x=2)}{f(x=2)} = \frac{f(y_i|x=2)}{1/3}, y=0,1,2$$

$$= \begin{cases} \frac{1/9}{1/3} & \text{for } y=0 \\ \frac{2/27}{1/3} = \frac{2}{9} & \text{for } y=1 \\ \frac{4/27}{1/3} = \frac{4}{9} & \text{for } y=2 \end{cases}$$

Hence $f(y) = f(y|x=2)$; therefore X & Y are independent.

Q7:

$y \backslash x$	1	2	3	$f_2(y)$
1	1/9	1/3	1/9	5/9 $= f_2(1)$
2	1/9	0	1/18	3/18 $= f_2(2)$
3	0	1/6	1/9	15/54 $= f_2(3)$
$f_1(x)$	2/9 $f_1(1)$	3/6 $f_1(2)$	5/18 $f_1(3)$	1

$f(1,1) = \frac{1}{9} \neq f_1(1) f_2(1) = (\frac{2}{9})(\frac{5}{9})$

$f(2,1) = \frac{1}{3} \neq f_1(2) f_2(1) = (\frac{3}{9})(\frac{5}{9})$

$f(3,1) = \frac{1}{9} \neq f_1(3) f_2(1) = (\frac{5}{18})(\frac{5}{9})$

$f(1,2) = \frac{1}{9} \neq f_1(1) f_2(2) =$

$\therefore$ X & Y are dependent.

Q8: Since X & Y are independent; then:

$f_1(x) f_2(y) = f(x,y)$

$$f_1(x) \cdot f_2(y) = \begin{cases} 6yx^2 & \text{for } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(xy) = \int_x \int_y xy f(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 (6yx^2)\, xy\, dy\, dx = \cdots = \frac{1}{2}.$$

sol.

Q10 $f_1(x) = \sum_{y=1}^{3} f(x,y)$

$$= \begin{cases} 9/15 & \text{for } x=1 \\ 6/15 & \text{for } x=2 \\ 0 & \text{o.w.} \end{cases}$$

J.p.f.

$f(x,y)$

y \ x	1	2	$f_2(y)$	
1	2/15	1/15	3/15	$f_2(1)$
2	4/15	1/15	5/15	$f_2(2)$
3	3/15	4/15	7/15	$f_2(3)$
$f_1(x)$	9/15	6/15	1	
	$f_1(1)$	$f_1(2)$		

$f_2(y) = \sum_{x=1}^{2} f(x,y)$

$$= \begin{cases} 3/15 & \text{for } y=1 \\ 5/15 & \text{for } y=2 \\ 7/15 & \text{for } y=3 \\ 0 & \text{o.w.} \end{cases}$$

$E(X) = \sum_{x=1}^{2} x f_1(x)$

$= (1)(9/15) + (2)(\frac{6}{15})$

$= \frac{21}{15}$

$E(Y) = \sum_{y=1}^{3} y f_2(y)$

$= (1)(\frac{3}{15}) + (2)(\frac{5}{15}) + (3)(\frac{7}{15})$

$= \frac{34}{15}$

$\rho_{xy} = Cov(x,y) / \sigma_x \sigma_y = ---$ اكمال الحل

Q12: ① $f(1,5) = f_1(1) f_2(5) = (.6)(.2) = .12$

$f(1,10) = f_1(1) f_2(10) = (.6)(.5) = .30$

$f(1,15) = f_1(1) f_2(15) = (.6)(.3) = .18$

$f(2,10) = f_1(2) f_2(10) = (.4)(.5) = .20$

$f(2,5) = f_1(2) f_2(5) = (.4)(.2) = .08$

$f(2,15) = f_1(2) f_2(15) = (.4)(.3) = .12$

Since (X & Y are independent r.v's)

$\Rightarrow f_1(x) f_2(y) = f(x,y)$ →

$$\therefore f(x,y) = \begin{cases} .12 & \text{for } x=1, y=5 \\ .08 & \text{for } x=2, y=5 \\ .30 & \text{for } x=1, y=10 \\ .20 & \text{for } x=2, y=10 \\ .18 & \text{for } x=1, y=15 \\ .12 & \text{for } x=2, y=15 \\ 0 & \text{o.w.} \end{cases}$$

② $E(XY) = \sum_X \sum_Y XY f(x,y)$

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Q13: Sol. $\circledast_1$ ⓐ $\int_0^y f(x|y)\,dx = 1$ (By cond. ② of Pr.f)

$C_1 \int_0^y \frac{x}{y^2}\,dx = \frac{C_1}{2y^2} x^2 \Big|_0^y = --- = 1 \Rightarrow C_1 = 2y^2$

$\frac{C_1}{2y^2}(y^2) = 1$ $\Rightarrow C_1 = 2$

$$\therefore f(x|y) = \begin{cases} \frac{2x}{y^2} & \text{for } 0<x<y<1 \\ 0 & \end{cases}$$

~~$0<x<1$~~, $0<y<1$

$0<x<y<1$, $0<x<1$

$\circledast_2$ ⓐ $\int_0^1 f_2(y)\,dy = 1$ (by Cond. ② of Pr.f.)

$C_2 \int_0^1 y^4\,dy = 1 \Rightarrow C_2 = 5$

$$\therefore f_2(y) = \begin{cases} 5y^4 & \text{for } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

ⓒ $f(x|y) = \frac{f(x,y)}{f_2(y)} \Rightarrow f(x,y) = f(x|y)\,f_2(y)$

$$= (2x)(5y^4) = \begin{cases} 10xy^4 & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

ⓑ $f_1(x) = \int_y f(x,y)\,dy = 10\int_0^1 xy^4\,dy = \begin{cases} 2x & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

ⓓ $P(-\frac{1}{2}<x<\frac{1}{2} \mid y=\frac{5}{8}) = \int_0^y f(x \mid y=\frac{5}{8})\,dx$

$= \int_0^{5/8} 10x(\frac{5}{8})^4\,dx = ---$

ⓔ $P(\frac{1}{4}<x<\frac{1}{2}) = \int_{1/4}^{1/2} f_1(x)\,dx = \int_{1/4}^{1/2} 2x\,dx = -- = \frac{3}{16}$

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محاضرات مادة

الإحصاء والاحتمالية

للعام الدراسي (2023-2024)

المرحلة الثالثة

Chapter Six

مدرس المادة: م.م هند إبراهيم محمد

Some Special Distribution:

Discrete Distribution:

1) Discrete uniform distribution:

The r.v. X is said to have an uniform distribution if P.m.f. is give by

$$f(X,N)=\begin{cases}\frac{1}{N} & ;\ x=1,2,3,\dots,N\\ 0 & ;o.w.\end{cases}\quad ;Where\ the\ parameter\ N\geq 1\ Natural\ numbere$$

And denoted by $X\sim D_u(N)$

Clear that

$$f(x)\geq 0 \qquad and \qquad \sum_{x=1}^{N} f(x)=1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x.

$$Note:\ the\ mean\ \mu=\frac{N+1}{2}\quad and\ \ the\ variance\ \sigma^2=\frac{N^2-1}{12}\quad (proof)$$

The moment generating function of uniform distribution is given by

$$M_x(t)=E(e^{tx})=\sum_{\forall x} e^{tx} f(x)\ =\sum_{\forall x} e^{tx}\,\frac{1}{N}=\frac{1}{N}\sum_{\forall x} e^{tx}$$

$$=\frac{1}{N}\,[\,e^t+\ e^{2t}+e^{3t}+\cdots+\ e^{Nt}]$$

$$=\frac{e^t}{N}\left[\,1+e^t+\ e^{2t}+\cdots+\ e^{(N-1)t}\right]$$

$$=\frac{e^t}{N}\left[\frac{1-\ e^{Nt}}{1-e^t}\right]$$

Example : If $X\sim D_u(6)$; find ❶ P(x > 1) ❷ P(x ≥ 3) ❸ mean and variance

Solution :

$$X\sim D_u(6) \;\; \Longmapsto \;\; f(x) = \begin{cases} \frac{1}{6} & ; \; x = 1,2,3,\dots,6 \\ 0 & ; o.w. \end{cases}$$

❶ P(x > 1) =1 – P(x ≤1)= 1- P(x=1)= 1 - 1/6 =5/6

❷ P(x ≥ 3) = P(x=3) + P(x=4)+ P(x=5)+ P(x=6) =1/6 + 1/6 +1/6 +1/6 =4/6

Or = 1- P(x<3) = 1- [P(x=1)+p(x=2)] = 1- [1/6 + 1/6]=1-2/6 =4/6

❸ Mean $\mu = \frac{N+1}{2} = \frac{6+1}{2} = \frac{7}{2}$

Variance $\sigma^2 = \frac{N^2-1}{12} = \frac{36-1}{12} = \frac{35}{12}$

2) Bernoulli distribution:

The r.v. X is said to have a Bernoulli distribution with parameter p if P.m.f. is give by

$$f(X,p) = \begin{cases} p^x \, q^{1-x} & ; \; x = 0,1 \\ 0 & ; o.w. \end{cases} \quad ; Where \text{ the } parameter \; 0 \le p \le 1 \, , q = 1 - p$$

And denoted by $X\sim Ber(p)$

Clear that

$$f(x) \ge 0 \quad and \quad \sum_{x=0}^{1} f(x) = p + q = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

$$Note:\ the\ mean\ \mu = p \quad and \quad the\ variance\ \sigma^2 = p\,q \quad (proof)$$

The moment generating function of uniform distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x)$$

$$= \sum_{\forall x} e^{tx}\ p^x q^{1-x} = \sum_{x=0}^{1} e^{tx}\ p^x q^{1-x}$$

$$= e^{0\,t}\ p^0 q^{1-0} + e^{1\,x} p^1 q^{1-1} = q + p\,e^t$$

3) Binomial distribution:

The r.v. X is said to have a Binomial distribution with two parameter n and p if

P.m.f. is give by

$$f(X,n,p) = \begin{cases} C_x^n\, p^x\, q^{n-x} & ;\ x = 0,1,2,\dots,n \\ 0 & ; o.w. \end{cases}$$

And denoted by $X \sim b(n,p)$

Where the two parameter n and p satisfy the following conditions:

1- n is a positive integer

2- $0 \le p \le 1$

3- $q = 1 - p$ or $p + q = 1$

Under the conditions, it is clear that

$$f(x) \ge 0\ \forall x\ \ and \quad \sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} C_x^n\, p^x\, q^{n-x} = (p+q)^n = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

Useful relationship: Binomial Formula is $\sum_{x=0}^{n} C_x^n a^x b^{n-x} = (a+b)^n$.

$Note: \ the\ mean\ \mu = n\,p \quad and \quad the\ variance\ \sigma^2 = n\,p\,q$

$Proof:$

$$\mu = E(x) = \sum_{\forall x} x\,f(x) = \sum_{x=0}^{n} x.C_x^n\ p^x\,q^{n-x}$$

$$\because\ C_x^n = \frac{n!}{x!\,(n-x)!}$$

$$\therefore \mu = \sum_{x=0}^{n} x.C_x^n\ p^x\,q^{n-x} = \sum_{x=0}^{n} x\frac{n!}{x!\,(n-x)!}\ p^x\,q^{n-x}$$

$$= \sum_{x=1}^{n} x\frac{n!}{x!\,(n-x)!}\ p^x\,q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!}\ p^x\,q^{n-x}$$

$$= np\sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!}\ p^{x-1}\,q^{n-x}$$

$let\ \ y = x-1 \quad and \quad m = n-1$

$$\Longmapsto \mu = np\sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!}\ p^y\,q^{m-y}$$

$$\therefore \mu = np\sum_{y=0}^{m} C_y^m\ p^y\,q^{m-y} = np(p+q)^m = np\ 1^m = np$$

$$\therefore\ \ \mu = np$$

$\because \quad \sigma^2 = var(x) = E(x^2) - E^2(x)$

We must compute $E(x^2)$ to that we compute $E[x(x-1)] = E(x^2) - E(x)$

$$E[x(x-1)] = \sum_{\forall x} x(x-1)\, f(x) = \sum_{x=0}^{n} x(x-1) C_x^n \; p^x \, q^{n-x}$$

$$\therefore E[x(x-1)] = \sum_{x=2}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} \; p^x \, q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!} \; p^x \, q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} \; p^{x-2} \, q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} C_{x-2}^{n-2} \; p^{x-2} \, q^{n-x}$$

$let \quad y = x - 2 \quad and \quad m = n - 2$

$$= n(n-1)p^2 \sum_{y=0}^{m} C_y^m \; p^y \, q^{m-y} = 1$$

$$E[x(x-1)] = E(x^2) - E(x) = n(n-1)p^2$$

$$\Longrightarrow E(x^2) = np + n^2p^2 - np^2$$

$$\therefore \quad \sigma^2 = E(x^2) - E^2(x) = np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np - np^2 = np(1-p) = npq$$

$$\therefore \quad \sigma^2 = npq$$

The moment generating function of Binomial distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x) = \sum_{\forall x} e^{tx} \; C_x^n \; p^x \, q^{n-x} = \sum_{x=0}^{n} C_x^n \, e^{tx} \; p^x \, q^{n-x}$$

$$= \sum_{x=0}^{n} C_x^n \, (e^t p)^x \, q^{n-x} = (q + p\, e^t)^n$$

The properties of binomial distribution is:

1- two possible outcome
2- n trials
3- independent trials

Example: Suppose X is binomially distributed with parameter n and p, further suppose

E(X) = 5 and var(X) = 4, find n and p.

Solution: $X\sim b(n,p) \Longmapsto E(x) = np = 5 \quad and \quad var(x) = npq = 4$

$$\Longmapsto q = \frac{4}{5} \Longmapsto p = 1 - \frac{4}{5} = \frac{1}{5}$$

$$\because \quad np = 5 \quad \Longmapsto n = \frac{5}{\frac{1}{5}} = 25$$

$$\therefore \quad X\sim b\left(25, \frac{1}{5}\right)$$

4) Poisson distribution:

The r.v. X is said to have a Poisson distribution with parameter λ if P.m.f. is give by

$$f(X,\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & ; \; x = 0,1,2,\dots,\infty \\ 0 & ; o.w. \end{cases} \quad , where \; \lambda > 0$$

And denoted by $X\sim Po(\lambda)$

$$Since \; >0 \Longmapsto f(x) \geq 0 \; ; \forall x$$

$$Since \quad \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda} \Longmapsto \sum_{\forall x} f(x) = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

$$Note: \ the\ Mean\ \mu = \lambda \quad and \quad the\ Variance\ \sigma^2 = \lambda$$

$Proof:$

$$Mean = E(X) = \sum_{x=0}^{\infty} x\, f(x;\lambda) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda \lambda^{x-1}}{x(x-1)!}$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \ldots\ldots\ldots + \infty\right]$$

$$= e^{-\lambda}\lambda e^{\lambda}$$

$$= \lambda$$

$\therefore$ Mean of the Poisson distribution is λ

And

Variance = Var(X) = $E(X^2) - [E(X)]^2$

$\because$ Mean of the Poisson distribution is λ From (1)

Var(X) = $E(X^2) - [\lambda]^2 \ldots\ldots\ldots (1)$

$$E(X^2) = \sum_{x=0}^{\infty} x^2\, f(x;\lambda) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$= E(x) = \lambda$

$$= e^{-\lambda} \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^2 \lambda^{x-2}}{x(x-1)(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\ldots\ldots + \infty \right] + \lambda$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} + \lambda$$

$$\Rightarrow \quad E(X^2) = \lambda^2 + \lambda \cdots\cdots\cdots\cdots (2)$$

Putting (2) in (1) we get

$$\text{Var(X)} = E(X^2) - [\lambda]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

$\therefore$ Variance of the Poisson distribution is λ

Note: In Poisson distribution the mean and variance are equal i.e. λ

The properties of Poisson distribution is:

1- The event occurring randomly in time.
2- The number of event in non-overlapping time period are independent.
3- $\lambda > 0$, rate of Poisson process is average number of events per unite time.

Example :Suppose the number of flaws in a 100–foot roll of paper is a Poisson random variable with λ = 10. Then the probability that there are eight flaws in a 100-foot roll is:

Solution:

$$P(X=8 \mid \lambda=10) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-10}(10)^8}{8!} = \frac{(2.71828)^{-10}(100{,}000{,}000)}{40{,}320} = .1126$$

The probability of seven flaws in a 50-foot roll is:

$$P(X=7 \mid \lambda=5) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-5}(5)^7}{7!} = \frac{(2.71828)^{-5}(78{,}125)}{5{,}040} = .1044$$

Note:

The Poisson distribution can be limiting case of a binomial distribution under certain conditions.

1. Number of trails i.e. n is indefinitely large i.e. $n \to \infty$
2. p, the probability of success in each trail is indefinitely small i.e $p \to 0$
3. $np = \lambda$ is finite.

Proof:(للاطلاع)

If X is a binomial distribution then the probability mass function is given by

$$P(X=x) = C_x^n \, p^x q^{n-x}, \; x = 0,1,2,........n$$

Under the above conditions

$$\lim_{n\to\infty} b(x;n,p) = \lim_{n\to\infty} C_x^n p^x q^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \qquad [\because np=\lambda]$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)..........(n-x+1)(n-x)!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty} \frac{n^x\left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)..................\left(1-\frac{x-1}{n}\right)\right]}{n^x} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty} 1 \, . \, \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \frac{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x} = \frac{\lambda^x}{x!}\frac{e^{-\lambda}}{1^x} = \frac{e^{-\lambda}\lambda^x}{x!}$$

Example : If n=5000 , p=0.001 and $X\sim b(\mathrm{n},\mathrm{p})$ find *P(x=7)=f(7)*

Solution : $X\sim b(\mathrm{n},\mathrm{p}) = X\sim Po(\lambda)$ λ= np=5000* 0.001=5

$$P(X=7|\lambda=5)=\frac{e^{-\lambda}\lambda^{x}}{x!}=\frac{e^{-5}(5)^{7}}{7!}=\frac{(2.71828)^{-5}(78,125)}{5,040}=.1044$$

$\therefore$ *f(7)=0.1044*

The moment generating function of Binomial distribution is given by

$$M_x(t)=e^{\lambda(e^t-1)}$$

Proof :

$$M_x(t)=E(e^{tx})=\sum_{\forall x} e^{tx}\, f(x) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

Example :Let X be r.v. whose M.g.f is give by e^{2e^t-2} $find\ P(X\geq 1)$

Solution : $X\sim Po(2)$

$$\therefore f(X;\lambda)=\begin{cases}\frac{e^{-2}\,2^x}{x!} & ;\ x=0,1,2,\dots,\infty\\ 0 & ;o.w.\end{cases}$$

$$\therefore P(X\geq 1)=1-P(X<1)$$

$$=1-P(X=0)$$

$$=1-\frac{e^{-2}\,2^0}{0!}=1-e^{-2}=0.864$$

Question: If x has Poisson distribution and P(X=0)=1/2 find E(x) ?

Solution :

$$\because\ X\sim Po(\lambda)$$

$$\therefore f(X,\lambda)=\begin{cases}\frac{e^{-\lambda}\,\lambda^x}{x!} & ;\ x=0,1,2,\dots,\infty\\ 0 & ;o.w.\end{cases}$$

[H. W.]

5) Negative Binomial distribution

Perform independent Bernoulli trials repeatedly until a given number of success are observed i.e. let x the total number of failure before the rth success, that means the r.v. X represent the number of failure Prior to the rth success the P.m.f of X is:

$$f(X;r,p) = \begin{cases} C_x^{x+r-1}\, p^r\, q^x & ; \ x = 0,1,2,\dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

And denoted by $X \sim Nb(r,p)$

Where the parameter r and p satisfy r=1, 2, 3, ……… and $0 \le p \le 1$ and $q = 1-p$

Since $0 \le p \le 1$

$$\therefore f(x) \ge 0 \ \forall x \ \ and \ \sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} C_x^{x+r-1}\, p^r\, q^x = p^r(1-q)^{-r} = 1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

Useful relationship: $\boxed{\sum_{x=0}^{\infty} C_x^{n+x-1} a^x = (1-a)^{-n}}$.

$$Note: \ the\ mean\ \mu = \frac{rq}{p} \quad and \quad the\ variance\ \sigma^2 = \frac{rq}{p^2}$$

$Proof:$

$$\mu = E(x) = \sum_{\forall x} x\, f(x) = \sum_{x=0}^{\infty} x.\ C_x^{x+r-1}\, p^r\, q^x$$

$$\because \ C_x^n = \frac{n!}{x!\,(n-x)!}$$

$$\therefore \mu = \sum_{x=1}^{\infty} x.C_x^{x+r-1}\ p^r\, q^x \ = \sum_{x=1}^{\infty} x\frac{(x+r-1)!}{x!\,(r-1)!}\ p^r\, q^x$$

$$= p^r \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(x-1)!\,(r-1)!}\, q^x$$

$$= r\, q\, p^r \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(x-1)!\, r!}\, q^{x-1}$$

$let \quad y = x - 1 \quad and \quad s = r + 1$

$$= r\, q\, p^r \sum_{y=0}^{\infty} \frac{(y+s-1)!}{y!\,(s-1)!}\, q^y$$

$$\Longrightarrow \mu = \frac{r\, q\, p^r}{(1-q)^s} = \frac{r\, q\, p^r}{p^{r+1}} = \frac{rq}{p}$$

$$\therefore \quad \mu = E(x) = \frac{rq}{p}$$

$$\because \quad \sigma^2 = var(x) = E(x^2) - E^2(x)$$

We must compute $E(x^2)$ to that we compute $E[x(x-1)] = E(x^2) - E(x)$

$$E[x(x-1)] = \sum_{\forall x} x(x-1)\, f(x) = \sum_{x=0}^{\infty} x(x-1)\, C_x^{x+r-1}\, p^r\, q^x$$

$$\therefore E[x(x-1)] = \sum_{x=2}^{\infty} x(x-1) \frac{(x+r-1)!}{x!\,(r-1)!}\, p^r\, q^x$$

$$= q^2\, p^r\, r(r+1) \sum_{x=2}^{\infty} \frac{(x+r-1)!}{(x-2)!\,(r+1)!}\, q^{x-2}$$

$let \quad y = x - 2 \quad and \quad s = r + 2$

$$= q^2\, p^r\, r(r+1) \sum_{y=0}^{\infty} \frac{(y+s-1)!}{y!\,(s-1)!}\, q^y$$

$$\Longrightarrow E(x^2) - E(x) = \frac{q^2\, p^r\, r(r+1)}{(1-q)^s} = \frac{q^2\, p^r\, r(r+1)}{p^{r+2}} = \frac{r(r+1)q^2}{p^2}$$

$$\Longrightarrow E(x^2) = \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + \frac{rq}{p}$$

$$\therefore \quad \sigma^2 = E(x^2) - E^2(x) = \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2}$$

$$= \frac{rq^2}{p^2} + \frac{rq}{p} = \frac{rq(q+p)}{p^2} = \frac{rq}{p^2}$$

$$\therefore \quad \sigma^2 = \frac{rq}{p^2}$$

The moment generating function of **Negative Binomial** distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x) = \sum_{\forall x} e^{tx}\ C_x^{x+r-1}\ p^r\ q^x$$

$$= p^r \sum_{x=0}^{\infty} C_x^{x+r-1}\ (e^t q)^x \quad = (\frac{p}{1 - q\,e^t})^r$$

Other formula of Negative Binomial distribution given by

$$f(X; r, p) = \begin{cases} C_{r-1}^{x+r-1}\ p^r\ q^x & ;\ x = 0,1,2,\dots\dots\dots \\ \\ 0 & ; o.w. \end{cases}$$

Where X denote the number of failures before we get the first r successes

Remark: If X denote the number of trials required to get a total of r successes then

$$f(X; r, p) = \begin{cases} C_{r-1}^{x-1}\ p^r\ q^{x-r} & ;\ x = r, r+1, r+2, \dots\dots\dots \\ \\ 0 & ; o.w. \end{cases}$$

6) Geometric Distribution

In case of the binomial distribution, the number of trials was predetermined. Sometimes, however, we wish to know the number of trials needed before a certain outcome occurs.

For example, we wish to play until we win; you roll dice until you get an 11; a mechanic waits for the first plane to arrive at the airport that needs repair; a basketball player shoots until he makes it. These situations fall under the ***Geometric distribution.*** **(Special case of Negative Binomial at r=1)**

The r.v. X is said to have a ***Geometric distribution*** with parameter p if

P.m.f. is given by

$$f(X;p)=\begin{cases} p\ q^x \ ;\ x=0,1,2,\dots\dots\dots \\ \\ 0 \quad ;o.w. \end{cases} \qquad where\ 0\le p\le 1;\ \ q=1-p$$

And denoted by $X\sim G(p)$

Since $0\le p\le 1$

$$\therefore f(x)\ge 0\ \ \forall x\ \ and\ \ \sum_{x=0}^{\infty} f(x)=\sum_{x=0}^{\infty} p\ q^x\ =\frac{p}{(1-q)}=1$$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

The moment generating function of Geometric distribution is given by

$$M_x(t)=E(e^{tx})=\sum_{\forall x} e^{tx} f(x)\ =\sum_{\forall x} e^{tx}\ \ p\ \ q^x$$

$$=p\sum_{x=0}^{\infty}(e^t q)^x\ \ =\frac{p}{1-q\,e^t}$$

$$Note:\ the\ mean\ \mu=\frac{q}{p}\qquad and\quad the\ variance\ \sigma^2=\frac{q}{p^2}$$

$Proof:$

$$M_x(t)=\frac{p}{1-q\,e^t}\Longmapsto M_x'(t)=\frac{p\,q\ e^t}{(1-q\,e^t)^2}\Longmapsto E(x)=\ M_x'(0)=\frac{p\,q\ e^0}{(1-q\,e^0)^2}=\frac{q}{p}$$

$$\therefore\quad \mu=E(x)=\frac{q}{p}$$

$$M_x''(t)=\frac{(1-q\,e^t)^2(p\,q\ e^t)+2q((1-q\,e^t)(p\,q\ e^t)}{(1-q\,e^t)^4}\Longmapsto E(x^2)=\ M_x''(0)$$

$$\therefore\ E(x^2) = \frac{(1-q\,e^0)^2(p\,q\ \ e^0) + 2q((1-q\,e^0)(p\,q\ \ e^0)}{(1-q\,e^0)^4}$$

$$\therefore\ E(x^2) = \frac{(1-q\,)^2(p\,q\ \ ) + 2q((1-q)(p\,q\,)}{(1-q\,)^4} = \frac{qp^3 + 2q^2p^2}{p^4} = \frac{qp\ + 2q^2}{p^2}$$

$$\sigma^2 = var(x) = E(x^2) - E^2(x) \Longrightarrow \sigma^2 = \frac{qp\ + 2q^2}{p^2} - \frac{q^2}{p^2} = \frac{q(p\ + q\,)}{p^2} = \frac{q}{p^2}$$

$$\therefore\quad \sigma^2 = \frac{q}{p^2}$$

The properties Geometric distribution

1. Each event falls into just one of two categories, which are generally referred to as a "success" or "failure."
2. The probability of success, call it *p*, is the same for each observation.
3. The observations are all independent.
4. The variable of interest is the number of trials required to obtain the **first** success.

Remark: If X denote the number of trials required to get a first success then the P.m.f is:

$$f(X;p) = \begin{cases} p\ \ q^{x-1} & ;\ x = 1,2,3,\dots\dots\dots \\ 0 & ;o.w. \end{cases}$$

$\boldsymbol{Note}$: $By\ this\ case\ the\ mean\ \mu = \frac{1}{p}\quad and\quad the\ variance\ \sigma^2 = \frac{1}{p^2}$

Proof (H.W.)

Example: On island of Oahu in the small village of Nanakuli, about 80% of the residents are of Hawaiian ancestry . Suppose you fly to Hawaii and visit Nanakuli.
What is the probability that the first villager you meet is Hawaiian?
What is the probability that you do not meet a Hawaiian until the third villager?

(على جزيرةِ أوهايو في القريةِ الصغيرةِ في نانا كولي، حوالي 80 % من السكّانِ مِنْ اصل هاواي. افترض انك سافرت إلى هاواي وزُرتُ نانا كولي. ما هو احتمال ان القرويِّ الأولِ الذي تلتقيه هاوايُ؟ ما هو احتمال بأنّك لا تلتقي باي هاواي حتى القرويِّ الثالثِ؟)

Solution :

$$f(X;p) = \begin{cases} p\ \ q^{x-1} & ;\ x = 1,2,3,\dots\dots\dots \\ 0 & ;o.w. \end{cases}$$

$p = 0.80 = .8 \quad ; \quad q = 1 - p = 1 - .8 = .2$

$f(1) = P(X = 1) = (1 - .8)^{1-1}(.80) = (.2)^0(.80) = .80$.

$f(3) = P(X = 3) = (.2)^2(.80) = .032$

What is the probability that you will meet at most three people from Hawaiian ancestry (اصل)?

$P(X \le 3) = P(X = 1) + P(X = 2) + P(X = 3)$

$= (.2)^0(.8) + (.2)^1(.8) + (.2)^2(.8)$

$= .8 + .16 + .032 = .992$

How many people should you expect to meet before you meet the first Hawaiian?

$\mu = \dfrac{1}{.80} = 1.25 \Longmapsto$ First integer number $\ge \mu \Longmapsto$ **Answer= 2**

Note : If X is a geometric random variable with probability p of success on each trial, the expected number of trials necessary to reach the first success is $\mu = E(x) = \dfrac{1}{P}$.

What is the probability that it takes more than three people before you meet a Hawaiian?

$P(X > 3) = (1 - .8)^4 = .0016$

Note, The probability that it takes more than n trials before we see the first success is $P(X > n) = (1 - p)^{n+1}$ and $P(X \ge n) = (1 - p)^n$. (Proof that H.W.)

Example: Suppose we flip a fair coin until we get a head. Let X be the number of trail to get a head. Find the probability mass function of X.

Solution:

$X \sim G(p) \quad p = 0.5 \quad and \quad q = 1 - p = 0.5$

$$P.m.f. \;\; is \quad f(X) = \begin{cases} (0.5)\;\;(0.5)^{x-1} = (0.5)^x & ;\; x = 1, 2, \dots \dots \\ 0 & ; o.w. \end{cases}$$

Question: A fair die cast on successive independent trail until the second six is observed.

Find the probability of observing exactly ten non-sixes before the second six is appear

Answer:

$$r = 2 \quad , \; p = \frac{1}{6} \qquad , \quad q = \frac{5}{6}$$

$$\therefore X \sim Nb(r,p) \;\; \Rrightarrow \;\; f(X;r,p) = \begin{cases} C_{r-1}^{x+r-1} \, p^r \, q^x & ; \; x = 0,1,2, \dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

$$\Rrightarrow \;\; f(x) = f\left(X; 2, \frac{1}{6}\right) = \begin{cases} C_1^{x+1} \, \left(\frac{1}{6}\right)^2 \, \left(\frac{5}{6}\right)^x & ; \; x = 0,1,2, \dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

$$\Rrightarrow f(10) = P(x = 10) = C_1^{11} \, \left(\frac{1}{6}\right)^2 \, \left(\frac{5}{6}\right)^{10}$$

$$= 10 \times (0.0278) \times (0.1615) = 0.0449$$

Or

$$f(X;r,p) = \begin{cases} C_x^{x+r-1} \, p^r \, q^x & ; \; x = 0,1,2, \dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

$$\Rrightarrow \;\; f(x) = f\left(X; 2, \frac{1}{6}\right) = \begin{cases} C_x^{x+1} \, \left(\frac{1}{6}\right)^2 \, \left(\frac{5}{6}\right)^x & ; \; x = 0,1,2, \dots\dots\dots \\ 0 & ; o.w. \end{cases}$$

$$\Rrightarrow f(10) = P(x = 10) = C_{10}^{11} \, \left(\frac{1}{6}\right)^2 \, \left(\frac{5}{6}\right)^{10}$$

$$= 10 \times (0.0278) \times (0.1615) = 0.0449$$

Theorem: (Memory Loss Property)

If X has the geometric distribution with parameter *p* then

$$P(X \geq i + j \mid X \geq i) = P(X \geq j) \quad ; \quad i\,, j = 1, 2, 3, \dots\dots\dots$$

Proof:

$$P(X \geq i + j \mid X \geq i) = \frac{P(X \geq i + j)}{P(X \geq i)} = \frac{(1-p)^{i+j}}{(1-p)^i} = (1-p)^j \; = P(X \geq j)$$

7) The hyper-geometric distribution :

Suppose that we have population of N objects; D of one type and N-D a second type. A random sample of size n is drawn from the population without replacement. Then if we let x denote the number of objects of the first type selected, we get first of all $\binom{D}{x}$ ways of choosing x object of the first type, $\binom{N-D}{n-x}$ ways of choosing $n-x$ object of the second type, $\binom{N}{n}$ ways of choosing a sample of size n from the population of N objects then the r.v. X has P.m.f is given by:

$$f(x) = f(x; N, D, n) = \begin{cases} \dfrac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} & ; x = a, a+1, a+2, \dots, b \\ 0 & ; otherwise \end{cases}$$

$\boldsymbol{where}$ ❶ $a = \max\{0\,, n + D - N\}$ and $b = \min\{n, D\}$

❷ $0 \le x \le D$ ❸ $0 \le n - x \le N - D$

And denoted by $X \sim hyp(N, D, n)$

$$clear\ that\ f(x) \ge 0\ \ \forall x$$

$$and \quad \sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} = \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} \binom{D}{x}\binom{N-D}{n-x} = \frac{\binom{N}{n}}{\binom{N}{n}} = 1$$

Useful relationship: $\displaystyle\sum_{k=0}^{m} \binom{A}{k}\binom{B}{m-k} = \binom{A+B}{m}$

Then f(x) satisfies the condition of begin a P.m.f. of discrete type of random variable x

$\boldsymbol{Theorem}$: $X \sim hyp(N, D, n)$ $then$ $\mu = n\dfrac{D}{N}$ and $\sigma^2 = n\dfrac{D}{N}\,\dfrac{N-n}{N-1}\,\dfrac{N-D}{N}$

$Proof$:

$$\mu = E(x) = \sum_{\forall x} x\, f(x) = \sum_{x=0}^{n} x.\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$

$$\because \; C_x^m = \frac{m!}{x!\,(m-x)!}$$

$$\therefore \mu = \sum_{x=1}^{n} x.\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} = \frac{1}{\binom{N}{n}} \sum_{x=1}^{n} x \frac{D!}{x!\,(D-x)!}\binom{N-D}{n-x}$$

$$= \frac{D}{\binom{N}{n}} \sum_{x=1}^{n} \frac{(D-1)!}{(x-1)!\,(D-x)!}\binom{N-D}{n-x}$$

$let \quad y = x-1 \;, \quad m = n-1 \quad and \quad A = D-1$

$$= \frac{D}{\binom{N}{n}} \sum_{y=0}^{m} \frac{A!}{y!\,(A-y)!}\binom{N-A-1}{m-y}$$

$$= \frac{D}{\binom{N}{n}} \sum_{y=0}^{m} \binom{A}{y}\binom{N-A-1}{m-y}$$

$$= \frac{D}{\binom{N}{n}} \binom{N-1}{m} = \frac{D}{\binom{N}{n}} \binom{N-1}{n-1} = \frac{D}{\cancel{\binom{N}{n}}} \, \frac{n}{N} \cancel{\binom{N}{n}}$$

$$\Longmapsto \mu = E(x) = n\frac{D}{N}$$

$$\because \quad \sigma^2 = var(x) = E(x^2) - E^2(x)$$

We must compute $E(x^2)$ to that we compute $E[x(x-1)] = E(x^2) - E(x)$

$$E[x(x-1)] = \sum_{\forall x} x(x-1)\,f(x) = \sum_{x=0}^{n} x(x-1)\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$

$$\therefore E[x(x-1)] = \frac{1}{\binom{N}{n}} \sum_{x=2}^{n} \frac{D!}{(x-2)!\,(D-x)!}\binom{N-D}{n-x}$$

$$= \frac{D(D-1)}{\binom{N}{n}} \sum_{x=2}^{n} \frac{(D-2)!}{(x-2)!\,(D-x)!}\binom{N-D}{n-x}$$

$$let \quad y = x-2 \ , \ m = n-2 \quad and \quad A = D-2$$

$$= \frac{D(D-1)}{\binom{N}{n}} \sum_{y=0}^{m} \frac{A!}{y!\,(A-y)!} \binom{N-A-2}{m-y}$$

$$= \frac{D(D-1)}{\binom{N}{n}} \sum_{y=0}^{m} \binom{A}{y} \binom{N-A-2}{m-y}$$

$$= \frac{D(D-1)}{\binom{N}{n}} \binom{N-2}{m} = \frac{D}{\binom{N}{n}} \binom{N-2}{n-2}$$

$$= \frac{D(D-1)}{\cancel{\binom{N}{n}}} \frac{n(n-1)}{N(N-1)} \cancel{\binom{N}{n}} = \frac{D(D-1)}{\mathrm{N}} \frac{n(n-1)}{(N-1)}$$

$$\Longmapsto E(x^2) = \frac{D(D-1)}{\mathrm{N}} \frac{n(n-1)}{(N-1)} + \frac{nD}{N}$$

$$\therefore \quad \sigma^2 = E(x^2) - E^2(x) = \frac{D(D-1)}{\mathrm{N}} \frac{n(n-1)}{(N-1)} + \frac{nD}{N} - \left(\frac{nD}{N}\right)^2$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{(D-1)(n-1)}{(N-1)} + 1 - \frac{nD}{\mathrm{N}}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{N(n-1)(D-1) + N(N-1) - nD(N-1)}{N(N-1)}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{\cancel{nND} - nN - ND \cancel{+N} + N^2 \cancel{-N} - \cancel{nND} + nD}{N(N-1)}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{N^2 - nN - ND + nD}{N(N-1)}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{N(N-n) - D(N-n)}{N(N-1)}\right]$$

$$= \frac{nD}{\mathrm{N}} \left[\frac{(N-n)(N-D)}{N(N-1)}\right]$$

$$\therefore \quad \sigma^2 = n \frac{D}{N} \; \frac{N-n}{N-1} \; \frac{N-D}{N}$$

The moment generating function of **Negative Binomial** distribution is given by

$$M_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} f(x) = \sum_{\forall x} e^{tx} \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$

$$= 1 + \sum_{k=1}^{\infty} [F(k+1) - 1] \frac{p}{k!}$$

Where: $F(k)$ is the sum of the first k-coefficient of the hyper-geometric series.

Note : If we set $p = \frac{D}{N}$ then the mean of the hyper-geometric distribution coincides with the mean of the binomial distribution, and the variance is $\frac{N-n}{N-1}$ times of the variance of binomial distribution.

Some continues Distribution:

1) Continues uniform distribution (Rectangular distribution) :

The r.v. X is said to have an uniform distribution if P.d.f. is give by

$$f(x,a,b) = \begin{cases} \dfrac{1}{b-a} & ;\ a \le x \le b \\ 0 & ;o.w. \end{cases} \quad ;Where\ a\ \ and\ \ b\ it\ satisfy - \infty < a < b < \infty$$

And denoted by $X \sim C_u(a,b)$

Clear that satisfy

$$f(x) \ge 0\ \ \forall\, x \quad and \quad \int_a^b f(x)\,dx = 1$$

Then f(x) satisfies the condition of begin a P.d.f. of continues type of random variable x.

$$Note:\ the\ mean\ \mu = \frac{a+b}{2} \quad and\ \ the\ variance\ \sigma^2 = \frac{(b-a)^2}{12} \quad (proof)$$

The moment generating function of uniform distribution is given by

$$M_x(t) = E(e^{tx}) = \int_a^b e^{tx}\, f(x)\, dx = \int_a^b e^{tx}\, \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\, dx \; = \; \frac{e^{bt} - e^{at}}{t(b-a)}$$

Example : If x is uniform distribution over (0 , 10) , calculate the probability that :
❶ *P(x< 3)* ❷ *P(3<x<8)* ❸ *P(x>6)* ❹The distribution function

Solution : $X \sim C_u(a,b) \qquad \therefore \; X \sim C_u(0{,}10)$

$$f(x,a,b) = \begin{cases} \dfrac{1}{b-a} & ;\; a \le x \le b \\ 0 & ; o.w. \end{cases} \Rightarrow f(x,0{,}10) = \begin{cases} \dfrac{1}{10} & ;\; 0 \le x \le 10 \\ 0 & ; o.w. \end{cases}$$

❶ $P(x < 3) = \int_0^3 \frac{1}{10}\, dx \; = \; \left[\frac{x}{10}\right]_0^3 = \; \frac{3}{10}$

❷ $P(x > 6) = \int_6^{10} \frac{1}{10}\, dx \; = \; \left[\frac{x}{10}\right]_6^{10} = \frac{10}{10} - \frac{6}{10} = \frac{4}{10}$

❸ $P(3 < x < 8) = \int_3^8 \frac{1}{10}\, dx \; = \; \left[\frac{x}{10}\right]_3^8 = \frac{8}{10} - \frac{3}{10} = \frac{5}{10}$

❹ $F(x) = P(X < x) = \int_{-\infty}^{x} f(u)\, du \; = \int_0^x \frac{1}{10}\, du \; = \; \left[\frac{u}{10}\right]_0^x = \frac{x}{10}$

$$\Longrightarrow \; F(x) = \begin{cases} 0 & ;\; x \le 0 \\ \dfrac{x}{10} & ; 0 < x < 10 \\ 1 & ; x \ge 10 \end{cases}$$

2) Normal distribution :

The normal distribution is one of the widely used in application of statistical methods the P.d.f. of X is give by:

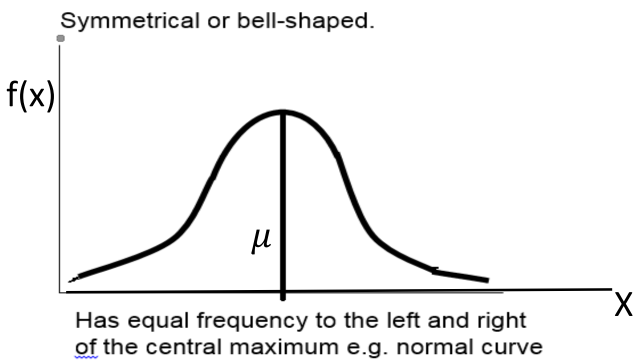
$$f(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad ;\ -\infty<x<\infty$$

Where the parameters μ and σ^2 satisfy $-\infty<\mu<\infty$ and $\sigma^2>0$,

And denoted by $X\sim N(\mu,\sigma^2)$

The properties of this p.d.f. are

1) It is a bell shaped curve is symmetric about μ



2) The two parameter that appear in the density μ and σ^2 represent the mean and the variance of the random variable X.

Now

Consider the integral

$$I=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$let\ y=\frac{x-\mu}{\sigma}\ \Longmapsto dy=\frac{dx}{\sigma}\ \Longmapsto\ dx=\sigma dy$$

$$\because\ \ -\infty<x<\infty\quad \therefore\ \ -\infty<y<\infty\quad \Longmapsto I=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}y^2}\,dy$$

$$\therefore\ I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx \quad and \quad I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dy$$

$$\therefore\ I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx \ .\ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \ .\ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dx\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} \ .\ e^{-\frac{1}{2}y^2}\, dx\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy$$

We change the variables to polar coordination by using

$x = r\cos\theta$ and $y = r\sin\theta$

$$\therefore\ I^2 = \frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{2\pi} r e^{-\frac{1}{2}r^2}\, d\theta\, dr = \frac{1}{2\pi}\int_{0}^{\infty} r e^{-\frac{1}{2}r^2} [\theta]_0^{2\pi}\, dr$$

$$= \int_{0}^{\infty} r e^{-\frac{1}{2}r^2} dr = \left[-e^{-\frac{1}{2}r^2}\right]_0^{\infty} = 1 \ \Longrightarrow I = 1$$

Then f(x) satisfies the condition of begin a P.d.f. of continues type to random variable x.

The moment generating function of uniform distribution is given by

$$M_x(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx}\, f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{tx} . e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[2x\sigma^2 t + x^2 - 2\mu x + \mu^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2x\left(\mu+\sigma^2 t\right) + \mu^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2x\left(\mu+\sigma^2 t\right) + \mu^2 + \left(\mu+\sigma^2 t\right)^2 - \left(\mu+\sigma^2 t\right)^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2x\left(\mu+\sigma^2 t\right) + \left(\mu+\sigma^2 t\right)^2\right]} e^{\frac{-\mu^2 + \left(\mu+\sigma^2 t\right)^2}{2\sigma^2}}\, dx$$

$$= \frac{e^{\frac{-\mu^2 + \left(\mu+\sigma^2 t\right)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2x\left(\mu+\sigma^2 t\right) + \left(\mu+\sigma^2 t\right)^2\right]}\, dx$$

$$= \frac{e^{\frac{-\mu^2 + \left(\mu+\sigma^2 t\right)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x - \left(\mu+\sigma^2 t\right)\right]^2}\, dx$$

$$If\ y = x - (\mu + \sigma^2 t) \Longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x - \left(\mu+\sigma^2 t\right)\right]^2}\, dx = I = 1$$

$$\therefore \quad M_x(t) = e^{\frac{-\mu^2+(\mu+\sigma^2 t)^2}{2\sigma^2}} = e^{\frac{-\mu^2+\mu^2+2\mu\sigma^2 t+(\sigma^2 t)^2}{2\sigma^2}}$$

$$\Longrightarrow \quad M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Note : The mean μ and the variance σ^2 of normal distribution will be calculated from $M_x(t)$ as following :

$$Since \quad M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \; E(x) = M'_x(0) \text{ and } E(x^2) = M''_x(0)$$

$$M'_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t) \Longrightarrow M'_x(0) = \mu$$

$$M''_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.\, \sigma^2 + e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)^2 \Longrightarrow M''_x(0) = \sigma^2 + \mu^2$$

$$\therefore \quad var(x) = E(x^2) - E^2(x) \Longrightarrow var(x) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

3) Standard normal distribution

If the normal random variable Z has mean zero and variance one instead of μ and σ^2 it is called standard normal distribution with P.d.f that:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad ; -\infty < z < \infty$$

And denoted by $Z \sim N(0,1)$

Theorem : If the r.v $X \sim N(\mu, \sigma^2), \sigma^2 > 0$; **then the r.v** $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Note : ❶ If the r.v $X \sim N(\mu, \sigma^2), \sigma^2 > 0$; then $P(a < x < b) = Z\left(\frac{b-\mu}{\sigma}\right) - Z\left(\frac{a-\mu}{\sigma}\right)$

❷ *We can write* $Z(x)$ *as the following forms* $N(x), \emptyset(x)$ *or* $F(x)$.

Remark: $N(-x) = 1 - N(x)$ *or* $Z(-x) = 1 - Z(x)$.

Note: $M_z(t) = e^{\frac{1}{2}t^2}$ and $Z(x) = N(x) = F(x) = Pr(Z \le x) = Pr(\frac{X-\mu}{\sigma} \le x)$

Example: Let $X \sim N(3,4)$. *Find* $P(X \le 4)$

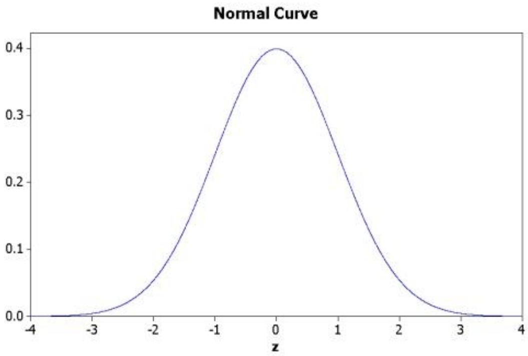
Solution :

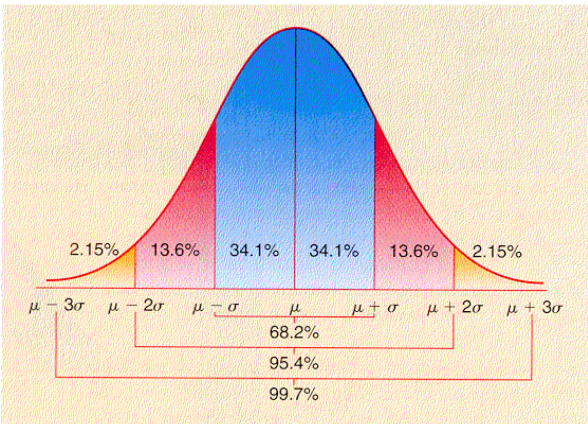
$$since\ X \sim N(3,4) \Longrightarrow \mu = 3 \ and\ \sigma^2 = 4 \Longrightarrow \sigma = 2$$

$$P(X \leq 4) = P\left(\frac{X-\mu}{\sigma} \leq \frac{4-\mu}{\sigma}\right) = P\left(Z \leq \frac{4-3}{2}\right) = Z\left(\frac{1}{2}\right) \; or \; \emptyset\left(\frac{1}{2}\right)$$

From Z table (standard normal table) we get $\emptyset\left(\frac{1}{2}\right) = Z\left(\frac{1}{2}\right) = 0.6915$

$\therefore \quad P(X \leq 4) = 0.6915$





Areas Under the Normal Curve