Chapter One

Reviewing of Matrices

Matrices

Matrix is an array of numbers, symbols, or expressions, arranged in rows and columns. A matrix is usually shown by capital letter. There are many things we can do with it.

Operations on Matrices

Adding

To add two matrices: add two numbers in the matching positions; note that the **two matrices must the same size**, where: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$A + B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Subtracting

To subtract two matrices: subtract two numbers in the matching positions; note that the **two matrices must the same size**, Subtracting is actually is addition of a negative matrix (*A+(-B)*),

where: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$A - B = A + (-B) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Multiply by a Constant (Scalar Multiplication)

We can multiply a matrix by a constant as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad 3 * A = 3 * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

H.W1: for the matrices $A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ find (a) ***3A***, (b) ***– B***, (c) ***3A-B***

Multiply by another Matrix

To multiply two matrices, we need to do the dot product. Note that the **number of columns for the first matrix must the same number of rows for the second one**.

where: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2*2}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}_{2*2}$

$$A * B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a*e + b*g & a*f + b*h \\ c*e + d*g & c*f + d*h \end{bmatrix}_{2*2}$$

While:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3*3} \quad , \quad B = \begin{bmatrix} j & k \\ l & m \\ n & o \end{bmatrix}_{3*2}$$

$$A * B = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix} * \begin{bmatrix} \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{l} & \boldsymbol{m} \\ \boldsymbol{n} & \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{a}*\boldsymbol{j} + \boldsymbol{b}*\boldsymbol{l} + \boldsymbol{c}*\boldsymbol{n}) & (\boldsymbol{a}*\boldsymbol{k} + \boldsymbol{b}*\boldsymbol{m} + \boldsymbol{c}*\boldsymbol{p}) \\ (\boldsymbol{d}*\boldsymbol{j} + \boldsymbol{e}*\boldsymbol{l} + \boldsymbol{f}*\boldsymbol{n}) & (\boldsymbol{d}*\boldsymbol{k} + \boldsymbol{e}*\boldsymbol{m} + \boldsymbol{f}*\boldsymbol{p}) \\ (\boldsymbol{g}*\boldsymbol{j} + \boldsymbol{h}*\boldsymbol{l} + \boldsymbol{i}*\boldsymbol{n}) & (\boldsymbol{g}*\boldsymbol{k} + \boldsymbol{h}*\boldsymbol{m} + \boldsymbol{i}*\boldsymbol{p}) \end{bmatrix}_{3*2}$$

H.W2: find the product ***A B***, where: $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}_{3*2}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}_{2*2}$

H.W3: find the product ***A B***, where: $A = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}_{3*1}$ and $B = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}_{1*3}$

H.W4: find the product ***A B***, where: $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}_{3*2}$ and $B = \begin{bmatrix} 1 & 4 \\ -2 & 5 \\ 0 & -1 \end{bmatrix}_{3*2}$

Transposing

To transpose the matrix, swap the rows and columns.

where: $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2*3}$, then $A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3*2}$

properties of Transposing:

1. $(A^T)^T = A$ **Transpose of transpose**
2. $(A+B)^T = A^T + B^T$ **Transpose of a sum**
3. $(c\,A)^T = c\,(A^T)$ **Transpose of scalar multiple**
4. $(A\,B)^T = B^T\,A^T$ **Transpose of a product**

H.W5: show that $(A\,B)^T$ and $B^T\,A^T$ are equal, where: $A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix}_{3*3}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}_{3*2}$

Trace of matrix

The trace of matrix A_{n*n} (**square** matrix) is denoted trace { } and is **equal to the sum of its diagonal elements**, where: $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}_{2*2}$, then tr(A) or $\{A\} = a + b$

H.W6:

$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}_{3*2}$, $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}_{2*2}$, $C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}_{2*3}$, $D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}_{3*3}$, $E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}_{3*3}$

Find: (1) ***tr(D)***, (2) ***tr(A)***, (3) ***tr (D – 3E)***, (4) ***tr (B C)***

(5) ***A B***, (6) ***B A***, (7) $\boldsymbol{C\,C^T}$, (8) $\boldsymbol{(D\,A)^T}$

(9) $\boldsymbol{2A^T + C}$, (10) $\boldsymbol{D^T - E^T}$, (11) $\boldsymbol{(D - E)^T}$

Determinant of a Matrix

The determinant of the matrix ***A*** denoted by **| |**, then the determent is **| *A* |**;

Determinant of a matrix of size 2*2:

where: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2*2}$, then $|\mathbf{A}| = a * d - b * c$

EX: find $|\boldsymbol{A}|$ for $\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}_{2*2}$

$|\mathbf{A}| = (2 * 1) - (0 * 0) = 2 - 0 = 2$

Determinant of a matrix of size 3*3:

where: $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3*3}$, then $|B| = +a(e*i - f*h) - b(d*i - f*g) + c(d*h - e*g)$

Ex: find $|\boldsymbol{B}|$ for $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix}_{3*3}$

$|B| = +1\big((-2*3) - (1*-1)\big) - 0\big((2*3) - (1*0)\big) + -3((2*-1) - (-2*0))$

$|B| = +1(-6+1) - 0(6-0) - 3(-2-0)$

$|B| = +1(-5) - 0 - 3(-2)$

$|B| = -5 + 6$

$|B| = 1$

Inverse of a Matrix

The inverse of a matrix $\boldsymbol{A}$ denoted as $\boldsymbol{A^{-1}}$; $\boldsymbol{A^{-1}} = \frac{adj\,(A)}{|\mathbf{A}|}$

Inverse of a matrix of size 2×2 defined as follow:

Where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2*2}$, then $|\mathbf{A}| = a*d - b*c$, $adj\,(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}_{2*2}$

$\boldsymbol{A^{-1}} = \frac{adj\,(A)}{|\mathbf{A}|} = \frac{1}{a*d-b*c}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}_{2*2}$

Inverse of a matrix of size 3×3 defined as follow:

Where $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_{3*3}$, then $\boldsymbol{A^{-1}} = \frac{adj\,(A)}{|\mathbf{A}|}$, $\boldsymbol{adj\,(A) = (Cof.\ A)^T}$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3*3}$

$|A| = \boldsymbol{+ a_{11}[(a_{22}*a_{33}) - (a_{23}*a_{32})] - a_{12}[(a_{21}*a_{33}) - (a_{23}*a_{31})] + a_{13}[(a_{21}*a_{32}) - (a_{22}*a_{31})]}$

$A_{11} = (-1)^{i+j} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (-1)^{i+j}\,[(a_{22}*a_{33}) - (a_{23}*a_{32})]$

$$A_{12} = (-1)^{i+j} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (-1)^{i+j} [(a_{21} * a_{33}) - (a_{23} * a_{31})]$$

$$A_{13} = (-1)^{i+j} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (-1)^{i+j} [(a_{21} * a_{32}) - (a_{22} * a_{31})]$$

$$A_{21} = (-1)^{i+j} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = (-1)^{i+j} [(a_{12} * a_{33}) - (a_{13} * a_{32})]$$

$$A_{22} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = (-1)^{i+j} [(a_{21} * a_{33}) - (a_{23} * a_{31})]$$

$$A_{23} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = (-1)^{i+j} [(a_{11} * a_{32}) - (a_{12} * a_{31})]$$

$$A_{31} = (-1)^{i+j} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (-1)^{i+j} [(a_{12} * a_{23}) - (a_{13} * a_{22})]$$

$$A_{32} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = (-1)^{i+j} [(a_{11} * a_{23}) - (a_{13} * a_{21})]$$

$$A_{33} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (-1)^{i+j} [(a_{11} * a_{22}) - (a_{12} * a_{21})]$$

$$\boldsymbol{Cof.\ A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_{3*3}, \qquad (\boldsymbol{Cof.\ A})^{\boldsymbol{T}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^{T}_{3*3}$$

$$\boldsymbol{adj}\ (\boldsymbol{A}) = (\boldsymbol{Cof.\ A})^{\boldsymbol{T}} \ , \ \boldsymbol{A}^{-1} = \frac{\mathbf{1}}{|\mathbf{A}|}(\boldsymbol{Cof.\ A})^{\boldsymbol{T}} \ ,$$

For check**:** $\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I}$

Note: For check the solution of inverse, the following condition must be met:

$\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I}$, where $\boldsymbol{I}$ is a unity matrix.

Note: A general ($n*n$) matrix can be inverted using methods such as (minors, cofactors and adjugate), the Gauss-Jorden elimination, Gaussian elimination, or LU decomposition. The square matrix is called invertible (or non-singular) if it have inverse & it is called singular if determinant of it equal zero.

Note: Let $\boldsymbol{A}$ and $\boldsymbol{D}$ two square matrices of same size,

$\boldsymbol{D}$ is called similar $\boldsymbol{A}$ to if there exists an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C}$, where $\boldsymbol{C}$ is called the modal matrix, The transformation $\boldsymbol{A}$ into $\boldsymbol{D}$ using $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\boldsymbol{C}$ is said to be similarity transformation.

H.W7: find $\boldsymbol{A^{-1}}$ for $\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}_{2*2}$

H.W8: find $\boldsymbol{B^{-1}}$ for $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix}_{3*3}$

H.W9: $\boldsymbol{D} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, find if $\boldsymbol{D}$ is similar to $\boldsymbol{A}$

Dividing

Dividing two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

$$\boldsymbol{A/B = A * B^{-1}}$$

Where $\boldsymbol{B^{-1}}$ is the inverse of a matrix $\mathbf{B}$.

Types of Matrices

Symmetric Matrix

A matrix is **symmetric** if it is equal to its own transpose; it is symmetric across the diagonal.

Ex: $A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$; $A = A^T$ Then A is a symmetric matrix.

Antisymmetric Matrix

Antisymmetric (skew-symmetric) matrix is a square matrix whose transpose equals its negative.

Ex: Where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $-A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A^T = -A$ Then A is antisymmetric matrix.

Orthogonal Matrix

A matrix is called **orthogonal** if $A^T = A^{-1}$ (i.e. $A\,A^T = A^T\,A = 1$).

H.W10: for the matrix $A = \begin{bmatrix} \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$, prove that A is orthogonal matrix.

Orthonormal Matrix

A matrix is called **orthonormal** if:

A is a square matrix, $A^T = A^{-1}$, $|A| = 1$

H.W11: for the matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ prove that A is orthonormal matrix.

Diagonal Matrix

A matrix is called a **diagonal** matrix in which the entries outside the main diagonal are all zero.

Lower Triangular Matrix

A square matrix is called lower triangular if all the entries above the main diagonal are zero.

Upper Triangular Matrix

A square matrix is called upper triangular if all the entries below the main diagonal are zero.

Hermitian transpose Matrix

$\boldsymbol{A^H}$ denotes to the conjugate transpose or **Hermitian transpose** by taking the transpose of and then complex conjugate of each entry,

(i.e. negating their imaginary parts but not their real parts).

Ex: Where $A = \begin{bmatrix} 1 & -2-i \\ 1+i & i \end{bmatrix}$, then $A^H = \begin{bmatrix} 1 & 1-i \\ -2+i & -i \end{bmatrix}$

Unitary Matrix

The square matrix $\boldsymbol{A}$ is called **unitary matrix** if $\boldsymbol{A^H = A^{-1}}$.

The unitary matrix leave the length of a complex vector unchanged, but for real matrix, unitary is the same as orthogonal.

Ex: $A = \begin{bmatrix} \boldsymbol{2^{-\frac{1}{2}}} & \boldsymbol{2^{-\frac{1}{2}}} & \boldsymbol{0} \\ \boldsymbol{-i2^{-\frac{1}{2}}} & \boldsymbol{i2^{-\frac{1}{2}}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{i} \end{bmatrix}$ is a unitary matrix.

Hermitian Matrix

Hermitian matrix (self-adjoint matrix) is a complex square matrix that equal to its own conjugate transpose.

Ex: $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$ is Hermitian matrix.

Skew-Hermitian Matrix

Skew-Hermitian matrices: $A^H = -A$.

Ex: $A = \begin{bmatrix} -i & 2+i \\ -2+i & 0 \end{bmatrix}$, prove that $\boldsymbol{A}$ is Skew-Hermitian matrix.

$$A^H = \begin{bmatrix} i & -2-i \\ 2-i & 0 \end{bmatrix}, \qquad -A = \begin{bmatrix} i & -2-i \\ 2-i & 0 \end{bmatrix}$$

$A^H = -A$ matrix is Skew-Hermitian

Normal Matrix

Normal matrices: $A^H A = A A^H$.

Special Matrices

Diagonally Dominant Matr

A matrix is called **diagonally dominant matrix** if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of magnitudes of all the other (non-diagonal) entries in that row $|\boldsymbol{a_{ii}}| \geq \sum \boldsymbol{i \neq j} |\boldsymbol{a_{ij}}|$.

However a matrix is called **strictly diagonally dominant** if $|\boldsymbol{a_{ii}}| > \sum \boldsymbol{i \neq j} |\boldsymbol{a_{ij}}|$ and strictly diagonally dominant is non- singular.

Ex:

$$\boldsymbol{A} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & 2 \\ -1 & 2 & 4 \end{bmatrix}_{3*3}$$ is diagonally dominant since $|3| \geq |-2| + |1|$, $|-3| \geq |1| + |2|$, $|4| \geq |-1| + |2|$

H.W 12: Classify the following matrices as diagonally dominant, strictly diagonally dominant or unknown:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} -6 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & -2 & 7 \end{bmatrix}.$$

Band Matrix

A band matrix is a spare matrix (i.e. a matrix in which most of the elements are zero) whose non-zero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on the other side.

Ex: $$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{5*5}$$ matrix is Band matrix.

Tridiagonal Matrix

A square matrix with non zero elements only on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal (i.e. along the subdiagonal and superdiagonal).

Ex: $A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}_{4*4}$ matrix is Tridiagonal.

The tridiagonal is not necessary symmetric and it is kind of band matrix.

Monotone Matrix

A monotonic matrix of order n is an (n*n) matrix in which every element is either zero or contains a number from the set {1, 2, ….,n}

(i.e. A is an (n*n) matrix is monotone if all elements of A^{-1} are nonnegative), Matrix is non-singular matrix.

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ These matrices are monotone

Pseudo Inverse of a Matrix

The matrix $\boldsymbol{(A^T A)^{-1} A^T}$ is called pseudo inverse of a matrix $\boldsymbol{A}$ and denoted by $\boldsymbol{pinv}(\boldsymbol{A})$.

The pseudo inverse can be expressed of a rectangular matrix, or not invertible square matrix

Ex: Find $\boldsymbol{A^{-1}}$ for the following matrix: $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}_{3*2}$

Solution: $A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}_{2*3} * \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}_{3*2}$

$$= \begin{bmatrix} (1*1)+(-1*-1)+(2*2) & (1*1)+(-1*1)+(2*3) \\ (1*1)+(1*-1)+(3*2) & (1*1)+(1*1)+(3*3) \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{adj\ (A^T A)}{|A^T A|}$$

$$|A^T A| = 6*11 - 6*6 = 66 - 36 = 30$$

$$adj\ (A^T A) = \begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{adj\ (A^T A)}{|A^T A|} = \frac{1}{30}\begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} \frac{11}{30} & -\frac{6}{30} \\ -\frac{6}{30} & \frac{6}{30} \end{bmatrix}$$

$$pinv(A) = (A^T A)^{-1}\, A^T = \frac{1}{30}\begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 5 & -17 & 4 \\ 0 & 12 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{-17}{30} & \frac{2}{15} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Chapter two

Eigenvalues, Eigenvectors and its Applications

Suppose that is a square $(\boldsymbol{n} * \boldsymbol{n})$ matrix. We say that a nonzero vector $\boldsymbol{v}$ is an eigenvector $(\boldsymbol{ev})$ and a scalar $\boldsymbol{\lambda}$ is its eigenvalue $(\boldsymbol{ew})$ if:

$$\boldsymbol{Av = \lambda v} \quad \text{.....(2-1)}$$

Geometrically this means that is in the same or appositive direction as $\boldsymbol{v}$, depending on the sign of λ.

Notice that Equation (2.1) can be rewritten as follows: $\boldsymbol{Av - \lambda v = 0}$

Since $\boldsymbol{Iv = v}$, we can do the following:

$$\boldsymbol{Av - \lambda v = \ Av - \lambda Iv = (A - \lambda I)v = 0}$$

If $\boldsymbol{v}$ is nonzero, then the matrix $(\boldsymbol{A - \lambda I})$ must be singular and $|\boldsymbol{A - \lambda I}|$

This is called the characteristic equation (or characteristic polynomial $\boldsymbol{p(\lambda)}$)

<u>Calculating Eigenvalues and Eigenvectors</u>

If is (2 × 2) or (3 × 3) matrix then we can find its eigenvalues and eigenvectors by hand.

<u>**Eigenspace of $\boldsymbol{\lambda}$**</u> is a space consist of set of all eigenvectors of a square $(n * n)$ matrix $(\boldsymbol{A})$ corresponding to $\boldsymbol{\lambda}$, together with zero vector is a subspace of $\boldsymbol{R^n}$

Ex: Find eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$

solution:

$$\boldsymbol{A - \lambda I} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} - \boldsymbol{\lambda} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda} & 0 \\ 0 & \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{1 - \lambda} & 4 \\ 3 & \boldsymbol{5 - \lambda} \end{bmatrix}$$

$|\boldsymbol{A - \lambda I}| = \boldsymbol{(1 - \lambda)(5 - \lambda) - (4 * 3) = (5 - \lambda - 5\lambda + \lambda^2) - 12 = (\lambda^2 - 6\lambda + 5) - 12 = \ \lambda^2 - 6\lambda - 7}$ (This called characteristic polynomial)

characteristic polynomial $\boldsymbol{\lambda^2 - 6\lambda - 7 = 0 \rightarrow \ (\lambda - 7)(\lambda + 1) = 0}$

$$\left.\begin{aligned} \boldsymbol{(\lambda - 7) = 0} \ &\boldsymbol{\rightarrow \lambda = 7} \\ \boldsymbol{(\lambda + 1) = 0} \ &\boldsymbol{\rightarrow \lambda = -1} \end{aligned}\right\} \boxed{\text{eigenvalues of } \boldsymbol{\lambda}}$$

Now find eigenvectors at the values of $\boldsymbol{\lambda}$; as following:

$\boldsymbol{\lambda = 7}$

$(\boldsymbol{A}-\boldsymbol{\lambda I})\boldsymbol{v} = \boldsymbol{0} \;\rightarrow\; (\boldsymbol{A}-\boldsymbol{7I})\boldsymbol{v} = \boldsymbol{0}$

$\left(\begin{bmatrix}1 & 4\\ 3 & 5\end{bmatrix} - \begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$

$\left(\begin{bmatrix}1-7 & 4-0\\ 3-0 & 5-7\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$

$\begin{bmatrix}-6 & 4\\ 3 & -2\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$

$\begin{bmatrix}-6x+4y\\ 3x-2y\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$

$-6x+4y = 0 \rightarrow \boldsymbol{-6x = -4y} \rightarrow \boldsymbol{3x = 2y}$

$3x-2y = 0 \rightarrow \boldsymbol{3x = 2y}$

Then ***(x, y)= (2,3)***

Thus the eigenvectors of $\boldsymbol{A}$ when $\boldsymbol{\lambda} = \boldsymbol{7}$ are non-zero vectors of form $\boldsymbol{r_1} = \begin{bmatrix}\boldsymbol{2}\\ \boldsymbol{3}\end{bmatrix}, \boldsymbol{r_1} \in \boldsymbol{R}\,\{0\}$

The $\boldsymbol{S_1} = \{\boldsymbol{r_1}\begin{bmatrix}\boldsymbol{2}\\ \boldsymbol{3}\end{bmatrix}, \boldsymbol{r_1} \in \boldsymbol{R}\}$ is a subspace of $\boldsymbol{R^2}$.

$\boldsymbol{\lambda = -1}$

$(\boldsymbol{A}-\boldsymbol{\lambda I})\boldsymbol{v} = \boldsymbol{0} \;\rightarrow\; (\boldsymbol{A}+\boldsymbol{1I})\boldsymbol{v} = \boldsymbol{0}$

$\left(\begin{bmatrix}1 & 4\\ 3 & 5\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$

$\left(\begin{bmatrix}1+1 & 4+0\\ 3+0 & 5+1\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$

$\begin{bmatrix}2 & 4\\ 3 & 6\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$

$\begin{bmatrix}2x+4y\\ 3x+6y\end{bmatrix} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}$

$2x+4y = 0 \rightarrow \boldsymbol{2x = -4y} \rightarrow \boldsymbol{x = -2y}$

$3x+6y = 0 \rightarrow \boldsymbol{3x = -6y} \rightarrow \boldsymbol{x = -2y}$

Then ***(x, y)= (-2,1)***

Thus the eigenvectors of $\boldsymbol{A}$ when $\boldsymbol{\lambda} = \boldsymbol{-1}$ are non-zero vectors of $\boldsymbol{\lambda}$ form $\boldsymbol{r_2} = \begin{bmatrix}\boldsymbol{-2}\\ \boldsymbol{1}\end{bmatrix}$

The $\boldsymbol{S_2} = \{\boldsymbol{r_2}\begin{bmatrix}\boldsymbol{-2}\\ \boldsymbol{1}\end{bmatrix}, \boldsymbol{r_2} \in \boldsymbol{R}\}$ is a subspace of $\boldsymbol{R^2}$.

Ex: Find eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix}$

solution:

$$\boldsymbol{A-\lambda I} = \begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix} - \boldsymbol{\lambda} * \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix} - \begin{bmatrix}\boldsymbol{\lambda} & 0 & 0\\ 0 & \boldsymbol{\lambda} & 0\\ 0 & 0 & \boldsymbol{\lambda}\end{bmatrix} = \begin{bmatrix}1-\boldsymbol{\lambda} & -3 & 3\\ 3 & -5-\boldsymbol{\lambda} & 3\\ 6 & -6 & 4-\boldsymbol{\lambda}\end{bmatrix}$$

$$|\boldsymbol{A-\lambda I}| = \{(\boldsymbol{1-\lambda})[(\boldsymbol{-5-\lambda})(\boldsymbol{4-\lambda}) - (\boldsymbol{3*-6})] - (\boldsymbol{-3})[(\boldsymbol{3})(\boldsymbol{4-\lambda}) - (\boldsymbol{3*6})] + (\boldsymbol{3})[(\boldsymbol{3*-6}) - (\boldsymbol{-5-\lambda})(\boldsymbol{6})]\}$$

$$= \{(\boldsymbol{1-\lambda})[(\boldsymbol{-5-\lambda})(\boldsymbol{4-\lambda}) - \boldsymbol{12}] + \boldsymbol{3}[(\boldsymbol{3})(\boldsymbol{4-\lambda}) - \boldsymbol{18}] + (\boldsymbol{3})[\boldsymbol{-18+30+6\lambda}]\}$$

$$= \{(\boldsymbol{1-\lambda})[(\boldsymbol{-5-\lambda})(\boldsymbol{4-\lambda}) - \boldsymbol{12}] + [(\boldsymbol{36-9\lambda}) - \boldsymbol{54}] + [\boldsymbol{-54+90+18\lambda}]\}$$

$$= \{(\boldsymbol{1-\lambda})[\boldsymbol{-20+5\lambda-4\lambda+\lambda^2-12}] + \boldsymbol{36-9\lambda-54-54+90+18\lambda}\}$$

$$= \boldsymbol{-\lambda^3+12\lambda+16}$$

$|A - \lambda I| = 0$

$-\lambda^3 + 12\lambda + 16 = 0 \Longrightarrow \lambda^3 - 4\lambda - 8\lambda - 16 = 0 \Longrightarrow (\lambda^3 - 4\lambda) - (8\lambda + 16) = 0$

$\lambda(\lambda^2 - 4) - 8(\lambda + 2) = 0 \Longrightarrow \lambda(\lambda + 2)(\lambda - 2) - 8(\lambda + 2) = 0$

$(\lambda + 2)(\lambda(\lambda - 2) - 8) = 0 \Longrightarrow (\lambda + 2)(\lambda^2 - 2\lambda - 8) = 0 \Longrightarrow (\lambda + 2)(\lambda + 2)(\lambda - 4) = 0$

$(\lambda + 2) = 0 \Longrightarrow (\lambda = -2)$, $(\lambda - 4) = 0 \Longrightarrow (\lambda = 4)$

Now find eigenvectors at the values of ; as following:

$\lambda = 4$

$(A - \lambda I)v = 0 \rightarrow (A - 4I)v = 0$

$$\left(\begin{bmatrix}1 & -3 & 3\\3 & -5 & 3\\6 & -6 & 4\end{bmatrix} - \begin{bmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 4\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left(\begin{bmatrix}-3 & -3 & 3\\3 & -9 & 3\\6 & -6 & 0\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-3x - 3y + 3z\\3x - 9y + 3z\\6x - 6y\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-3x - 3y + 3z = 0$

$3x - 9y + 3z = 0$

$6x - 6y = 0$

Then $(x, y, z) = (\frac{1}{2}, \frac{1}{2}, 1)$

Thus the eigenvectors of A when $\lambda = 4$ are non-zero vectors of form $r_1 = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix}, r_1 \in R\,\{0\}$

The $S_1 = \{r_1 \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix}, r_1 \in R\}$ is a subspace of R^3.

$\lambda = -2$

$(A - \lambda I)v = 0 \rightarrow (A + 2I)v = 0$

$$\left(\begin{bmatrix}1 & -3 & 3\\3 & -5 & 3\\6 & -6 & 4\end{bmatrix} - \begin{bmatrix}-2 & 0 & 0\\0 & -2 & 0\\0 & 0 & -2\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left(\begin{bmatrix}3 & -3 & 3\\3 & -3 & 3\\6 & -6 & 6\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}3x - 3y + 3z\\3x - 3y + 3z\\6x - 6y + 6z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$3x - 3y + 3z = 0$

$3x - 3y + 3z = 0$

$6x - 6y + 6z = 0$

Then $(x, y, z) = (0,1,1)$

Thus the eigenvectors of A when $\lambda = -2$ are non-zero vectors of λ form $r_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, r_2 \in R\,\{0\}$

The $S_2 = \{r_2 \begin{bmatrix}0\\1\\1\end{bmatrix}, r_2 \in R\}$ is a subspace of R^3.

Complex Eigenvalues

It turns out that the eigenvalues of some matrices are complex numbers, even when the matrix only contains real numbers. When this happens the complex ew's must occur in conjugate pairs, i.e., $\lambda_{1,2} = \alpha \pm \beta i$

The corresponding ev's must also come in conjugate pairs: $w = u \pm vi$

Ex: Find eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

solution: $A - \lambda I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$|A - \lambda I| = (-\lambda)(-\lambda) - (1 * -1) = \lambda^2 + 1$ (This called characteristic polynomial)

characteristic polynomial $\lambda^2 + 1 = 0 \rightarrow \lambda^2 = -1 = \pm i$

$$\left.\begin{matrix} \lambda = +i \\ \lambda = -i \end{matrix}\right\} \boxed{\text{eigenvalues of } \lambda}$$

Now find eigenvectors at the values of λ ; as following:

$\lambda = i$

$(A - \lambda I)v = 0 \rightarrow (A - iI)v = 0$

$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} i & -1 \\ 1 & -i \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -ix - y \\ x - iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-ix - y = 0$

$x - iy = 0$

Then $(x, y) = (1, -i)$

Thus the eigenvectors of A when $\lambda = i$ are non-zero vectors of form $r_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, r_1 \in R\,\{0\}$

The eigenspace $= \{\begin{bmatrix} Z_1 \\ -iZ_2 \end{bmatrix}, Z_1, Z_2 \in R\}$

$\lambda = -i$

$(A - \lambda I)v = 0 \rightarrow (A - -iI)v = 0$

$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} ix - y \\ x + iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$ix - y = 0$

$x + iy = 0$

Then $(x, y) = (1, i)$

Thus the eigenvectors of A when $\lambda = i$ are non-zero vectors of form $r_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}, r_2 \in R\,\{0\}$

The eigenspace $= \{\begin{bmatrix} Z_1 \\ iZ_2 \end{bmatrix}, Z_1, Z_2 \in R\}$

Complex Eigenvalues

Notes:

- An eigenvalue of a A_{n*n} is a root of the characteristic polynomial.
- There are n distinct eigenvalues λ of A because $det(A-\lambda I)=0$.
- If A be an upper triangular or lower triangular or diagonal matrix then its eigenvalues are the diagonal elements.
- If A be a square matrix then A and A^T have the same eigenvalues.
- If A be a square matrix then $|A|$ is equal to the product of all eigenvalues of A.
- The set of all the eigenvalues of A is referred to as the spectrum of A and denoted by (A) and the maximum modulus of the eigenvalues is called spectral radius and denoted by $\rho(A)$:

$$\rho(A)=max|\lambda|, \lambda\epsilon v(A)$$

H.W1/ Find all eigenvalues and their eigenspace for $A=\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}2 & -1\\ -4 & 2\end{bmatrix}$?

H.W 2/ Given that 2 is an eigenvalue for $=\begin{bmatrix}4 & -1 & 6\\ 2 & 1 & 6\\ 2 & -1 & 8\end{bmatrix}$, Find a basis of its eigenspace?

H.W 3/ Find eigenvalue and a basis of each eigenspace for: (a) $A=\begin{bmatrix}4 & -2\\ -3 & 9\end{bmatrix}$, (b) $B=\begin{bmatrix}7 & 4 & 6\\ -3 & -1 & -8\\ 0 & 0 & 1\end{bmatrix}$.

Cayley-Hamilton Theorem

Arthur Cayley (16 August 1821-26 January 1895) was a British mathematician.

Let A be a square $(n\times n)$ matrix with characteristic polynomial

$p\ (\lambda)=\ \lambda^n+c_1\lambda^{n-1}+\cdots+c_{n-1}\lambda+c_n$ and $\lambda^n+c_1\lambda^{n-1}+\cdots+C_{n-1}\lambda+c_n=0$

Then $A^n+c_1A^{n-1}+\cdots+c_{n-1}A+c_nI_n=0$

Theorem: Let A be an $(n\times n)$ matrix with a characteristic equation

$$|(A-\lambda I)|=\boldsymbol{\Delta}(\boldsymbol{\lambda})$$

$$=(-\lambda)^n+C_{n-1}\lambda^{n-1}+\cdots+c_1\lambda+c_0$$

$$=0$$

$$\Delta(A)=(-1)^nA^n+c_{n-1}A^{n-1}+\cdots+c_1A+c_0I$$

$$= 0$$

Ex 4: Apply Cayley-Hamilton Theorem on matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$?

solution: $P(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6$

$$= 2 - \lambda - 2\lambda + \lambda^2 - 6$$

نعوض المصفوفة (A) في متعدد الحدود (λ) لتكن P(A)

$$p(A) = A^2 - 3A - 4I$$

$$p(A) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$p(A) = \begin{bmatrix} 1+6 & 2+4 \\ 3+6 & 6+4 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$p(A) = \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Ex 5: Apply Cayley-Hamilton Theorem on matrix $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$?

solution: $\lambda^2 - (tr.A)\lambda + det.A = 0$

$$\lambda^2 - 3\lambda + [(0 \times 3) - (-1 \times 2)] = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$A^2 - 3A + 2I = 0$$

$$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

H.W4 / Apply Cayley-Hamilton Theorem on matrix for: (a) $A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$, (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

Eigenvalues and Eigenvectors of symmetric matrix

A matrix is symmetric if it is equal to its own transpose, in symmetric matrix the upper right left and the lower left half of the matrix are mirror images of each other about the diagonal. A$(n \times n)$ symmetric matrix not only has a nice structure, but it also satisfied the following:

- It has exactly n eigenvalues (not necessary distinct).
- There exists a set of n eigenvectors, one for each eigenvalue, that are mutually orthogonal.
- A symmetric matrix has n eigenvalues and there exist linearly independent eigenvectors (because of orthogonal) even if the eigenvalues are not distinct.

Ex6: Find eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$?

solution: $\boldsymbol{A - \lambda I} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \boldsymbol{\lambda} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}$

$$|A - \lambda I| = (5-\lambda)(5-\lambda) - (3*3) = \lambda^2 - 10\lambda + 25 - 9 = \lambda^2 - 10\lambda + 16$$

$$= (\lambda - 8)(\lambda - 2) = 0 \rightarrow \lambda = 8, 2$$

Now find eigenvectors at the values of λ ; as following:

$\lambda = 8$

$(A - \lambda I)v = 0 \rightarrow (A - 8I)v = 0$

$\left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -3x + 3y \\ 3x - 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-3x + 3y = 0$

$3x - 3y = 0$

Then $(x, y) = (1, 1)$

Thus the eigenvectors of A when $\lambda = 8$ are non-zero vectors of form $r_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, r_1 \in R/\{0\}$

The eigenspace $= \{\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, Z_1, Z_2 \in R\}$

$\lambda = 2$

$(A - \lambda I)v = 0 \rightarrow (A - 2I)v = 0$

$\left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3x + 3y \\ 3x + 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x + 3y = 0$

$3x + 3y = 0$

Then $(x, y) = (1, -1)$

Thus the eigenvectors of A when $\lambda = 2$ are non-zero vectors of form $r_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, r_2 \in R/\{0\}$

The eigenspace $= \{\begin{bmatrix} Z_1 \\ -1 \end{bmatrix}, Z_1, Z_2 \in R\}$

Thus we have two orthogonal eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (linearly independent).

Ex7: Apply Cayley-Hamilton Theorem on $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$ $and\ then\ find\ A^{-1}\ and\ A^3$?

solution:

$$P(\lambda) = \lambda^3 - (trace\ of\ A)\,\lambda^2 + (sum\ of\ minors\ of\ diagonals\ of\ A)\lambda - det.A = 0$$

$(trace\ of\ A) = 4 + 2 + 1 = 7$

$(sum\ of\ minors\ of\ diagonals\ of\ A)$ = $\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix}$=2+3+2=7

$det.A$ =4[(2*1)-(0*-2)] -3[(2*1) - (1*-2)] + 1[(2*0)-(2*1)]=4[2] – 3[4] +[-2]=8-12-2 = - 6

$$P(\lambda) = \lambda^3 - 7\lambda^2 + 7\lambda + 6 = 0$$

By Cayley-Hamilton Theorem:

$A^3 - 7A^2 + 7A + 6 = 0$

Find A^{-1} multiply by A^{-1}

$A^2 - 7A + 7 + 6A^{-1} = 0$

$A^{-1} = \frac{-1}{6}(A^2 - 7A + 7I)$

$$A^2 = A * A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 18 & -1 \\ 10 & 10 & -4 \\ 5 & 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{6}\left(\begin{bmatrix} 23 & 18 & -1 \\ 10 & 10 & -4 \\ 5 & 3 & 2 \end{bmatrix} - 7\begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \frac{-1}{6}\begin{bmatrix} 2 & -3 & -8 \\ -4 & -11 & 10 \\ -2 & 3 & 2 \end{bmatrix}$$

Find A^3

$A^3 - 7A^2 + 7A + 6 = 0$

$A^3 = 7A^2 - 7A - 6I$

$$A^3 = 7\begin{bmatrix} 23 & 18 & -1 \\ 10 & 10 & -4 \\ 5 & 3 & 2 \end{bmatrix} - 7\begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 161 & 126 & -7 \\ 70 & 70 & -28 \\ 35 & 21 & 14 \end{bmatrix} - \begin{bmatrix} 28 & 21 & 7 \\ 14 & 14 & -14 \\ 7 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 127 & 105 & -14 \\ 56 & 50 & -14 \\ 28 & 21 & 1 \end{bmatrix}$$

Ex8: Use the cayley-Hamilton theorem to calculate A^5 for $A = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}$?

solution:

$$\lambda^2 - (tr.A)\lambda + det.A = 0$$

$$\lambda^2 - [5 + 2]\lambda + [(5 * 2) - (-6 * -3)] = 0$$

$$\lambda^2 - 7\lambda - 8 = 0$$

$$A^2 - 7A - 8I = 0$$

$$A^2 = 7A + 8I$$

$$A^4 = (A^2)^2 = (7A + 8I)^2$$

$$A^4 = 49A^2 + 2(7A)(8I) + 64I^2$$

$$A^4 = 49\,A^2 + 112A + 64I^2$$

$$A^4 = 49(7A + 8I) + 112A + 64I^2$$

$$A^4 = 343A + 392I + 112A + 64I^2$$

$$A^4 = 455A + 456I$$

$$A^5 = A^4A = 455A^4 + 456A$$

$$A^5 = 455(7A + 8I) + 456A$$

$$A^5 = 3641A + 3640I$$

$$A^5 = 3641\begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix} + 3640\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 18205 + 3640 & -10923 + 0 \\ -21846 & 7282 + 3640 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 21845 & -10923 \\ -21846 & 10922 \end{bmatrix}$$

H.W.5: Using Cayley-Hamilton Theorem to $find\ A^{-1}\ and\ A^3$ of $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$?

Applications of Eigenvalues and Eigenvectors

In this section some applications of eigenvalues and eigenvectors are introduce to illustrate the importance of topic.

Positive Definite Matrices

A symmetric ($n \times n$) real matrix $\boldsymbol{A}$ is said to be positive definite if the scalar($v^T Av$) is positive for every nonzero column vector v of n real numbers. However, a Hermitian matrix $\boldsymbol{A}$ is said to be positive definite if the scalar ($v^T Av$) is real and positive for all nonzero column vector of complex numbers.

Ex10: Find if the following matrices are positive definite:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution:

The identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is positive definite because a real matrix, it is symmetric, and for any non-zero column vector v with real entries a and ,

$$v^T Av = [a \quad b] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^2 + b^2$$

is positive but as a complex matrix, for any non-zero column vector v with complex entries a and b,

$$v^H Av = [a^* \quad b^*] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = aa^* + bb^* = |a^2||b^2|$$

is positive and one of $a$and b is not zero.

The symmetric matrix $B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite because a real matrix, and for any nonzero column vector v with real entries a , b and c.

$$v^T A v = [a \quad b \quad c]\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= [(2a-b)(-a+2b-c)(-b+2c)]\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= 2a^2 - ab - ab + 2b^2 - bc - bc$$

$$= a^2 + (a^2 - 2ab + b^2) + (b^2 - 2ac + c^2) + c^2 = a^2 + (a+b)^2 + (b-c)^2 + c^2$$

So, this is positive.

Other way to knowing the matrix is positive definite or not illustrated in the following definition.

Definition

A symmetric $(n \times n\ )$ real matrix A is said to be positive definite if all the eigenvalues of the matrix A is positive.

Ex11: The following matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ is positive definite, by using eigenvalue method.

Solution:

$$|\boldsymbol{A} - \boldsymbol{\lambda I}| = \mathbf{0} \rightarrow \begin{vmatrix} \mathbf{1-\lambda} & \mathbf{2} \\ \mathbf{2} & \mathbf{1-\lambda} \end{vmatrix} = \mathbf{0} \rightarrow (\mathbf{1-\lambda})(\mathbf{1-\lambda}) - \mathbf{4} = \mathbf{0}$$

$$\mathbf{1 - \lambda - \lambda + \lambda^2 - 4 = 0 \rightarrow -3 - 2\lambda + \lambda^2 = 0}$$

$$\mathbf{\lambda^2 - 2\lambda - 3 = 0}$$

$$\mathbf{(\lambda - 3)(\lambda + 1) = 0}$$

So, the eigenvalue is: $\boldsymbol{\lambda = 3\ , -1}$

So, the matrix (A) is not positive definite matrix.

Positive Semi Definite Matrices

A symmetric $(n \times n)$ real matrix is said to be positive semi definite if the scalar $(v^T Av \ \geq 0)$ i.e. (non- negative) for every nonzero column vector v of real n numbers but If A is complex matrix A then is said to be positive semi definite if the scalar $(v^H Av \ \geq 0)$.

Ex12: The following matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is positive semi definite?

Solution:

اولا: نتأكد ان المصفوفة متماثلة من خلال $A = A^T$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v^T Av = [a \quad b] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \geq 0 = [0+0 \quad 0+b] \begin{bmatrix} a \\ b \end{bmatrix}$$

$$0 + b^2 = b^2 \geq 0$$

Yes, the matrix A is positive semi definite.

Other way to knowing the matrix is positive semi definite or not illustrated in the following definition.

Definition

A symmetric $(n \times n\ )$ real matrix is said to be positive semi definite if all the eigenvalues λ of the matrix A is as $\lambda \geq 0$.

Negative Definite Matrices

A symmetric $(n \times n)$ real matrix A is said to be negative definite if the scalar $(v^T Av)$ is negative for every nonzero column vector v of real n numbers. However, a Hermitian matrix A is said to be negative definite if the scalar $(v^H Av)$ is real and negative for all nonzero column vector v of n complex numbers.

Ex12: If the following matrix $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is Negative definite?

Solution:

$$A^T = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow A = A^T$$

$$v^T A v = [a \quad b \quad c] \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [-3a \quad -2b \quad -c] \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= -3a^2 - 2b^2 - c^2 < 0$$

Yes, the matrix A is Negative definite matrix.

Other way the eigenvalue of , by eigenvalues λ:

$$|\boldsymbol{A} - \boldsymbol{\lambda I}| = \mathbf{0} \rightarrow \begin{vmatrix} \mathbf{-3-\lambda} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2-\lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1-\lambda} \end{vmatrix} = \mathbf{0}$$

$$= (\mathbf{-3-\lambda})[(\mathbf{-2-\lambda})(\mathbf{-1-\lambda}) - \mathbf{0}] - \mathbf{0} + \mathbf{0} = \mathbf{0}$$

$$(\mathbf{-3-\lambda})(\mathbf{-2-\lambda})(\mathbf{-1-\lambda}) = \mathbf{0}$$

$$\lambda = (-3, -2, -1) \ < 0$$

So, the matrix A is Negative definite matrix.

Negative Semi Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be negative semi definite if the scalar $(v^T A v \ \leq 0)$ (i.e. non- positive) for every nonzero column vector $\boldsymbol{v}$ of real $\boldsymbol{n}$ numbers but If $\boldsymbol{A}$ is complex matrix then $\boldsymbol{A}$ is said to be negative semi definite if the scalar $(v^H A v \ \leq 0)$.

Ex13: If the following matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ is Negative semi definite?

Solution:

$$A^T = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow A = A^T$$

$$v^T A v = [a \quad b] \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = [0 \quad -b] \begin{bmatrix} a \\ b \end{bmatrix} = -b^2 \leq 0$$

<u>Notes</u>

- The example **13** above shows that a matrix in which some elements are negative may still be positive definite. Conversely, a matrix whose entries are positive is not necessary positive definite and the following example as,

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \; If\ the\ \ v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$v^T C v = [-1 \quad 1] \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= [-1+2 \quad -2+1] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = [1 \quad -1] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1-1 = -2 \; < 0$$

- For any real invertible matrix $\boldsymbol{A}$, the $\boldsymbol{A^T A}$ is a positive definite.

- A symmetric matrix is a positive definite ↔ all eigenvalues are positive.

- A symmetric matrix is a negative definite all ↔ eigenvalues are negative.

H.W.5/ Classify the following matrices as positive definite, negative definite, positive semi definite or negative semi definite:

$$A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix},\quad B=\begin{bmatrix}-4 & -6\\ -3 & -5\end{bmatrix},\quad C=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix},\quad D=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix},\quad E=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix},$$

$$F=\begin{bmatrix}-3 & 0\\ 0 & 0\end{bmatrix},\quad G=\begin{bmatrix}2 & -4\\ -1 & 2\end{bmatrix},\quad H=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},\quad K=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\quad L=\begin{bmatrix}5 & -3\\ 3 & -1\end{bmatrix},$$

$$M=\begin{bmatrix}5 & 4 & 1\\ 4 & 5 & 1\\ 1 & 1 & 2\end{bmatrix},\quad N=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 2\\ -2 & 0 & 3\end{bmatrix},\quad Q=\begin{bmatrix}-5 & -2\\ -4 & -1\\ -3 & 0\end{bmatrix}$$

Diagonalization of a Matrix with Distinct Eigenvalues

Diagonalizable matrix $\boldsymbol{A}$ is a square matrix when an invertible matrix $\boldsymbol{C}$ exists such that $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\,\boldsymbol{C}$ is a diagonal matrix.

Ex14: Prove that the matrix $A = \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix}$ is diagonalizable.

Solution:

$\lambda_1 = 2$ and $v_{1=} r_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$ and $v_{2=} r_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

There exists $\boldsymbol{C} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $\boldsymbol{C}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\,\boldsymbol{C} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} * \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} * \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is a diagonal matrix.

Ex15: Prove that the matrix $A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ is not diagonalizable.

Solution:

$\lambda_1 = \lambda_2 = 2$ (A repeated root) and $v_{1=}\, v_{2=}\, r \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a matrix does not diagonalizable because it has not two distinct eigenvalues.

Ex16: Let $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$:

1. Prove that $\boldsymbol{A}$ is diagonalizable.
2. Find the diagonal matrix $\boldsymbol{D}$ similar to $\boldsymbol{A}$
3. Find $\boldsymbol{A}^{5}$

Solution:

$\lambda_1 = 2$ and $v_{1=} r_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ and $v_{2=} r_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Since has two distinct eigenvalues then $\boldsymbol{A}$ diagonalizable.

$C = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, C^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

$D = C^{-1} A\, D = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} * \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ is a diagonal matrix.

$D^5 = \begin{bmatrix} 2^5 & 0 \\ 0 & (-1)^5 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix}$

$A^5 = C\, D^5\, C^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} * \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -30 & -66 \\ 33 & 65 \end{bmatrix}$.

Notes

- The product $\boldsymbol{D = C^{-1} A\, C}$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}$ (distinct eigenvalues).

  Such that for $\boldsymbol{A_{2*2}}$ and $\boldsymbol{\lambda_1 \neq \lambda_2}$ there is diagonal matrix $\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
- $\boldsymbol{A}$ is a diagonalizable if it has linearly independent the eigenvectors.
- $\boldsymbol{A}$ is similar to a diagonal matrix $\boldsymbol{D = C^{-1} A\, C}$ then $\boldsymbol{A^k = C\, D^k\, C^{-1}}$
- If $\boldsymbol{A}$ is a symmetric matrix then eigenvectors that associated to distinct eigenvalues of $\boldsymbol{A}$ are orthogonal.

Orthogonally and Diagonalizable of a Matrix

Orthogonally Diagonalizable matrix $\boldsymbol{A}$ is a square matrix when an orthogonal matrix $\boldsymbol{C}$ exists such that $\boldsymbol{D} = \boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A}\,\boldsymbol{C}$ is a diagonal matrix.

Notes:

- $\boldsymbol{A}$ square matrix is said to be orthogonally diagonalizable if $\boldsymbol{A}$ is a symmetric matrix.
- $\boldsymbol{A}$ square matrix $\boldsymbol{C}$ is said to be orthogonal if the columns (rows) of $\boldsymbol{C}$ is an orthonormal set.
- The eigenvalues of a square matrix $\boldsymbol{A}$ lies on the main diagonal of $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{A}\,\boldsymbol{C} = \boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A}\,\boldsymbol{C}$, where $\boldsymbol{C}$ is an orthogonal matrix.
- Norm of vector $\boldsymbol{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is denoted and define as follows:

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 + \cdots .. + \boldsymbol{x}_n^2}$$

Ex17: Prove that the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ is orthogonally diagonalizable.

Solution:

$\lambda_1 = -2$ and $v_{1=} r_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\|v_1\| = \sqrt{0^2 + 1^2 + 0^2} = 1$$

$\lambda_2 = 4$ and $v_{2=} r_2 \begin{bmatrix} \frac{-1}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$$\|v_2\| = \sqrt{(-1)^2 + 0^2 + 2^2} = \sqrt{5}$$

$$v_{22} = \frac{v_2}{\|v_2\|} = \frac{v_2}{\sqrt{5}}$$

$$v_{22} = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$\lambda_3 = -1$ and $v_{3=} r_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$\|v_3\| = \sqrt{(2)^2 + 0^2 + 1^2} = \sqrt{5}$$

$$v_{33} = \frac{v_3}{\|v_3\|} = \frac{v_3}{\sqrt{5}}$$

$$v_{33} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

There exists $C = \begin{bmatrix} 0 & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$, $C^T = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$,

$\boldsymbol{D} = \boldsymbol{C^T A\, C} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} * \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix} * \begin{bmatrix} 0 & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is diagonal matrix

then $\boldsymbol{A}$ is orthogonally diagonalizable.

H.W.6 Find orthogonally diagonalizable for the following matrices:

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

REFERENCES:

- ***Applied mathematics lectures / Mathematics Department / Ibn Al-Haytham College of Education / University of Baghdad.***

- ***Muir, T. (1906). The theory of determinants in the historical order of development (Vol. 1). Macmillan and Company, limited.***

- ***Vinod, H. D. (2011). Hands-on matrix algebra using R: Active and motivated learning with applications. World Scientific Publishing Company.***