Chapter two :- IDEALS المثاليات

Defention:-

let $(R, +, \cdot)$ be aring and S a subring of R:

1. S is called a right ideal of $R \Leftrightarrow a \cdot r \in S$ $\forall\, a \in S ; \forall\, r \in R$ ...

2. S is called a left ideal of $R \Leftrightarrow r \cdot a \in S$ $\forall\, a \in S ; \forall\, r \in R$.

3. S is called left and right ideals of R (or two sided ideals) $\Leftrightarrow a \cdot r \in S$ and $r \cdot a \in S$, $\forall\, a \in S ; r \in R$

Remark:- In Comm. ring every left ideal is right ideal and Conversely.

Example:- The subring $(Z_e, +, \cdot)$ is an ideal of $(\mathbb{Z}, +, \cdot)$

* since if $a = 2n \in Z_e$ and $r \in \mathbb{Z}$, then

$r \cdot a = a \cdot r = (2n) \cdot r = 2(nr) \in Z_e$

Theorem:- let $(R, +, \cdot)$ be a ring, and let $\emptyset \neq S \subseteq R$ Then S is an ideal of R Iff:-

1. $a - b \in S$ $\forall\, a, b \in S$.

2. $a \cdot r \in S$ and $r \cdot a \in S$. $\forall\, a \in S ; r \in R$

Examples:-

① Consider $(\mathbb{R},+,\cdot)$, $\emptyset \neq \mathbb{Z} \subseteq \mathbb{R}$

here we have $5 \in \mathbb{Z}$, $\frac{1}{3} \in \mathbb{R}$, but $5 \cdot \frac{1}{3} = \frac{5}{3} \notin \mathbb{Z}$

$\therefore$ $\mathbb{Z}$ is not an ideal of $\mathbb{R}$

② $\emptyset \neq \mathbb{Q} \subseteq \mathbb{R}$, $\frac{1}{2} \in \mathbb{Q}$, $\sqrt{3} \in \mathbb{R}$ but $\frac{\sqrt{3}}{2} \notin \mathbb{Q}$

Then $\mathbb{Q}$ is not ideal in $\mathbb{R}$

③ Find all ideals of $(\mathbb{Z}_6, +, \cdot)$.

$I_1 = (1) = \mathbb{Z}_6$, $I_2 = \{0\}$, $I_3 = \{0, 2, 4\}$, $I_4 = \{0, 3\}$

ملاحظة: جميع الحلقات الجزئية من الحلقة $(\mathbb{Z}_n,+,\cdot)$ هي مثاليات
فقط في الحلقة $(\mathbb{Z}_n,+,\cdot)$. اي انه بصورة عامة ليس كل Subring
هي تمثل Ideal

Example:- Consider the ring $(M_2(\mathbb{Z}), +, \cdot)$ and let:

1. $S_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$; Is S_1 left (right) ideal of $M_2(\mathbb{Z})$?

2. $S_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$; Is S_2 Left (right) ideal of $(M_2(\mathbb{Z}))$?

3. $S_3 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$; Is S_3 Left (right) ideal of $M_2(\mathbb{Z})$?

Solution:-

1. $\emptyset \neq S_1 \subseteq M_2(\mathbb{Z})$

Let: $\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} \in S_1$; $a_1, a_2, b_1, b_2 \in \mathbb{Z}$

$\therefore \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} - \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & 0 \\ 0 & b_1 - b_2 \end{pmatrix} \in S_1$.

Let $\begin{pmatrix} x & y \\ t & w \end{pmatrix} \in M_2(\mathbb{Z})$; $x, y, t, w \in \mathbb{Z}$.

$\therefore \begin{pmatrix} x & y \\ t & w \end{pmatrix} \cdot \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} = \begin{pmatrix} xa_1 & yb_1 \\ ta_1 & wb_1 \end{pmatrix} \notin S_1$.

$\therefore$ S_1 is not Left ideal.

Now for the (right):-

$\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ t & w \end{pmatrix} = \begin{pmatrix} a_1 x & a_1 y \\ b_1 t & b_1 w \end{pmatrix} \notin S_1$.

$\therefore$ S_1 is not right ideal.

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2. $\emptyset \neq S_2 \subseteq M_2(\mathbb{Z})$

Let $\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix} \in S_2$; $a_1, a_2, b_1, b_2 \in \mathbb{Z}$

$\therefore \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ 0 & 0 \end{pmatrix} \in S_2$

and Let $\begin{pmatrix} x & y \\ t & w \end{pmatrix} \in M_2(\mathbb{Z})$; $x, y, t, w \in \mathbb{Z}$

$$\therefore \begin{pmatrix} x & y \\ t & w \end{pmatrix} \cdot \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x a_1 & x b_1 \\ t a_1 & t b_1 \end{pmatrix} \notin S_2$$

$\therefore$ S_2 is not left ideal.

Now for the (right) :-

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ t & w \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 t & a_1 y + b_1 w \\ 0 & 0 \end{pmatrix} \in S_2$$

$\therefore$ S_2 is a right ideal of $M_2(\mathbb{Z})$

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3. For S_3 is H.W.

Example:- let $R = (\mathbb{Z} \times \mathbb{Z}, \oplus, \odot)$

And let:- $S_1 = \{(a,b);\ a \in \mathbb{Z}, b \in \mathbb{Z}\}$

$S_2 = \{(a,b);\ a \in \mathbb{Z}, b \in \mathbb{Z}_e\}$

$S_3 = \{(a,b);\ a\ \mathbb{Z}_e, b \in \mathbb{Z}_e\}$

$S_4 = \{(a,0);\ a \in \mathbb{Z}\}$.

IS S_1, S_2, S_3, S_4 an ideal of R.

Direct sum of Rings.

Note :- Definition: let $(R, +, \cdot)$ and $(R', +', \cdot')$ be any two rings

let $X = R \times R' = \{(a,b) : a \in R, b \in R'\}$ we define $\oplus, \odot$ on X by

$(a,b) \oplus (c,d) = (a + c, b +' d)$

$(a,b) \odot (c,d) = (a \cdot c, b \cdot' d)$

Then $(X, \oplus, \odot)$ is a ring.

Solution :- 2 $\quad \emptyset \neq S_2 \subseteq R$

Let $(a_1, b_1), (a_2, b_2) \in S_2$; $a_1, a_2 \in \mathbb{Z}, b_1, b_2 \in \mathbb{Z}_e$

$\therefore (a_1, b_1) - (a_2, b_2) = (a_1 - a_2, b_1 - b_2) \in S_2$

since $a_1, a_2 \in \mathbb{Z} \Rightarrow a_1 - a_2 \in \mathbb{Z}$

$b_1, b_2 \in \mathbb{Z}_e \Rightarrow b_1 - b_2 \in \mathbb{Z}_e$

let $(x, y) \in R = \mathbb{Z} \times \mathbb{Z}$

$(x, y) \odot (a_1, b_1) = (x a_1, y b_1) \in S_2$

since $x \in \mathbb{Z}, a_1 \in \mathbb{Z} \Rightarrow x a_1 \in \mathbb{Z}$

$y \in \mathbb{Z}, b_1 \in \mathbb{Z}_e \Rightarrow y b_1 \in \mathbb{Z}_e$

and we have $(a_1, b_1) \odot (x, y) = (a_1 x, b_1 y) \in S_2$.

$\therefore$ S_2 is an ideal (Left, Right) of $R = \mathbb{Z} \times \mathbb{Z}$.

Solution: 4 $\quad \emptyset \neq S_4 \subseteq R$

Let $(a_1, 0), (a_2, 0) \in S_4$; $a_1, a_2 \in \mathbb{Z}$

$\Rightarrow (a_1, 0) - (a_2, 0) = (a_1 - a_2, 0) \in S_4$

since $a_1, a_2 \in \mathbb{Z} \Rightarrow a_1 - a_2 \in \mathbb{Z}$

let $(x, y) \in R = \mathbb{Z} \times \mathbb{Z}$

$\therefore (x, y) \odot (a_1, 0) = (x a_1, 0) \in S_4$

since $x, a_1 \in \mathbb{Z} \Rightarrow x \cdot a_1 \in \mathbb{Z}$

and we have $(a_1, 0) \odot (x, y) = (a_1 x, 0) \in S_4$

$\because a_1, x \in \mathbb{Z} \Rightarrow a_1 \cdot x \in \mathbb{Z}$

$\therefore$ S_4 is an ideal (left, Right) of $R = \mathbb{Z} \times \mathbb{Z}$

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1 and 3 is a H.W.

Remarks :-

1. Let $(R,+,\cdot)$ be a ring with unity 1, I be an ideal of R. If $1 \in I$, then $I = R$.

Proof :- $I \subseteq R$ (by def. of ideal).

and $\forall\, r \in R \Rightarrow r = r \cdot 1$ (1 is the identity of R)

but $r \in R,\ 1 \in I \Rightarrow r \cdot 1 \in I$ (I is an ideal)

$\therefore\ r \in I \Rightarrow R \subseteq I$, Thus $R = I$.

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② let R be a ring with unity 1 and I is an ideal of R $a \in R$. If a is an invertible element and $a \in I$, then $I = R$.

Proof :- let $a \in I$ and a is an invertible element.

Then $\exists\, a^{-1} \in R$ s.t $a a^{-1} = 1$.

Therefore $1 = a \cdot a^{-1} \in I$ (I is an ideal of R)

$\Rightarrow\ 1 \in I$ (by using no. 1)

Theorem:- let $(R,+,\cdot)$ be a ring and let I, J be two ideals of R. Then $(I \cap J, +, \cdot)$ is an ideal of R

Proof: $\emptyset \neq I \cap J \subseteq R$ (since $0 \in I,\ 0 \in J \Rightarrow 0 \in I \cap J$).

let $a, b \in I \cap J$, $r \in R$

$\because\ a, b \in I \cap J \Rightarrow a \in I \wedge b \in I,\ a \in J \wedge b \in J$

$\therefore\ a - b \in I \ \wedge\ a - b \in J$

and $a \cdot r \in I \ \wedge\ r \cdot a \in I$ (since I is an ideal of R)

$a \cdot r \in J \ \wedge\ r \cdot a \in J$ (since J is an ideal of R)

Thus $a - b \in I \cap J$ and $r \cdot a \in I \cap J$, $a \cdot r \in I \cap J$.

$\therefore\ I \cap J$ is an ideal of R.

Remark:- The union of two ideals of ring R is not necessary an ideal of R, for example:

$(\mathbb{Z}_6, +, \cdot)$, (2) and (3) are ideals of $\mathbb{Z}_6$

But $(2) \cup (3) = \{0, 2, 3, 4\}$ not ideal of $\mathbb{Z}_6$

Since $4, 3 \in (2) \cup (3) \Rightarrow 4 - 3 \notin (2) \cup (3)$.

Definition:- let R be a comm. ring with unity and $a \in R$ let $I = \{r.a : r \in R\}$. Then I is an ideal of R and this ideal is called a principle ideal generated by a and denoted by (a), $id(a)$, $\langle a \rangle$.

Example: ① Consider the ring $(\mathbb{Z}, +, \cdot)$

$id(2) = \{r.2 : r \in \mathbb{Z}\} = \{0, \pm 2, \pm 4, \dots\}$

$id(-2) = \{r.(-2) : r \in \mathbb{Z}\} = \{0, \mp 2, \mp 4, \dots\}$

$\therefore\ id(a) = id(-a)$ (بصورة عامة)

and $id(1) = \mathbb{Z}$, $id(0) = \{0\}$

② Consider the ring $(\mathbb{Z}_6, +, \cdot)$

$id(1) = \{r.1, r \in \mathbb{Z}\} = \mathbb{Z}_6$

$id(0) = \{0\}$

$id(2) = \{r.2 : r \in \mathbb{Z}\}$

$= \{0.2, 1.2, 2.2, 3.2, 4.2, 5.2\}$

$= \{0, 2, 4, 0, 2, 4\} = \{0, 2, 4\}$

Some Operations on ideals:-

Definition: let $(R,+,\cdot)$ be a ring and I, J be two ideals of R. Define $I+J = \{x+y : x \in I \wedge y \in J\}$. Then we have $I+J = \{x+y : x \in I \wedge y \in J\}$ is called the sum of I and J

Remark: $I+J$ is an ideal of a ring $(R,+,\cdot)$, where I, J are ideals of R.

Proof:- since $I+J \subseteq R$ (by Def^n of ideal)

and $I+J \neq \emptyset$, since $0 = 0+0 \in I+J$

Now, Let x_1+y_1 and $x_2+y_2 \in I+J$

s.t $x_1, x_2 \in I$ and $y_1, y_2 \in J$

$$(x_1+y_1)-(x_2+y_2) = (x_1-x_2)+(y_1-y_2) \in I+J$$

Since, $x_1, x_2 \in I \Rightarrow (x_1-x_2) \in I$

$y_1, y_2 \in J \Rightarrow (y_1-y_2) \in J$ $\{\because I, J$ are two ideals of $R\}$

let $r \in R$, $r\cdot(x_1+y_1) = rx_1 + ry_1 \in I+J$

$x_1 \in I \Rightarrow r\cdot x_1 \in I$

$y_1 \in J \Rightarrow r\cdot y_1 \in J$ $\{\because I \wedge J$ are two ideals of $R\}$

Similarly: $(x_1+y_1)\cdot r \in I+J$

Then $I+J$ is an ideal of R.

Remark:- let $(R, +, \cdot)$ be a Comm. ring and $a, b \in R$

Then $id(a, b) = id(a) + id(b)$.

Example:- ① In $(\mathbb{Z}, +, \cdot)$

* $id(3) + id(4) = id(3, 4) = id(1) = \mathbb{Z}$
* $id(2) + id(4) = id(2, 4) = id(2) = \mathbb{Z}_e$

② Consider $(\mathbb{Z}_{12}, +, \cdot)$ ring.

let $I = id(2)$, $J = id(3)$, $K = id(4)$, $L = id(6)$

then:

* $K + L = id(4) + id(6) = \{0, 4, 8\} + \{0, 6\}$

$= \{0+0, 0+6, 4+0, 4+6, 8+0, 8+6\}$

$= \{0, 6, 4, 10, 8, 2\} = id(2)$

and $L + L = id(6) + id(6) = \{0, 6\} + \{0, 6\} = \{0, 6\} = L$

$J + L = \{0, 3, 6, 9\}$

$I + L = \{0, 2, 4, 6, 8, 10\} = id(2)$

Remark:- let R be a ring, I be an ideal of R

Then $I + I = I$

Proof:- It's clear that $I \subseteq I + I$

Now to prove $I + I \subseteq I$:

For all $x + y \in I + I$; $x \in I \wedge y \in I$

$\therefore$ $x + y \in I$ (I is an ideal of R)

Thus $I + I \subseteq I$, and then $I + I = I$

Definition:- let $(R, +, \cdot)$ be a ring, I, J be two ideals of R. Let $I \cdot J = \{ \sum_{f.\text{sum}} a_i b_i : a_i \in I, b_i \in J \}$

then $I \cdot J$ Called the Product of I and J.

Remark:- $IJ \subseteq I$ and $IJ \subseteq J$

Proof: To Prove that $IJ \subseteq I$

let $x \in IJ \Rightarrow x = \sum_{f.\text{sum}} a_i b_i$, $a_i \in I$, $b_i \in J$

$\therefore$ $x = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n$, for som $n \in \mathbb{Z}^+$

but $a_i \in I$ $\forall i = 1, 2, 3, \dots, n$

$\Rightarrow a_i b_i \in I$ $\forall i = 1, 2, 3, \dots, n$

$\Rightarrow x \in I$ (since $+$ is closed on I)

$\Rightarrow I \cdot J \subseteq I$

by same way we have: $IJ \subseteq J$.

Example:- Consider $(\mathbb{Z}_6, +, \cdot)$, let $I = \{0, 2, 4\}$, $J = \{0, 3\}$

Then $a_i \cdot b_i = 0$ $\forall a_i \in I$ and $b_i \in J$

$\therefore$ $I \cdot J = \{0\}$

Definition:- let $(R, +, \cdot)$ be a Comm.ring with unity 1 Then $(R, +, \cdot)$ is Called field $\Leftrightarrow \forall a \in R, a \neq 0$, we have a is an invertible element.

Example:-

1- $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are fields.

2- $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Z}_e, +, \cdot)$ is not field.

3- direct sum of $(\mathbb{R}, +, \cdot)$ with $(\mathbb{R}, +, \cdot)$ is not a field.

- since all elements of the form $(a, 0)$ and $(0, b)$ $a \neq 0$, $b \neq 0$ are not invertible elements.

Example:- Is $(\mathbb{Z}[\sqrt{3}], +, \cdot)$ field ?

Solution:- (The ring $(\mathbb{Z}[\sqrt{3}], +, \cdot)$ it has element of the form $a + b\sqrt{3}$, such that $a, b \in \mathbb{Z}$)

Now let $x = 1 + 3\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$

$$\therefore \frac{1}{x} = \frac{1}{1+3\sqrt{3}} = \frac{1-3\sqrt{3}}{(1+3\sqrt{3})\cdot(1-3\sqrt{3})} = \frac{1-3\sqrt{3}}{-26}$$

$$\therefore \frac{1}{x} = \frac{-1}{26} + \frac{3}{26}\sqrt{3} \notin \mathbb{Z}[\sqrt{3}]$$

$\therefore \mathbb{Z}[\sqrt{3}]$ is not field.

Definition:- let $(F, +, \cdot)$, $(K, +, \cdot)$ be two fields s.t $F \subseteq K$, Then F is called a subfield of K.

Remark: let $(F, +, \cdot)$ be a field and let $\phi \neq S \subseteq F$ Then $(S, +, \cdot)$ is a subfield of F iff:-

1) $a - b \in S \quad \forall\, a, b \in S$

2) $a \cdot b^{-1} \in S \quad \forall\, a, b \in S$

Example:- $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$

Then $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ is a sub field of $\mathbb{R}$.

Sol^n:- let $a + b\sqrt{2}$, $c + d\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$, Then

1. $(a + b\sqrt{2}) - (c + d\sqrt{2}) = (a - c) + (b - d)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

2. $(a + b\sqrt{2}) \cdot (c + d\sqrt{2})^{-1} = (a + b\sqrt{2}) \cdot \dfrac{1}{(c + d\sqrt{2})} = \dfrac{(a + b\sqrt{2})}{(c + d\sqrt{2})}$

$$= \frac{(a + b\sqrt{2})}{(c + d\sqrt{2})} \cdot \frac{c - d\sqrt{2}}{c - d\sqrt{2}}$$

$$= \frac{ac + ad\sqrt{2} + cb\sqrt{2} - 2bd}{c^2 - 2d^2} = \frac{(ac - 2bd) + (ad + cb)\sqrt{2}}{c^2 - 2d^2}$$

$$= \frac{ac - 2bd}{c^2 - 2d^2} + \frac{ad + cb}{c^2 - 2d^2}\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

$\therefore\ \mathbb{Q}[\sqrt{2}]$ is sub field of $\mathbb{R}$