جامعة ديالى
كلية التربية المقداد

التفاضل والتكامل

قسم الرياضيات - المرحلة الاولى

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CHAPTER ONE:- The Real Numbers *R*

1. Sets:

- Natural Numbers $N = \{1, 2, 3, \cdots\}$
- Integers Numbers $Z = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} = Z^- \cup \{0\} \cup Z^+$
- Rational Numbers $Q = \{\frac{a}{b}$: a, b are integers numbers and $b \neq 0\}$.
- Irrational Numbers I: Such as $\sqrt{2}$ and π are numbers which are not rational.
- Real Numbers R: The set of rational and irrational numbers ($R = Q \cup I$).
- Complex Numbers $C = \{x + yi : x, y$ are real numbers and $i = \sqrt{-1}\}$.

Clearly, $N \subseteq Z \subseteq Q \subseteq R \subseteq C$.

2. Operations With Real Numbers:

If a, b and c are real numbers, then:

1) $a + b \in R$ and $a \times b \in R$ (Closure law)
2) $a + b = b + a$ (Commutative law of addition)
3) $a \times b = b \times a$ (Commutative law of multiplication)
4) $a + (b + c) = (a + b) + c$ (associative law of addition)
5) $a \times (b \times c) = (a \times b) \times c$ (associative law of multiplication)
6) $a \times (b + c) = a \times b + a \times c$ (distributive law)
7) $a + 0 = 0 + a = a$ (0 is called the identity with respect to addition)
   $a \times 1 = 1 \times a = a$ (1 is called the identity with respect to multiplication)
8) For any a there is a number $x \in R$ such that $x + a = a + x = 0$, x is called the inverse of a with respect to addition and is denoted by $-a$.
9) For any $a \neq 0$ there is a number $x \in R$ such that $x \times a = a \times x = 1$, x is called the inverse of a with respect to multiplication and is denoted by a^{-1} or $\frac{1}{a}$.

3. Types of Intervals: (انواع الفترات)

Interval Notation	Set definition	Name	Region on the Real Number Line
(a, b)	$\{x : a < x < b\}$	Open	
$[a, b]$	$\{x : a \leq x \leq b\}$	Closed	
$[a, b)$	$\{x : a \leq x < b\}$	Half Open	
$(a, b]$	$\{x : a < x \leq b\}$	Half Open	
(a, ∞)	$\{x : x > a\}$	Open	
$[a, \infty)$	$\{x : x \geq a\}$	Closed	
$(-\infty, b)$	$\{x : x < b\}$	Open	
$(-\infty, b]$	$\{x : x \leq b\}$	Closed	
$(-\infty, \infty)$	R	Open and Closed	

4. Inequalities: (المتباينات)

If $a - b$ is a nonnegative number, we say that a is greater than or equal to b or b is less than or equal to a, and write, respectively $a \geq b$ or $b \leq a$. If there is no possibility that $a = b$, we write $a > b$ or $b < a$.

Theorem (4.1):

If a, b, c and d are any real numbers, then:

1) If $a < b$ and $b < c$, then $a < c$

e.g., $3 < 5$ and $5 < 7 \Rightarrow 3 < 7$

2) If $a < b$, then $a \pm c < b \pm c$

e.g., $10 < 13 \Rightarrow 10 + 3 < 13 + 3$ and $10 - 3 < 13 - 3$

3) If a < b, then $\begin{cases} a \times c < b \times c \\ \frac{a}{c} < \frac{b}{c} \end{cases}$ when c > 0

e.g., $10 < 20 \Rightarrow 10 \times 3 < 20 \times 3 \Rightarrow 30 < 60$

4) If a < b, then $\begin{cases} a \times c > b \times c \\ \frac{a}{c} > \frac{b}{c} \end{cases}$ when c < 0

e.g., $10 < 20 \Rightarrow 10 \times -2 > 20 \times -2 \Rightarrow -20 > -40$

$$\Rightarrow \frac{10}{-2} > \frac{20}{-2} \Rightarrow -5 > -10$$

5) If a < b, then $\frac{1}{a} > \frac{1}{b}$

e.g., $3 < 5 \Rightarrow \frac{1}{3} > \frac{1}{5}$

6) If a < b and c < d, then a + c < b + d e.g., 3 < 5 and 6 < 9 ⇒ 3 + 6 < 5 + 9

$$\Rightarrow \cancel{3} + 2x - \cancel{3} < 7 - 3 \Rightarrow 2x < 4 \Rightarrow \frac{\cancel{2}x}{\cancel{2}} < \frac{4}{2} \Rightarrow x < 2$$

Example (4.1): Find the solution set of the following inequalities.

1) $3 + 2x < 7$

Solution:

$$\Rightarrow \cancel{3} + 2x - \cancel{3} < 7 - 3 \Rightarrow 2x < 4 \Rightarrow \frac{\cancel{2}x}{\cancel{2}} < \frac{4}{2} \Rightarrow x < 2$$

$\therefore$ The solution $= \{x : x < 2\} = (-\infty, 2)$

2) $2 - 3x < 4 + 2x$

Solution:

$2 - 3x + 3x < 4 + 2x + 3x$ (adding to both sides $+3x$)

$\Rightarrow 2 < 4 + 5x \Rightarrow 2 - 4 < 4 + 5\ x - 4$ (adding to both sides -4)

$-2 < 5x \Rightarrow \frac{-2}{5} < \frac{5x}{5}$ (Dividing both sides by 5)

$\Rightarrow \frac{-2}{5} < x$

$\therefore$The solution $\{x : x > \frac{-2}{5}\} = (\frac{-2}{5}, \infty)$

3) $2 < 3x - 1 \leq 11$

Solution: $\Rightarrow 2 + 1 < 3x - 1 + 1 \leq 11 + 1$

$\Rightarrow 3 < 3x \leq 12 \Rightarrow \frac{3}{3} < \frac{3x}{3} \leq \frac{12}{3} \Rightarrow 1 < x \leq 4$

$\therefore$ The solution $= \{\ x: 1 < x \leq 4\} = (1,4]$

4) $\frac{2}{x} < \frac{1}{4}\ , x \neq 0$

Solution:

x may be positive or negative.

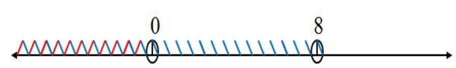
Case 1: If $x > 0$

$\Rightarrow \frac{2}{x} \times x < \frac{1}{4} \times x \Rightarrow 2 < \frac{x}{4} \Rightarrow 2 \times 4 < \frac{x}{4} \times 4 \Rightarrow 8 < x$

$\therefore$ The solution $= \{x: x > 8\} = (8, \infty)$

Case 2: *If* $x < 0$

$\Rightarrow \frac{2}{x} \times x > \frac{1}{4} \times x \Rightarrow 2 > \frac{x}{4} \Rightarrow 2 \times 4 > \frac{x}{4} \times 4 \Rightarrow 8 > x$



$\therefore$ The solution $= \{x: x < 0\} = (-\infty, 0)$

$\therefore$ The general solution is $(-\infty, 0) \cup (8, \infty)$

5) $\frac{x-7}{x+3} > 2\ \ , x \neq -3$

Solution:

Case 1: *If* $x + 3 > 0 \Rightarrow x > -3$

$\frac{x-7}{x+3}(x+3) > 2(x+3) \Rightarrow x-7 > 2x+6 \Rightarrow x-2x > 6+7 \Rightarrow -x > 13 \Rightarrow x < -13$ this is false.

Case 2: If $x+3<0 \Rightarrow x<-3$

$\frac{x-7}{x+3}(x+3) < 2(x+3) \Rightarrow x-7 < 2x+6 \Rightarrow x-2x < 6+7 \Rightarrow -x < 13 \Rightarrow x > -13$

$\therefore$ The solution is $= \{x: -13 < x < -3\} = (-13,-3)$

$\therefore$ The general solution is $= \{x: -13 < x < -3\} = (-13,-3)$

Exercises (4): Solve the following inequalities:

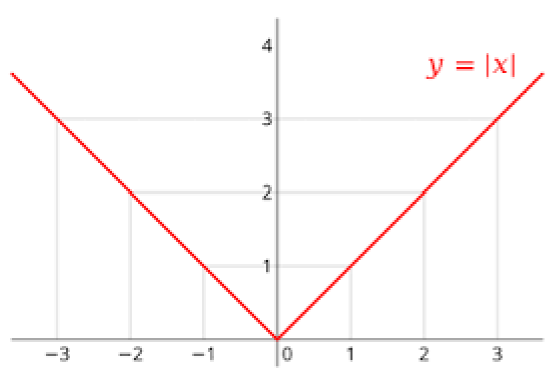
1) $\frac{x+4}{x-3} < 2$ 2) $\frac{-x}{x+5} < 1$ 3) $x^2 - 6x + 5 > 0$

4) (x – 1)2(x + 4) < 0 5) 5x – 2x2 > 0

5. Absolute Value (“magnitude”):

Definition (5.1): *If* x and y any real numbers, then:

$$|x| = \begin{cases} x \text{ if} & x > 0 \\ 0 \text{ if} & x = 0 \\ -x \text{ if} & x < 0 \end{cases}$$



Properties:

1) $|-x| = |x|$
2) $|xy| = |x||y|$
3) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $y \neq 0$
4) $|x + y| \leq |x| + |y|$
5) $|x - y| \geq |x| - |y|$
6) $-|a| \leq a \leq |a|$
7) *If* $|x| \leq a$, then $-a \leq x \leq a$
8) *If* $|x| \geq a$, then $x \leq -a$ or $x \geq a$

Example (5.1):

1) $|4 - 8| = |-4| = 4$
2) $|4| + |-3| = 4 + 3 = 7$

Example (5.2): $|2x - 4| \leq 8$

Solution: $-8 \leq 2x - 4 \leq 8$

$\Rightarrow -8 + 4 \leq 2x - 4 + 4 \leq 8 + 4$ (adding to the each sides +4)

$\Rightarrow -4 \leq 2x \leq 12$ $\Rightarrow$ $-4\backslash 2 \leq 2x\backslash 2 \leq 12\backslash 2$ (Dividing each sides by 2)

$\Rightarrow -2 \leq x \leq 6$

∴ The solution is the set $x = [\,-2, 6\,]$

Example (5.3): $|2x - 4| \geq 8$

Solution: $2x - 4 \leq -8$

$\Rightarrow 2x \leq -8 + 4 \Rightarrow 2x \leq -4$

$\Rightarrow 2x\backslash 2 \leq -4\backslash 2 \Rightarrow x \leq -2$

$x = (-\infty, -2]$

or

$2x - 4 \geq 8$

$\Rightarrow 2x \geq 8 + 4 \Rightarrow 2x \geq 12$

$\Rightarrow 2x\backslash 2 \geq 12\backslash 2 \Rightarrow x \geq 6$

$x = [6, \infty)$

∴ The solution is the set $x = (-\infty, -2] \cup [6, \infty)$

Example (5.4): Solve $\left|x+\frac{1}{x}\right|>2\ ,x\neq 0$

Solution: $\Rightarrow \left|\frac{x^2+1}{x}\right|>2 \Rightarrow \frac{|x^2+1|}{|x|}>$ 2 (Since $x^2+1) > 0$

$\Rightarrow \frac{x^2+1}{|x|}>2 \Rightarrow x^2+1>2|x| \Rightarrow x^2-2|x|+1>0$

$\Rightarrow |x|^2 - 2|x| + 1 > 0$ (Since $x^2 = |x|^2$)

$\Rightarrow (|x| - 1)^2 > 0, \qquad |x| \neq 1$

∴ The solution is the set of real number except $x = 1, x = -1$ and $x = 0$
∴ The solution is $= (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$

Example (5.5): Solve $|x + 3| \leq 5$
Solution:
$|x + 3| \leq 5$ if and only if $-5 \leq x + 3 \leq 5$
$\Rightarrow -5 - 3 \leq x+3 - 3 \leq 5 - 3 \Rightarrow -8 \leq x \leq 2$

∴ The solution is $= \{x: -8 \leq x \leq 2\} = [-8,2]$

Exercises (5): Solve the following inequalities:
1) $|2x - 3| < |x + 2|$

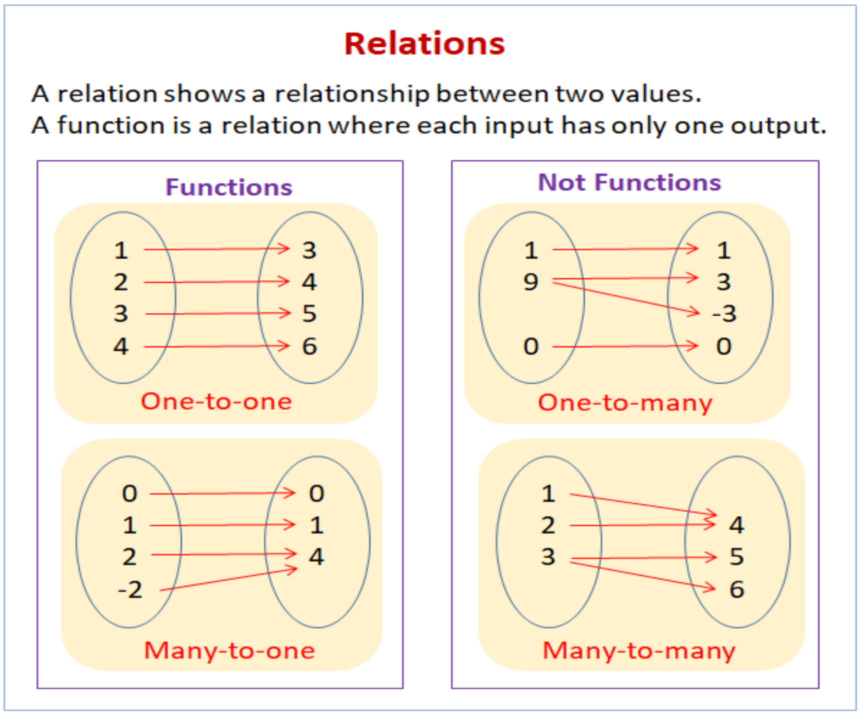
2) $|2x + 1| > 2$

3) $|5 - 3x| < 2$

CHAPTER TWO: The Functions

1. Functions: -

Definition: A relation $f : X \to Y$ is called function if and only if for each element $x \in X$, there exist a unique element $y \in Y$ such that $y = f(x)$.



- The variable x in a function $y = f(x)$ is called the **independent** variable of the function f. The variable y whose value **dependent on x**, is called dependent variable of the function f.

- If $y = f(x)$, then the set of all possible inputs ($x - values$) is called the domain of f and denoted by D_f or ***Domain of (f)***.
  And the set of outputs ($y - values$) that result when x varies over the domain is called the range of f and denoted by R_f or ***Range of (f)***.

Example (1.1): Find the domain and range of the following functions:

1) $f(x) = x - 2$ 2) $f(x) = x^2 - 4$ 3) $f(x) = \sqrt{x - 2}$

4) $f(x) = |x|$ 5) $f(x) = \dfrac{x^2-4}{x-2}$ 6) $f(x) = \dfrac{1}{(x-2)(x-3)}$

Solution:

1) $D_f = \mathbb{R}$ and $R_f = \mathbb{R}$

2) $D_f = \mathrm{R}$

Let $y = x^2 - 4 \Rightarrow x^2 = y + 4 \Rightarrow x = \mp\sqrt{x+4}$

If $y + 4 \geq 0 \Rightarrow y \geq -4$

$\therefore R_f = [-4, \infty)$

3) $x - 2 \geq 0 \Rightarrow x \geq 2$

$\therefore \mathrm{D}_f = [2, \infty)$ and $\mathrm{R}_f = [0, \infty)$

4) $\mathrm{D}_f = \mathrm{R}$ and $\mathrm{R}_f = [0, \infty)$

5) $x + 2 = 0 \Rightarrow x = -2$

$\therefore \mathrm{D}_f = (-\infty, -2) \cup (-2, \infty)$

Since $f(x) = \frac{x2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = x - 2$ for $x \neq -2$

$\Rightarrow f(x) = x - 2$

$\Rightarrow y = x - 2 \Rightarrow y = -2 - 2 = -4$

$\therefore \mathrm{R}_f = (-\infty, -4) \cup (-4, \infty)$

6) $\mathrm{D}_f = (-\infty, 2) \cup (2, 3) \cup (3, \infty)$

Let $y = \dfrac{1}{(x-2)(x-3)} \Rightarrow y = \dfrac{1}{x^2 - 5x + 6} \Rightarrow y(x^2 - 5x + 6) = 1$

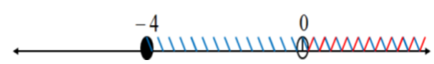
$$\Rightarrow x^2 - 5x + 6 = \frac{1}{y} \Rightarrow x^2 - 5x + \left(6 - \frac{1}{y}\right) = 0$$

$$\Rightarrow x = \frac{5 \pm \sqrt{25 - 4\left(6 - \frac{1}{y}\right)}}{2} = \frac{5 \pm \sqrt{25 - 24 + \frac{4}{y}}}{2} = \frac{5 \pm \sqrt{1 + \frac{4}{y}}}{2}$$

If $1 + \dfrac{4}{y} \geq 0 \Rightarrow \dfrac{4}{y} \geq -1$

Case 1: If $y > 0$

$\Rightarrow 4 \geq -y \Rightarrow y \geq -4 \Rightarrow (0, \infty)$



Case 2: If $y < 0$

$\Rightarrow 4 \leq -y \Rightarrow y \leq -4 \Rightarrow (-\infty, -4]$

$\therefore R_f = (-\infty, -4] \cup (0, \infty)$

Example (1.2): Find the domain of the following functions:

1) $f(x) = \dfrac{3x}{x^2 - 4x - 12}$ **2)** $f(x) = \dfrac{\sqrt{x-1}}{x^2+4}$ **3)** $f(x) = \dfrac{1}{\sqrt{x^2-4}}$

Solution:

1) $x^2 - 4x - 12 = 0 \Rightarrow (x-6)(x+2) = 0 \Rightarrow x = 6, x = -2$

$\therefore D_f = (-\infty, -2) \cup (-2, 6) \cup (6, \infty)$

2) $x - 1 \geq 0 \Rightarrow x \geq 1$

$\therefore D_f = [1, \infty)$

3) $x^2 - 4 > 0 \Rightarrow x^2 > 4$ this is true if $x < -2$ or $x > 2$

$\therefore D_f = (-\infty, -2) \cup (2, \infty)$

Exercises (6.1): Find D_f and R_f of the following functions:

1) $f(x) = -x^2 + 4$ 2) $f(x) = \sqrt{x}$ 3) $f(x) = \sin(x)$

4) $f(x) = \cos^2(x)$

Exercises (1.2): Find D_f of the following functions:

1) $f(x) = \begin{cases} 3 - x & if \quad x \leq 1 \\ 5x - 3 & if \quad x > 1 \end{cases}$ 2) $f(x) = \dfrac{|x|}{x}$

Definition (1.2): Let $f(x)$ be a function with domain D_f and $g(x)$ be a function with domain D_g and define:

$$D = D_f \cap D_g, \text{ then:}$$

1) $(f + g)(x) = f(x) + g(x)$ with domain D

2) $(f - g)(x) = f(x) - g(x)$ with domain D

3) $(f.g)(x) = f(x).g(x)$ with domain D

4) $(f/g)(x) = f(x)/g(x)$ with domain D and $g(x) \neq 0$

Example (1.3): Let $f(x) = 1+\sqrt{x-2}$ and $g(x) = x - 3$, find $(f+g)(x)$, $(f-g)(x)$, $(f.g)(x)$, $(f/g)(x)$ and state the domain of $f+g$, $f-g$, $f.g$, f/g.

1) $(f+g)(x) = f(x) + g(x) = 1+\sqrt{x-2} + x - 3 = x - 2 + \sqrt{x-2}$

2) $(f-g)(x) = f(x) - g(x) = 1+\sqrt{x-2} - (x-3) = 4 - x + \sqrt{x-2}$

3) $(f.g)(x) = f(x).g(x) = (1+\sqrt{x-2})(x-3) = x - 3 + (x-3)\sqrt{x-2}$

4) $(f/g)(x) = f(x)/g(x) = \dfrac{1+\sqrt{x-2}}{x-3}$

$\because f(x) = 1+\sqrt{x-2} \Rightarrow x - 2 \geq 0 \Rightarrow x \geq 2$

$\therefore Df = [2, \infty)$

$\because g(x) = x - 3$

$\therefore Dg = (-\infty, \infty)$

$\therefore D = Df \cap Dg = [2, \infty) \cap (-\infty, \infty) = [2, \infty)$

$\therefore \text{Dom}(f+g, f-g, f.g) = D = [2, \infty)$

$\text{Dom}(f/g) = [2, 3) \cup (3, \infty)$

Exercises (1.3): Let $f(x) = 2\sqrt{x-2}$ and $g(x) = \sqrt{x-2}$, find the domain of $\boldsymbol{f+g, f-g,}$ $\boldsymbol{f.g}$**, and** $\boldsymbol{f/g}$**.**

2.Composition of Function:

Definition (2.1): The composition function $(f \circ g)$ defined by $(f \circ g)(x) = f(g(x))$ the notation $(f \circ g)$ is read (f follows g or the composition of f and g).

$f : X \longrightarrow Y , g : Y \longrightarrow Z \Rightarrow f \circ g : X \longrightarrow Z$

Example (2.1): Let $f(x) = 2x + 1$ and $g(x) = x^2 - x$ find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution:

1) $(f \circ g)(x) = f(g(x)) = f\left(x^2 - x\right) = 2(x^2 - x) + 1 = 2x^2 - 2x + 1$

2) $(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - (2x + 1)$

Example (2.2): Let $f(x) = \sqrt{x - 3}$ and $g(x) = \sqrt{x^2 + 3}$ find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution:

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x^2+3}\right) = \sqrt{\sqrt{x^2+3} - 3}$$

$$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x-3}\right) = \sqrt{(\sqrt{x-3})^2 + 3} = \sqrt{x \cancel{-3} \cancel{+3}} = \sqrt{x}$$

Exercises (2.1): Find $(f \circ g)(x)$ and $(g \circ f)(x)$ for the following

1) $f(x) = x^2 \ , \ g(x) = \sqrt{1-x}$

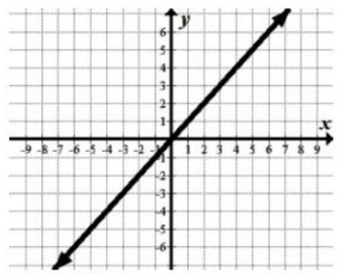
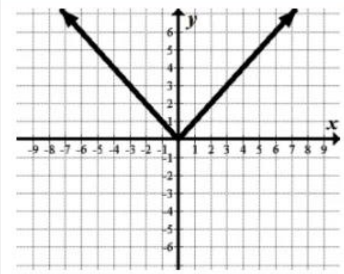
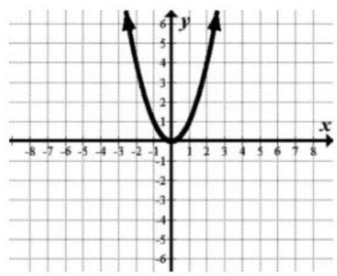
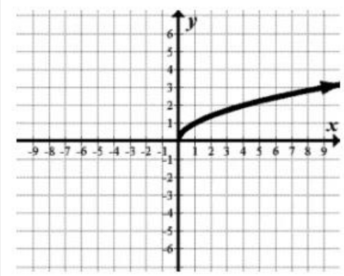
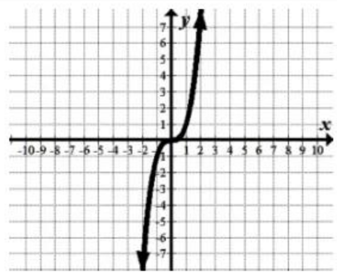
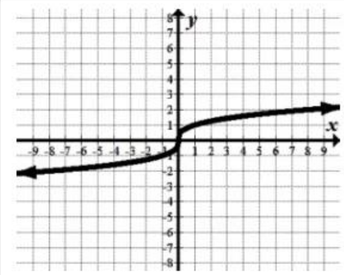
2) $f(x) = \dfrac{1+x}{1-x} \ , \ g(x) = \dfrac{x}{1-x}$

3) $f(x) = \dfrac{x}{1+x^2} \ , \ g(x) = \dfrac{1}{x}$

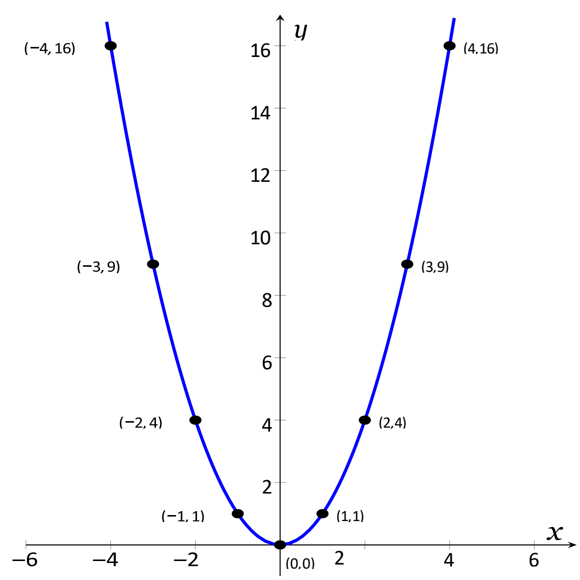
3. Graph of a Function:

A function f establishes a set of ordered pairs (x, y) of real number. The plot of these pairs $(x, f(x))$ in a coordinate system is the graph of f.

Graph of functions: Graph of familiar function

Parent Function	Graph	Parent Function	Graph
$y = x$ **Linear**, Odd Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$ End Behavior**: $x \to -\infty,\ y \to -\infty$ $x \to \infty,\ y \to \infty$ Critical points: $(-1,-1), (0,0), (1,1)$		$y = \lvert x \rvert$ **Absolute Value**, Even Domain: $(-\infty, \infty)$ Range: $[0, \infty)$ End Behavior: $x \to -\infty,\ y \to \infty$ $x \to \infty,\ y \to \infty$ Critical points: $(-1,1), (0,0), (1,1)$	
$y = x^2$ **Quadratic**, Even Domain: $(-\infty, \infty)$ Range: $[0, \infty)$ End Behavior: $x \to -\infty,\ y \to \infty$ $x \to \infty,\ y \to \infty$ Critical points: $(-1,1), (0,0), (1,1)$		$y = \sqrt{x}$ **Radical** (Square Root), Neither Domain: $[0, \infty)$ Range: $[0, \infty)$ End Behavior: $x \to 0,\ y \to 0$ $x \to \infty,\ y \to \infty$ Critical points: $(0,0), (1,1), (4,2)$	
$y = x^3$ **Cubic**, Odd Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$ End Behavior: $x \to -\infty,\ y \to -\infty$ $x \to \infty,\ y \to \infty$ Critical points: $(-1,-1), (0,0), (1,1)$		$y = \sqrt[3]{x}$ **Cube Root**, Odd Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$ End Behavior: $x \to -\infty,\ y \to -\infty$ $x \to \infty,\ y \to \infty$ Critical points: $(-1,-1), (0,0), (1,1)$	

Example (3.1): Sketch a graph of the function $f(x) = x^2$



Solution:

$D_f = \mathrm{R}$

Make a table values of x from the domain.

x	-4	-3	-2	-1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

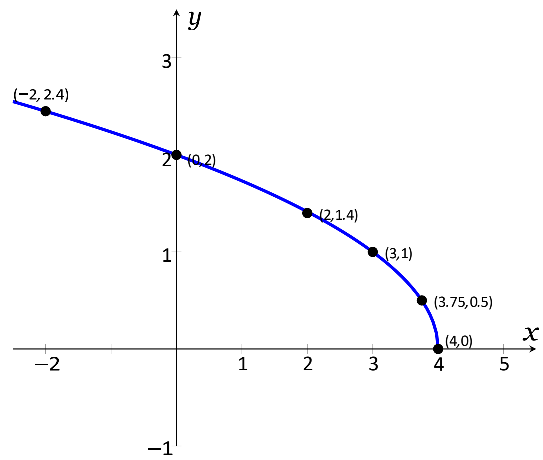
Example (3.2): Sketch a graph of the function $f(x) = \sqrt{4 - x}$

Solution:

$$4 - x \geq 0 \Rightarrow 4 \geq x \Rightarrow D_f = (-\infty, 4]$$

Make a table values of x from the domain.

x	4	3.75	3	2	0	-2
y	0	0.5	1	1.4	2	2.4



Example (3.3): Sketch a graph of the

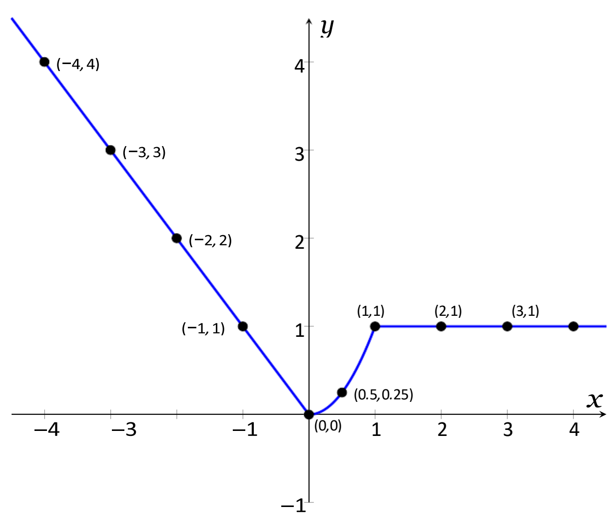
$$f(x) = \begin{cases} -x & if \quad x < 0 \\ x^2 & if \quad 0 \leq x \leq 1 \\ 1 & if \quad x > 1 \end{cases}$$

Solution:

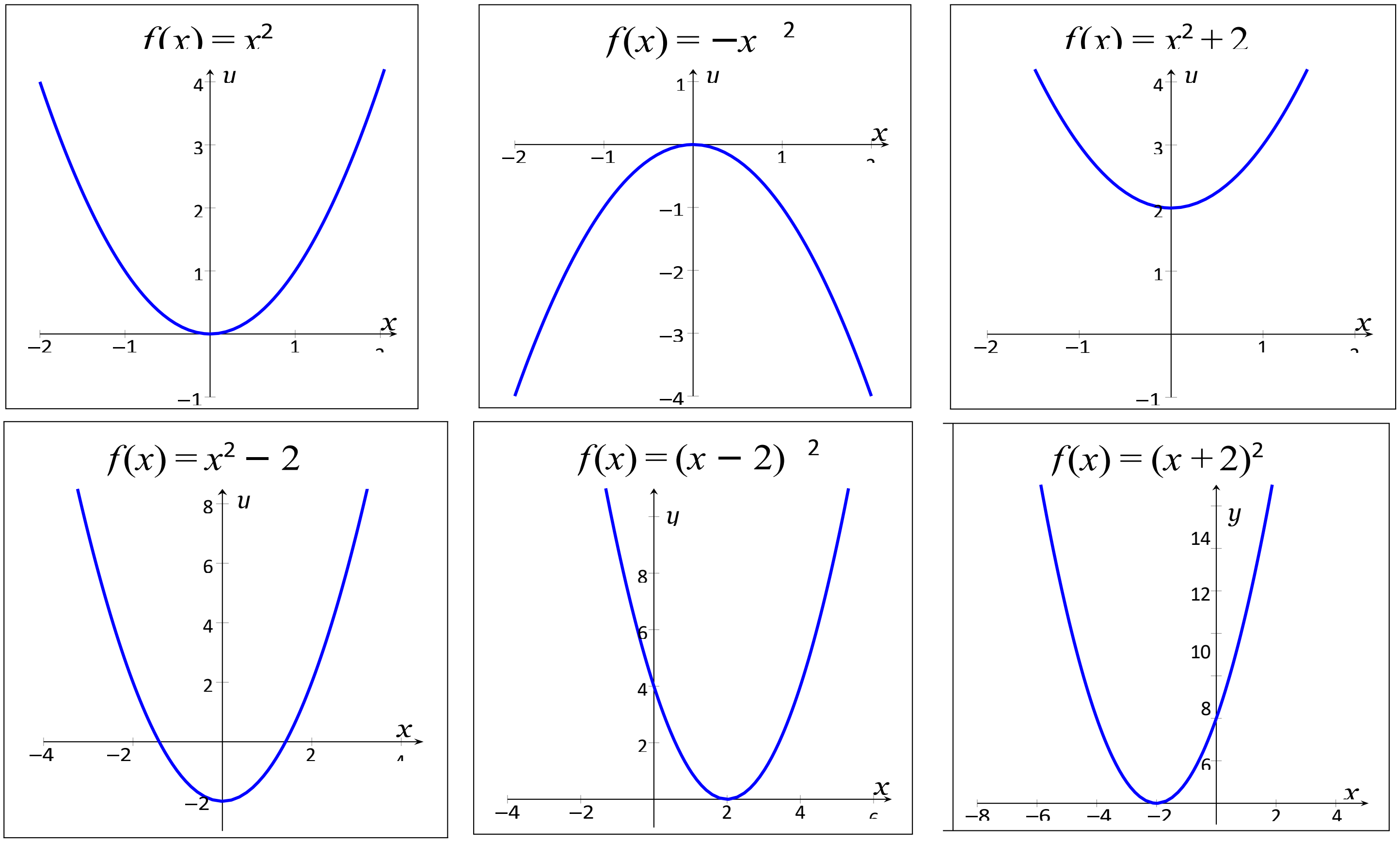
$D_f = R$

Make a table value of x from the domain.

x	-4	-3	-2	-1	0	0.5	1	2	3	4
y	4	3	2	1	0	0.25	1	1	1	1



Remark (3.1):



4. Even Functions and Odd Functions:

Definition (4.1): A function $y = f(x)$ is an even function of x if $f(-x) = f(x)$ for every x in the function's domain. It is odd function of x if $f(-x) = -f(x)$ for every x in the function's domain.

Example (4.1): $f(x) = x^2$ is even function since $f(-x) = (-x)^2 = x^2 = f(x)$

$f(x) = x^3$ is odd function since $f(-x) = (-x)^3 = -x^3 = -f(x)$

Example (4.2): even function (x^2 , x^4, $\frac{1}{x^2}$)

Odd function (x^3, x^5, $\sqrt[3]{x}$, x , $\frac{1}{x}$)

Notes: –

1) The graph of even function is symmetric about the y-axis.
2) The graph of odd function is symmetric about the origin.

5. Test of Symmetric:

To test for various kinds of symmetry we state the following rules:

i. about $x - axis$

replace y by $-y$ $(-y \longrightarrow y)$ in its equation produces an equivalent equation.

ii. about $y - axis$

replace x by $-x$ $(-x \longrightarrow x)$ in its equation produces an equivalent equation.

iii. about the origin point

replace x by $-x$ and y by $-y$ $(-x \longrightarrow x \wedge -y \longrightarrow y)$ in its equation produces an equivalent equation.

Definition (5.1): A line $y = b$ is a horizontal asymptote of the graph of the relation if the distance between the curve and the line $y = b$ tends to zero as the curve continuous upwards beyond all bound.

Definition (5.2): A line $x = a$ is a vertical asymptote of the graph of the relation if the distance between the curve and the line $x = a$ tends to zero as the curve continuous upwards beyond all bound.

* To test a horizontal asymptote, we flow the following:

1- We solve x in terms of y.

2- If x is given of form $x = \frac{r(y)}{t(y)}$ and find all those values of y for which t(y) = 0 and

$r(y) \neq 0$ then the values of y which satisfy $t(y) = 0$ are horizontal asymptotes of the graph.

* To test a vertical asymptote, we flow the following:

1- We solve y in terms of x.

2- If y is given of form $y = \frac{g(x)}{h(x)}$ and find all those values of x for which $h(x) = 0$ and $g(x) \neq 0$ then the values of x which satisfy $h(x) = 0$ are vertical asymptotes of the graph.

1) If y is given of form $y = \frac{g(x)}{h(x)}$ and find all those values of x for which $h(x) = 0$

and $g(x) \neq 0$ then the values of x which satisfy $h(x) = 0$ are vertical asymptotes of the graph.

Example (10.1): Sketch a graph of the following functions:

1) $(x^2 - 4)y^2 = 1$ 2) $x^2y = x - 3$ $(H.W)$

Solution 1: $Dom = (-\infty, -2) \cup (2, \infty)$

Test of Symmetric:

i. about $x-axis$ $(-y \longrightarrow y)$

$\Rightarrow (x^2-4)(-y)^2=1 \Rightarrow (x^2-4)y^2=1$

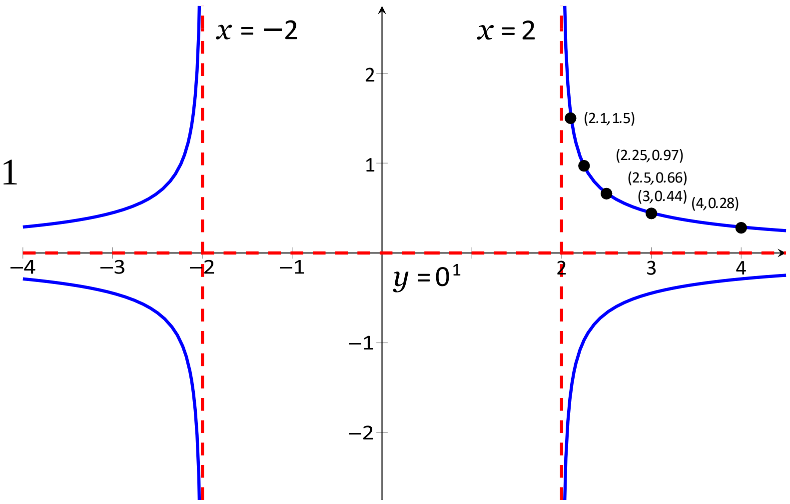
$\therefore$ Symmetric about $x-axis$.

ii. about $y-axis$ $(-x \longrightarrow x)$

$\Rightarrow ((-x)^2-4)y^2=1 \Rightarrow (x^2-4)y^2=1$

$\therefore$ Symmetric about $y-axis$.

iii. From (i) and (ii) we get symmetric about the origin point.



Test of Asymptotes:

1) $(x^2-4)y^2=1 \Rightarrow x^2y^2-4y^2=1 \Rightarrow x^2y^2=1+4y^2 \Rightarrow x=\pm\frac{\sqrt{1+4y^2}}{y}$

$\Rightarrow y=0$ is a horizontal asymptote.

2) $(x^2-4)y^2=1 \Rightarrow y=\pm\sqrt{\frac{1}{x^2-4}} \Rightarrow$ If $\sqrt{x^2-4}=0 \Rightarrow x^2-4=0 \Rightarrow x=\pm 2$

$\therefore x=2$ and $x=-2$ are vertical asymptotes.

x	-4	-3	-2.5	-2.25	-2.1	2.1	2.25	2.5	3	4
y	±0.28	±0.44	±0.66	±0.97	±1.5	±1.5	±0.97	±0.66	±0.44	±0.28

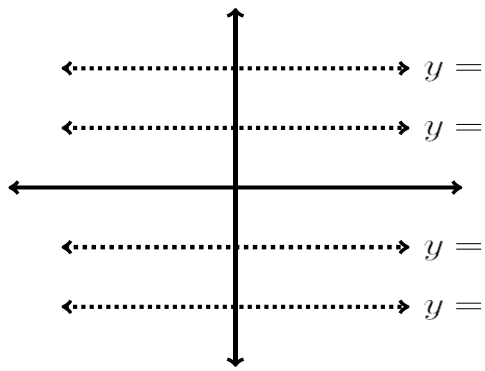
6. Families of a Function:

1. Constant function

Definition: A function f whose values are all the same.

Example: Find the domain and the range for $f(x) = -2$, in this function the Range R_f Is always -2 for all $x \in D_f$.

x	$f(x) = -2$
-2	-2
-1	-2
0	-2
2	-2

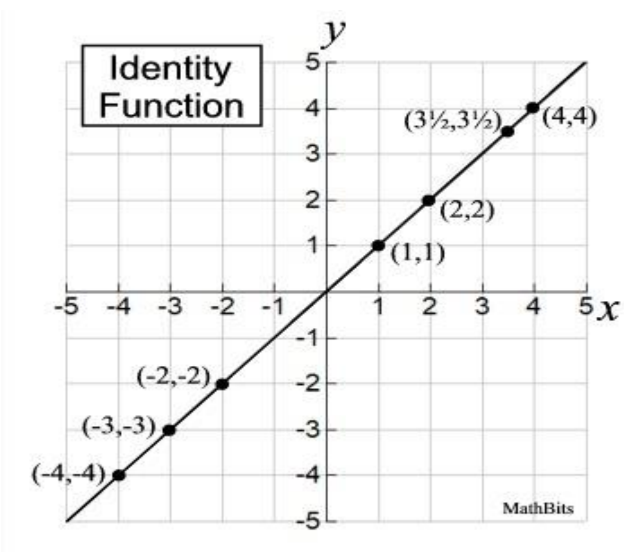


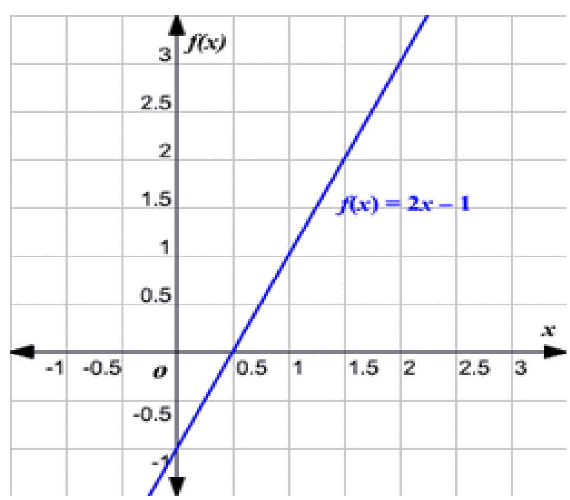
$D_f = \mathbb{R}, \; R_f = \{y : y = -2\}$

2. Linear function

Definition: is a function whose graph is a straight line.

$f(x) = mx + b$



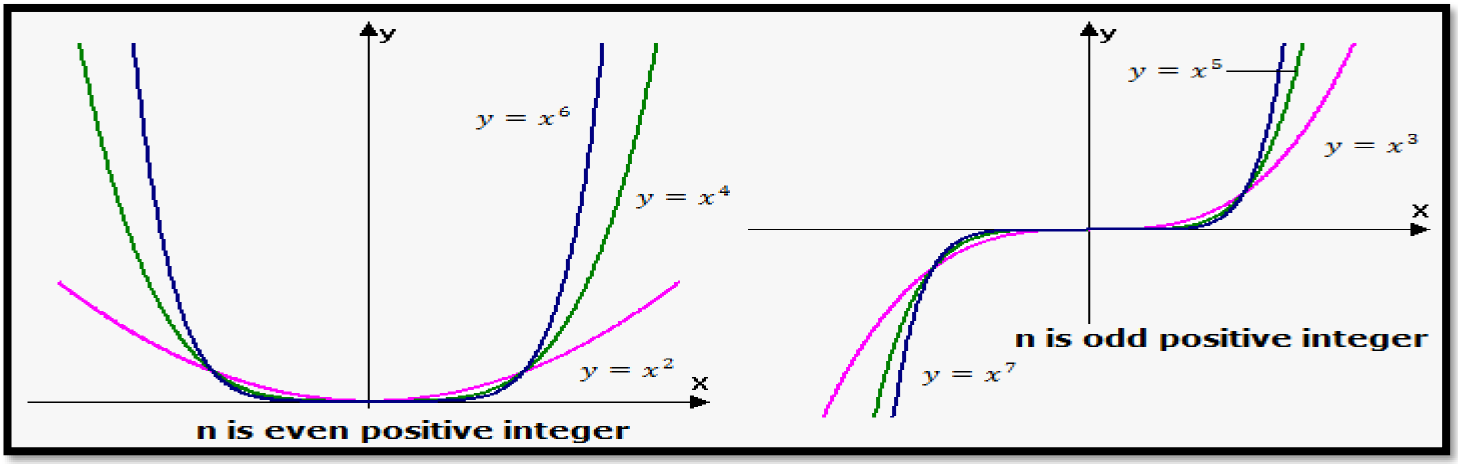


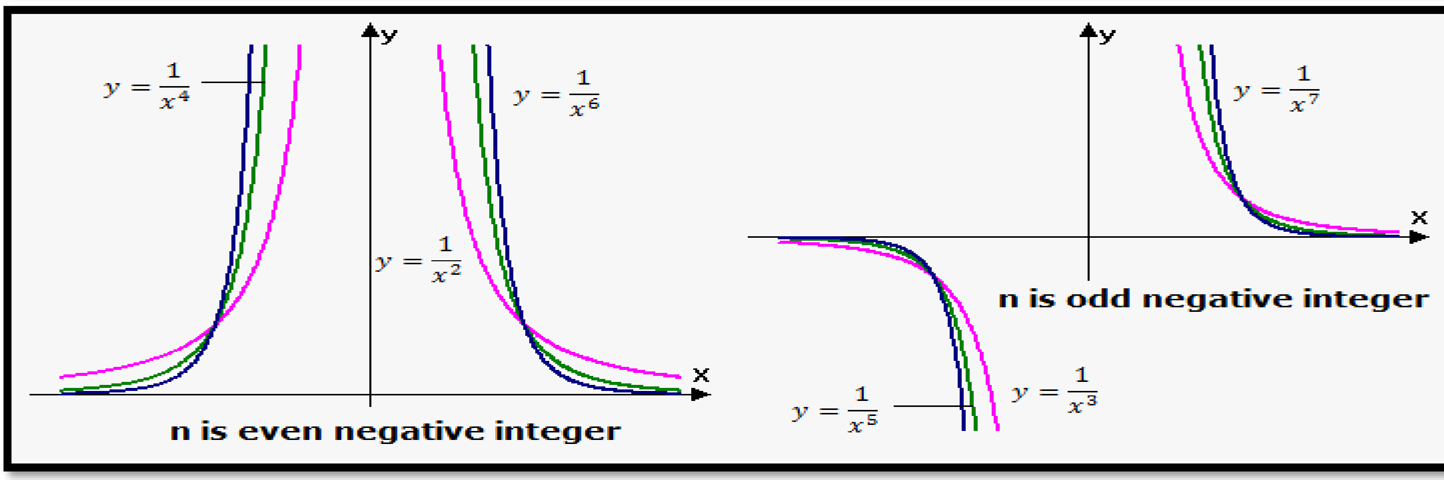
$D_f = \mathbb{R}$

$R_f = \mathbb{R}$

3. Power function

Definition: Function of the form $f(x) = x^n$ where n is constant.





4. Polynomials function

Definition: is a function that involves only non-negative integer powers or only positive integer exponents of a variable in an equation **like the quadratic equation, cubic equation, etc.**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0$$

a: Constant n:1,2,3,…….

Example:

$3+5x$ .... , **degree 1**

x^2-3x+1 ...., **degree 2**

$7- x^2- x^4$ ...., **degree 4**

5. **Rational function**

Definition: is any function that can be defined by a rational fraction, which is an algebraic fraction such that both the numerator and the denominator are polynomials.

$$f(x)=\frac{P(x)}{Q(x)}$$

Example:

$$f(x)=\frac{3x^2-3}{x^2+2x-1} \quad , \quad g(x)=\frac{2x^3}{x}$$

6. **algebraic function**

Definition: is a function that can be defined as the root of an irreducible polynomial equation. Algebraic functions are often algebraic expressions using a finite number of terms, involving only the algebraic operations addition, subtraction, multiplication, division, and raising to a fractional power.

Example:

$$f(x)=\sqrt{x^2+4} \quad , \quad g(x)=3\sqrt[3]{x}$$

7. **trigonometric function**

Definition: are real functions which relate an angle of a right-angled triangle to ratios of two side lengths**.**

α°	0°	30°	45°	60°	90°	180°	270°	360°
α rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan\alpha$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	0	–	0
$\cot\alpha$	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	–	0	–

1) $\sin(\theta) = \frac{y}{r}$

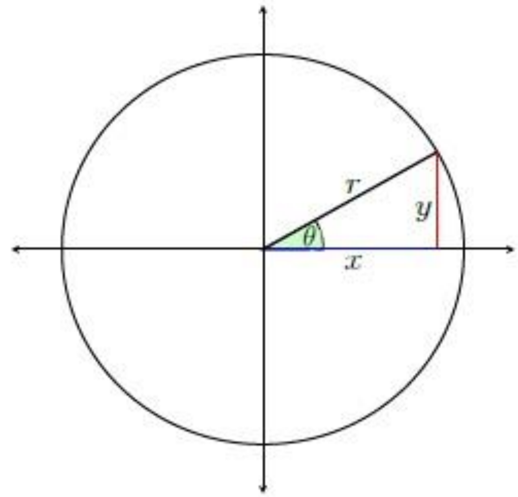
2) $\cos(\theta) = \frac{x}{r}$

3) $\tan(\theta) = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin(\theta)}{\cos(\theta)}$

4) $\cot(\theta) = \frac{x}{y} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{\cos(\theta)}{\sin(\theta)}$

5) $\sec(\theta) = \frac{r}{x} = \frac{1}{\cos(\theta)}$

6) $\csc(\theta) = \frac{r}{y} = \frac{1}{\sin(\theta)}$



7) $\because x^2 + y^2 = r^2 \Rightarrow \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \Rightarrow \cos^2(\theta) + \sin^2(\theta) = 1$

8) $\because x^2 + y^2 = r^2 \Rightarrow \frac{x^2}{y^2} + 1 = \frac{r^2}{y^2} \Rightarrow \cot^2(\theta) + 1 = \csc^2(\theta)$

9) $\because x^2 + y^2 = r^2 \Rightarrow 1 + \frac{y^2}{x^2} = \frac{r^2}{x^2} \Rightarrow 1 + \tan^2(\theta) = \sec^2(\theta)$

Properties of Trigonometric Functions:

1) $\sin(\theta + \frac{\pi}{2}) = \cos(\theta)$

2) $\cos(\theta + \frac{\pi}{2}) = -\sin(\theta)$

3) $\sin(x \mp y) = \sin(x)\cos(y) \mp \sin(y)\cos(x)$

4) $\cos(x \mp y) = \cos(x)\cos(y) \pm \sin(x)\sin(y)$

5) $\sin(2x) = 2\sin(x)\cos(x)$

6) $\cos(2x) = \cos^2(x) - \sin^2(x)$

7) $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

8) $\tan(x \mp y) = \dfrac{\tan(x) \mp \tan(y)}{1 \pm \tan(x)\tan(y)}$

9) $\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

10) $\cos(x)\cos(y) = \frac{1}{2}(\cos(x+y) + \cos(x-y))$

11) $\sin(x)\cos(y) = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

Chapter three

1.limits

If the values of a function $f(x)$ approach the value L as x approaches c, we say f has limit L as x approaches c and we write $\lim_{x\to c} f(x) = L$

Example: Find $\lim_{x\to -2} \frac{4}{x^2}$

$$\lim_{x\to -2} \frac{4}{x^2} = \frac{4}{(-2)^2} = 1$$

x	-2.1	-2.01	-2.001	-2.0001	$\cdots$	-2	$\cdots$	-1.999	-1.99	-1.9
$f(x)$	0.90702	0.99007	0.99900	0.99990	$\cdots$	1	$\cdots$	1.0010	1.0100	1.1080

left side $\longrightarrow$ $\longleftarrow$ right side

Theorem (1):

If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, then

1) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$

2) $\lim_{x\to a}(f(x) \times g(x)) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x) = A \times B$

3) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$, $B \neq 0$

4) $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) = kA$, k is constant

5) $\lim_{x\to a} k = k$, where k is constant

6) $\lim_{x\to a} x = a$

7) $\lim_{x\to a} x^n = \left(\lim_{x\to a} x\right)^n = a^n$

8) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{A}$, $A > 0$ if n is even

Example (1.1): Find $\lim_{x\to 5}(x^2 - 4x + 3)$

Solution:

$\lim_{x\to 5}(x^2 - 4x + 3) = \lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3 = \lim_{x\to 5} x^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3 = 25 - 20 + 3 = 8$

Example (1.2): Find $\lim_{x\to 2}(\frac{5x^3+4}{x-3})$

Solution:

$\lim_{x\to 2}(\frac{5x^3+4}{x-3}) = \frac{\lim_{x\to 2} 5x^3+4}{\lim_{x\to 2} x-3} = \frac{40+4}{2-3} = \frac{44}{-1} = -44$

Example (1.3): Find $\lim_{x\to 5}(\frac{x^2-25}{x-5})$

Solution:

$\lim_{x\to 5}(\frac{x^2-25}{x-5}) = \lim_{x\to 5}(\frac{(x-5)(x+5)}{x-5}) = \lim_{x\to 5}(x+5) = 5+5 = 10$

Exercises (1): Find the following limits:

1) $\lim_{x\to 1}\frac{x^3-1}{x-1}$
2) $\lim_{x\to 0}\frac{5x^2-4}{x+1}$
3) $\lim_{x\to 4}\frac{x^2-x-12}{x-4}$
4) $\lim_{x\to 3}\frac{x^2-6x+9}{x-3}$
5) $\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}$
6) $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$

1.1 Right-Hand and Left-Hand Limits:

Let *f(x)* be a function then the right-hand limit defined as = $\lim_{x\to a+} f(x)$ (the limit of *f*(*x*) as *x* approaches a from the right). and the left-hand limit defend as $\lim_{x\to a-} f(x)$ (the limit of *f(x)* as *x* approaches a from the left).

Remark (1.1):

$\lim_{x\to a} f(x)$= L if and only if $\lim_{x\to a-} f(x)$ = L = $\lim_{x\to a+} f(x)$

Example (1) Find $\lim_{x\to 3}[x]$

Solution:

$$\lim_{x\to 3^-}[x] = 2 \text{ and } \lim_{x\to 3^+}[x] = 3 \Rightarrow \lim_{x\to 3^-}[x] \neq \lim_{x\to 3^+}[x]$$

$\therefore$ the limit dose not exists.

Example (2) : If $f(x) = \begin{cases} 4 - x^2 & \text{if} \quad x \leq 1 \\ 2 + x^2 & \text{if} \quad x > 1 \end{cases}$ find $\lim_{x\to 1} f(x)$

Solution:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(4 - x^2) = 4 - 1 = 3$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2 + x^2) = 2 + 1 = 3$$

$$\because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 3 \Rightarrow \lim_{x\to 1} f(x) = 3$$

Theorem (2):

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

Example (2.1): Find $\lim_{x\to 0}\frac{1-\cos(x)}{x}$

Solution

$$\lim_{x\to 0}\frac{1-\cos(x)}{x} \times \frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0}\frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x\to 0}\frac{\sin(x)}{x} \times \lim_{x\to 0}\frac{\sin(x)}{1+\cos(x)} = 0$$

Example (2.2): Find $\lim_{x\to 0}\frac{15\sin(x)}{7x}$

Solution

$$\lim_{x\to 0}\frac{15\sin(x)}{7x} \times \frac{15}{15} = \frac{15}{7}\lim_{x\to 0}\frac{\sin(15x)}{15x} = \frac{15}{7} \times 1 = \frac{15}{7}$$

Exercises (2.1): Find the following limits:

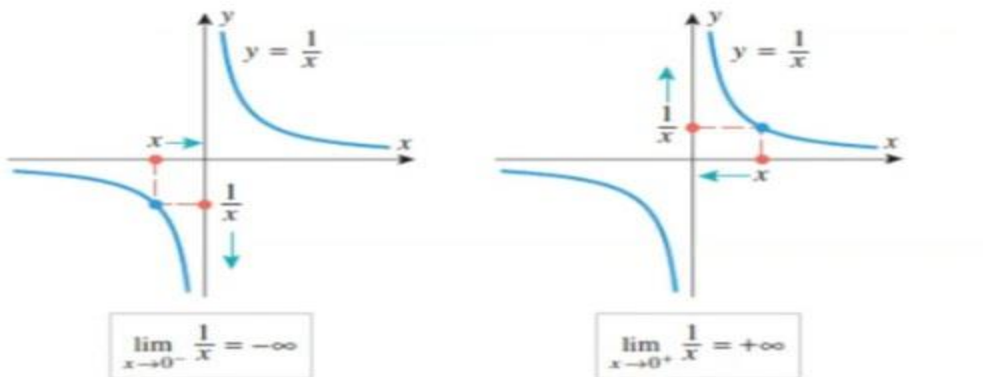
1) $\lim_{x\to 0}\frac{\sin(x)}{\sqrt{x}}$　　2) $\lim_{x\to 0} x\cot(x)$　　3) $\lim_{y\to 0}\frac{1-\cos(y)}{y^2}$

4) $\lim_{t\to 0}\frac{\tan(t)}{2t}$　　5) $\lim_{x\to 0}\frac{2x+1-\cos(x)}{3x}$　　6) $\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}$

1.2 Limits at Infinity

⊛ We say that $\lim_{x\to+\infty} f(x) = L$ if for any positive number ϵ we can find a positive number N such that $|f(x) - L| < \epsilon$ for all $x > N$.

⊛ We say that $\lim_{x\to-\infty} f(x) = L$ if for any positive number ϵ we can find a positive number N such that $|f(x) - L| < \epsilon$ for all $x < -N$.



Theorem (3):

1) $\lim_{x\to+\infty} \frac{1}{x} = 0$ 2) $\lim_{x\to-\infty} \frac{1}{x} = 0$

Theorem (4):

$\lim_{x\to+\infty} x^n = +\infty$, $n = 1, 2, 3,..$

$$\lim_{x\to-\infty} x^n = \begin{cases} -\infty & n = 1,3,5,.. \\ +\infty & n = 2,4,6,.. \end{cases}$$

Example (4.1): Find $\lim_{x\to-\infty} \frac{x^2}{2x^2+1}$

Solution:

$$\lim_{x\to-\infty} \frac{\frac{x^2}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}} = \lim_{x\to-\infty} \frac{1}{2+\frac{1}{x^2}} = \frac{1}{2+\frac{1}{\infty}} = \frac{1}{2+0} = \frac{1}{2}$$

Example (4.2): Find $\lim_{x\to\infty} \sqrt[3]{\frac{3x+5}{6x\ \ -8}}$

Solution:

$$\lim_{x\to\infty} \sqrt[3]{\frac{3x+5}{6x-8}} = \sqrt[3]{\lim_{x\to\infty} \frac{3x+5}{6x-8}} = \sqrt[3]{\lim_{x\to\infty} \frac{3+\frac{5}{x}}{6-\frac{8}{x}}} = \sqrt[3]{\frac{1}{2}}$$

Exercises (4): Find the following limits:

1) $\lim_{x\to\infty} (\sqrt{x^6+5} - x^3)$　　2) $\lim_{x\to\infty} \frac{7x-4}{\sqrt{x^3+5}}$　　3) $\lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5}$

4) $\lim_{x\to\infty} \frac{\sqrt{5x^2-2}}{x+3}$　　5) $\lim_{x\to-\infty} -4x^8$

Theorem (5):

If $g(x) \le f(x) \le h(x)$ for all x such that $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$, where L is constant then $\lim_{x\to\infty} f(x) = L$

Example (5.1): prove $\lim_{x\to\infty} \frac{sin\ (x)}{x} = 0$

Proof:

Since $-1 \le \sin(x) \le 1 \Rightarrow \frac{-1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$

$\because \lim_{x\to\infty} \frac{-1}{x} = 0$ and $\lim_{x\to\infty} \frac{1}{x} = 0$

$\therefore \lim_{x\to\infty} \frac{\sin(x)}{x} = 0$

Example (5.2): : Find $\lim_{x\to\infty} \frac{\cos^2(2x)}{4x^2}$　　$(\mathcal{H}.\mathcal{W})$

Example (5.3): Find $\lim_{x\to-\infty} \left(1+\frac{2}{x}\right) \cos\left(\frac{1}{x}\right)$

Solution:

$$\lim_{x\to-\infty} \left(1+\frac{2}{x}\right) \cos\left(\frac{1}{x}\right) = 1$$

Example (5.4): Find $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Solution:

Let $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$, at $x \to \infty$ then $y \to 0$

$\therefore \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{y\to 0} \frac{\sin(y)}{y} = 1$

2. Continuity

Definition (2.1): A function *f* is said to be continuous at *x* = *c* provided the following conditions are satisfied:

i. $f(c)$ is defined

ii. $\lim_{x\to c} f(x)$ exists

iii. $\lim_{x\to c} f(x) = f(c)$

Example (2.1): Determine whether the following functions are continuous or not at *x* = 2.

$1)\ f(x) = \frac{x^2-4}{x-2}$ $\qquad 2)\ g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if} \quad x \neq 2 \\ 3 & \text{if} \quad x = 2 \end{cases}$

$3)\ h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if} \quad x \neq 2 \\ 4 & \text{if} \quad x = 2 \end{cases}$

Solution:

$1)\ f(2) = \frac{4-4}{2-2} = \frac{0}{0}$ not defined

$\therefore f(x)$ is discontinuous

2)

i. $g(2) = 3$

ii. $\lim\limits_{x \to 2} g(x) = \lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x-2)(x+2)}{x-2} = 4$ exists

iii. $\lim\limits_{x \to 2} g(x) \neq g(2)$

$\therefore g(x)$ is discontinuous

3)

i. $h(2) = 4$

ii. $\lim\limits_{x \to 2} h(x) = \lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x-2)(x+2)}{x-2} = 4$ exist

iii. $\lim\limits_{x \to 2} h(x) = h(2)$

$\therefore h(x)$ is continuous

Theorem (2.1):

Every polynomial functions are continuous.

Theorem (2.2):

A Rational functions are continuous at every number where the denominator is non zero.

Theorem (2.3):

If the functions f and g are continuous at c, then:

1) $f \mp g$ is continuous at c
2) $f.g$ is continuous at c
3) f/g is continuous at c *if* $g(c) \neq 0$

Example (2.2): Show that whether the function $f(x) = \frac{x^2-9}{x^2-5x+6}$ continuous or not?

Solution:

$x^2 - 5x + 6 = 0 \Rightarrow (x-3)(x-2) = 0 \Rightarrow x = 3,\ x = 2$

$\therefore f(x)$ continuous at every points except $x = 3$ and $x = 2$

Exercises (2): Show that whether the following functions are continuous or not?

1) $g(x) = |x|$ at $x = 0$

2) $f(x) = \begin{cases} x^2 + 2x + 1 & \text{if} \quad x \geq 1 \\ 3x + 1 & \text{if} \quad x < 1 \end{cases}$ at $x = 1$

Theorem (2.4):
The functions *sin(x)* and *cos(x)* are continuous functions.

Theorem (2.5):
i. If the function *g(x)* is continuous at *c*, and *f(x)* continuous at *g(c)*, then $f \circ g$ is continuous at *c*.

ii. If the function *g* is continuous everywhere and the function *f* is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.

Example (2.3): Show that the function $h(x) = \left(\frac{x \sin(x)}{x^2 + 2}\right)^2$ is continuous at every value of *x*.

Solution:

$f(x) = x^2$ and $g(x) = \frac{x \sin(x)}{x^2 + 2}$

$g_1(x) = \frac{x}{x^2 + 2}$ and $g_2(x) = \sin(x)$

∵ *f(x)* is continuous (*by* Theorem (2.1))

Since *g1(x)* is continuous (*by* Theorem (2.2))

and *g2(x)* is continuous (*by* Theorem (2.4))

∴ *g(x)* is continuous (*by* Theorem (2.3))

∴ $(f \circ g)(x) = \left(\frac{x \sin(x)}{x^2 + 2}\right)^2$ (by Theorem (2.5))

∴ *h(x)* is continuous.

Chapter four

1. Derivative:

The derivative of a function *f* is the function *f′* whose value at *x* is defined *by* the equation:

$$\frac{df}{dx} = \frac{d}{dx}f(x) = \frac{dy}{dx} = y' = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Definition (1.1):

A function that has a derivative at a point *x* is said to be differentiable at *x*.

Definition (1.2):

A function that is differentiable at every point of its domain is called differentiable.

Definition (1.3):

When the number *f′(x)* exists it is called the slope of the curve *y = f(x) at x*.

The line through the point (*x*, *f(x)*) with *slope f′(x)* is the tangent to the curve at *x*.

Example (1.1): Find $\frac{dy}{dx}$ by definition for the following functions:

1) $y = x^2 + 2x + 1$ 2) $y = \sqrt{x^2+3}$

Solution: 1

$$y' = \frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2+2(x+h)+1-x^2-2x-1}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2+2x+2h+1-x^2-2x-1}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h+2)}{h} = 2x+2$$

Solution: 2

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{(x+h)^2+3}-\sqrt{x^2+3}}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^2+3-(x^2+3)}{h(\sqrt{(x+h)^2+3}+\sqrt{x^2+3})} = \lim_{h\to 0}\frac{x^2+2xh+h^2+3-x^2-3}{h(\sqrt{(x+h)^2+3}+\sqrt{x^2+3})}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h(\sqrt{(x+h)^2+3}+\sqrt{x^2+3})} = \frac{2x}{\sqrt{x^2+3}+\sqrt{x^2+3}} = \frac{2x}{2\sqrt{x^2+3}} = \frac{x}{\sqrt{x^2+3}}$$

Exercises (1.1): Find $\dfrac{dy}{dx}$ by definition for the following functions:

1) $y = x^3 + 3$ 2) $y = \dfrac{x+1}{x-1}$ 3) $y = \sqrt{x}$ 4) $y = \dfrac{1}{x}$ 5) $y = \dfrac{1}{\sqrt{x+1}}$

Differentiation Theorem:

1) $\dfrac{d}{dx}(c) = 0$, c is constant.

2) $\dfrac{d}{dx}(cf(x)) = c\dfrac{d}{dx}(f(x))$

3) $\dfrac{d}{dx}(f(x) \mp g(x)) = \dfrac{d}{dx}(f(x)) \mp \dfrac{d}{dx}(g(x))$

4) $\dfrac{d}{dx}(f(x) \times g(x)) = f(x) \times \dfrac{d}{dx}(g(x)) + g(x) \times \dfrac{d}{dx}(f(x))$

5) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x) \times \frac{d}{dx}(f(x)) - f(x) \times \frac{d}{dx}(g(x))}{(g(x))^2}$, $g(x) \neq 0$

6) $\dfrac{d}{dx}(f(x))^n = n \times (f(x))^{n-1} \times \dfrac{d}{dx}(f(x))$

7) $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Example (1.2): Find f' by definition for the following functions:

1) $f(x) = x + \dfrac{1}{x^2} + 3$ 2) $f(x) = \sqrt{x^3 - 2} + \dfrac{1}{\sqrt{x+1}}$ 3) $f(x) = (x^2+1)^3(x^3-1)^2$

Solution: 1

$$f'(x) = 1 - \frac{2x}{x^4} = 1 - \frac{2}{x^3}$$

Solution: 2

$$f(x) = (x^3-2)^{\frac{1}{2}} + (x+1)^{-\frac{1}{2}}$$

$$\therefore f'(x) = \frac{1}{2}(x^3-2)^{-\frac{1}{2}} \times 3x^2 - \frac{1}{2}(x+1)^{-\frac{3}{2}} \times 1 = \frac{3}{2}\frac{x^2}{\sqrt{x^3-2}} - \frac{1}{2}\frac{1}{\sqrt{(x+1)^3}}$$

Solution: 3

$$f'(x) = (x^2+1)^3 \times 2(x^3-1) \times 3x^2 + (x^3-1)^2 \times 3(x^2+1)^2 \times 2x$$

$$= 6x^2(x^2+1)^3(x^3-1) + 6x(x^3-1)^2(x^2+1)^2$$

Exercises (1.2): Find $f'$by definition for the following functions:

1) $f(x) = \left(\dfrac{x+1}{x^2-2}\right)^3$ 2) $f(x) = x^2 + \dfrac{1}{x^2}$ 3) $f(x) = \dfrac{x^2+1}{x^2-1}$, $x^2 \neq 1$

4) $f(x) = (x-1)^3(x+2)^4$ 5) $f(x) = \dfrac{x^3-1}{\sqrt{x+1}}$ 6) $f(x) = (x^2+1)^8$

7) $f(x) = (x+1)^2(x^2+1)^{-3}$

1.1 Second and Higher-Order Derivative:

If the derivative f' of a function f itself differentiable then the derivative of f' is denoted by f'' and is called the second derivative of f.

$$i.e: f'(x) = \frac{d}{dx}(f(x))$$

$$f''(x) = \frac{d^2}{dx^2}(f(x)) = \frac{d}{dx}\left[\frac{d}{dx}(f(x))\right]$$

$$f'''(x) = \frac{d^3}{dx^3}(f(x)) = \frac{d^2}{dx^2}\left[\frac{d}{dx}(f(x))\right]$$

$$\vdots$$

$$f^{(n)}(x) = \frac{d^n}{dx^n}(f(x))$$

Example (1.3): Find $f^{(5)}(x)$ where $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$

Solution:

$f'(x) = 12x^3 - 6x^2 + 2x - 4$

$f''(x) = 36x^2 - 12x + 2$

$f'''(x) = 72x - 12$

$f^{(4)}(x) = 72$

$f^{(5)}(x) = 0$

Exercises (1.3): Find $\dfrac{d^4y}{dx^4}$ where $y = \dfrac{3}{x^3}$

Theorem (1.1):

If f has a derivative at $x = c$, then f is continuous at c.

1.2 Chain Rule:

i. If y is a differentiable function of u and u is a differentiable function of x then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

ii. If y is a differentiable function of u and x is a differentiable function of u then,

$$\frac{dy}{dx} = \frac{dy/du}{dx/du}$$

Example (1.4): If $y = t^4 + 2t + 3$, $x = t^2 + 1$ find $\frac{dy}{dx}$

Solution:

$\frac{dy}{dt} = 4t^3 + 2$ and $\frac{dx}{dt} = 2t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3+2}{2t} = \frac{2t^3+1}{t} = \frac{2(\sqrt{x-1})^3+1}{\sqrt{x-1}} = \frac{2(x-1)^{\frac{3}{2}}+1}{\sqrt{x-1}}$$

Example (1.5): If $y = \frac{u^3+1}{u^3-2}$, $u = \sqrt{x+1}$ find $\frac{dy}{dx}$

Solution:

$$\frac{dy}{du} = \frac{(u^3-2).3u^2 - (u^3+1).3u^2}{(u^3-2)^2} = \frac{3u^5 - 6u^2 - 3u^5 - 3u^2}{(u^3-2)^2} = \frac{-9u^2}{(u^3-2)^2}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-9u^2}{(u^3-2)^2} \times \frac{1}{2\sqrt{x+1}} = \frac{-9(x+1)}{\left((x+1)^{\frac{3}{2}}-2\right)^2} \times \frac{1}{2\sqrt{x+1}}$$

$$= \frac{-9\sqrt{x+1}}{2\left((x+1)^{\frac{3}{2}}-2\right)^2}$$

Exercises (1.4):

1) If $y = t^2 + 2t$, $t = \dfrac{x-2}{3-x}$ find $\dfrac{dy}{dx}$

2) If $y = \sqrt{t} + \dfrac{1}{\sqrt{t}}$, $x = t^2 + 2t$ find $\dfrac{dy}{dx}$

3) If $x = \sqrt{t} - t^3$, $y = t^{\frac{2}{3}} + t^2$ find $\dfrac{dy}{dx}$

4) If $y = s^2$, $s = r + 1$, $r = t^2 - 5$, $t = w + 3$, $w = x^2$ find $\dfrac{dy}{dx}$

1.3 Implicit differentiation:

If y can not be written in the form $y = f(x)$ then to find $\dfrac{dy}{dx}$:

i. Differentiable both sides with respect to x.

ii. Solve the result for $\dfrac{dy}{dx}$.

Example (1.6): Find $\dfrac{dy}{dx}$ for the functions

1) $x^3 + y^3 = 3xy$

Solution:

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y \Rightarrow 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

$$\Rightarrow \frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$$

2) $xy + y^2x + 3y - 2x = 0$

Solution:

$$\Rightarrow x\frac{dy}{dx} + y + y^2 + 2yx\frac{dy}{dx} + 3\frac{dy}{dx} - 2 = 0$$
$$\therefore \frac{dy}{dx} = \frac{2 - y - y^2}{x + 2xy + 3}$$

3) $\dfrac{1}{yx^2} + \dfrac{1}{yx} = y + x$

Solution:

$$\Rightarrow (yx^2)^{-1} + (yx)^{-1} = y + x$$
$$\Rightarrow -(yx^2)^{-2}\left(2yx + x^2\frac{dy}{dx}\right) - (yx)^{-2}\left(y + x\frac{dy}{dx}\right) = \frac{dy}{dx} + 1$$
$$\Rightarrow -2yx(yx^2)^{-2} - x^2(yx^2)^{-2}\frac{dy}{dx} - y(yx)^{-2} - x(yx)^{-2}\frac{dy}{dx} = \frac{dy}{dx} + 1$$
$$\Rightarrow \frac{dy}{dx}\left(-x^2(yx^2)^{-2} - x(yx)^{-2} - 1\right) = 2yx(yx^2)^{-2} + y(yx)^{-2} + 1$$
$$\therefore \frac{dy}{dx} = \frac{2yx(yx^2)^{-2} + y(yx)^{-2} + 1}{-x^2(yx^2)^{-2} - x(yx)^{-2} - 1}$$

Exercises (1.5): Find $\dfrac{dy}{dx}$ if

1) $x^2y^2 + \dfrac{x}{y} = 0$　　　2) $\dfrac{x^2y}{x-y} = \dfrac{3x}{4+y}$

3) $\dfrac{1}{x} + \dfrac{1}{y} = 1$　　　4) $xy^2 = \dfrac{x+y}{x-y}$

5) $y = \sqrt{\sqrt{x} + \sqrt{x^2 + \sqrt{x}}}$

1.4 Derivatives of Trigonometric Functions:

1) $\dfrac{d}{dx}(\sin(u)) = \cos(u).\dfrac{du}{dx}$

2) $\frac{d}{dx}(\cos(u)) = -\sin(u).\frac{du}{dx}$

3) $\frac{d}{dx}(\tan(u)) = \sec^2(u).\frac{du}{dx}$

4) $\frac{d}{dx}(\cot(u)) = -\csc^2(u).\frac{du}{dx}$

5) $\frac{d}{dx}(\sec(u)) = \sec(u)\tan(u).\frac{du}{dx}$

6) $\frac{d}{dx}(\csc(u)) = -\csc(u)\cot(u).\frac{du}{dx}$

Example (1.7): Find $\frac{dy}{dx}$ or $f'(x)$ if

1) $f(x) = \tan(3x^2)$

Solution:

$f'(x) = 6x\sec^2(3x^2)$

2) $y = \sin(2x) + \sec(3x)$

Solution:

$\Rightarrow \frac{dy}{dx} = 2\cos(2x) + 3\sec(3x)\tan(3x)$

3) $y = \cos(\sqrt{x})$

Solution:

$\Rightarrow \frac{dy}{dx} = -\sin(\sqrt{x}).\frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x}}\sin(\sqrt{x})$

4) $y^2 = x^2 + \sin(xy)$

Solution:

$2y\frac{dy}{dx} = 2x + \cos(xy)\left(x\frac{dy}{dx} + y\right) \Rightarrow 2y\frac{dy}{dx} = 2x + x\cos(xy)\frac{dy}{dx} + y\cos(xy)$

$\Rightarrow 2y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = 2x + y\cos(xy)$

$\therefore \frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$

5) $xy = \csc(x-y)$

Solution:

$$x\frac{dy}{dx} + y = -\csc(x-y)\cot(x-y)\left(1 - \frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dy}{dx} = \frac{-y - \csc(x-y)\cot(x-y)}{x - \csc(x-y)\cot(x-y)}$$

Exercises (1.6): Find $\frac{dy}{dx}$ by definition for the following functions:

1) $y^2x = \cos^3(x-y)^2$

2) $y = x^2\tan(x^2)$

3) $y = \cot\left(\frac{\sin^2(x)}{\tan(x)}\right)$

4) $yx^2 = \sin^4(x^3)$

5) $y = \tan^2(x)\cot^2(1-x)$

6) $y = \tan^2(x)\cot^2(x)$

1.5 Derivatives of the Inverse Trigonometric Functions:

1) $\frac{d}{dx}(\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}}.\frac{du}{dx}$

2) $\frac{d}{dx}(\cos^{-1}(u)) = \frac{-1}{\sqrt{1-u^2}}.\frac{du}{dx}$

3) $\frac{d}{dx}(\tan^{-1}(u)) = \frac{1}{1+u^2}.\frac{du}{dx}$

4) $\frac{d}{dx}(\cot^{-1}(u)) = \frac{-1}{1+u^2}.\frac{du}{dx}$

5) $\frac{d}{dx}(\sec^{-1}(u)) = \frac{1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$

6) $\frac{d}{dx}(\csc^{-1}(u)) = \frac{-1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$

Proof: 1

$$\because \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let $y = \sin^{-1}(u) \Rightarrow \sin(y) = u$

$$\Rightarrow \cos(y).\frac{dy}{dx}=\frac{du}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{\cos(y)}.\frac{du}{dx}$$

$$\because \sin(y)=u\Rightarrow \sin^2(y)=u^2\Rightarrow 1-\sin^2(y)=1-u^2\Rightarrow \cos^2(y)=1-u^2$$

$$\Rightarrow \sqrt{\cos^2(y)}=\sqrt{1-u^2}\Rightarrow \cos(y)=\sqrt{1-u^2}$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\sin^{-1}(u))=\frac{1}{\sqrt{1-u^2}}.\frac{du}{dx}$$

Example (1.8): Find $\frac{dy}{dx}$ if

1) $y=\sin^{-1}(3x^2)$

Solution:

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-(3x^2)^2}}.6x=\frac{6x}{\sqrt{1-9x^4}}$$

2) $y=\tan^{-1}(3\tan(x))$

Solution:

$$\Rightarrow \frac{dy}{dx}=\frac{1}{1+(3\tan(x))^2}.3\sec^2(x)=\frac{3\sec^2(x)}{1+9\tan^2(x)}$$

3) $y=\sec^{-1}(2x^2)$

Solution:

$$\Rightarrow \frac{dy}{dx}=\frac{1}{|2x^2|\sqrt{(2x^2)^2-1}}.4x=\frac{4x}{2x^2\sqrt{4x^4-1}}$$

Exercises (1.7): Find $\frac{dy}{dx}$ for the following functions:

1) $y=\tan^{-1}\left(\sqrt{x+1}\right)$

2) $y=x\cos^{-1}(3x)$

3) $y=\cot^{-1}\left(\frac{x}{2}\right)+\tan^{-1}\left(\frac{x}{2}\right)$

4) $y=\cot^{-1}\left(\frac{x+1}{1-x}\right)$

1.6 Derivatives of the Logarithmic and Exponential Functions:

$$1)\ \frac{d}{dx}(\ln(u)) = \frac{1}{u}.\frac{du}{dx}$$

$$2)\ \frac{d}{dx}(\log_a(u)) = \frac{1}{u\ln(a)}.\frac{du}{dx}$$

$$3)\ \frac{d}{dx}(a^u) = a^u\ln(a).\frac{du}{dx}$$

$$4)\ \frac{d}{dx}(e^u) = e^u.\frac{du}{dx}$$

Example (1.9): Find $\frac{dy}{dx}$ by definition for the following functions:

1) $y = \ln(x^3)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x^3}.3x^2 = \frac{3}{x}$$

2) $y = \ln(\sin^{-1}(2x))$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin^{-1}(2x)}.\frac{2}{\sqrt{1-4x^2}} = \frac{2}{\sin^{-1}(2x)\sqrt{1-4x^2}}$$

3) $y = (100)^{x^2+2x}$

$$\Rightarrow \frac{dy}{dx} = (100)^{x^2+2x}\ln(100).(2x+2)$$

4) $y = e^{\sin(x)}$

$$\Rightarrow \frac{dy}{dx} = e^{\sin(x)}\cos(x) = \cos(x)e^{\sin(x)}$$

5) $y = x\log_3 x$

$$\Rightarrow \frac{dy}{dx} = \cancel{x}.\left(\frac{1}{\cancel{x}\ln(3)}\right) + \log_3 x = \frac{1}{\ln(3)} + \log_3 x$$

6) $y = e^{\ln(x)+x}$

$$\Rightarrow \frac{dy}{dx} = e^{\ln(x)+x}.\left(\frac{1}{x}+1\right) = e^{\ln(x)}e^x\left(\frac{1+x}{x}\right)$$

$$= \cancel{x}e^x\left(\frac{1+x}{\cancel{x}}\right) = e^x(1+x)$$

Exercises (1.8): Find $\frac{dy}{dx}$ by definition for the following functions:

1) $y = e^{\ln(x)-\ln(1+x)}$ 2) $y = \ln\left(\frac{1}{x}\right)$ 3) $y = e^{\ln\left(\frac{1}{x^2}\right)}$ 4) $y = \frac{\log_3 x^2}{\log_2 x}$

Example (1.10): Find $\frac{dy}{dx}$ by definition for the following functions:

1) $y = (\sin(x))^{\cos(x)}$

Solution:

$$\Rightarrow \ln(y) = \ln(\sin(x))^{\cos(x)}$$

$$\Rightarrow \ln(y) = \cos(x)\ln(\sin(x))$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos(x).\frac{1}{\sin(x)}.\cos(x) + \ln(\sin(x)).(-\sin(x))$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos(x)\cot(x) - \sin(x)\ln(\sin(x))$$

$$\Rightarrow \frac{dy}{dx} = y(\cos(x)\cot(x) - \sin(x)\ln(\sin(x)))$$

$$= (\sin(x))^{\cos(x)}(\cos(x)\cot(x) - \sin(x)\ln(\sin(x)))$$

2) $y^x = x^y$

Solution:

$$\ln(y^x) = \ln(x^y) \Rightarrow x\ln(y) = y\ln(x) \Rightarrow x.\frac{1}{y}\frac{dy}{dx} + \ln(y) = y\frac{1}{x} + \ln(x)\frac{dy}{dx}$$

$$\frac{x}{y}\frac{dy}{dx} - \ln(x)\frac{dy}{dx} = \frac{y}{x} - \ln(y) \Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x} - \ln(y)}{\frac{x}{y} - \ln(x)}$$

Exercises (1.9): Find $\frac{dy}{dx}$ by definition for the following functions:

1) $y = (x)^{\sin(x)}$ 2) $y = x^x$ 3) $y = x^{x^2}$

2. L'Hôpital's Rule:

Suppose that $f(x_\circ) = g(x_\circ)$ and that the functions f and g are both differentiable on an open interval (a, b) that contains the point $x_\circ$.

Suppose also $g' \neq 0$ at every point in (a, b) except possibly $x_\circ$, then

$$\lim_{x \to x_\circ} \frac{f(x)}{g(x)} = \lim_{x \to x_\circ} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

i. The Form $\left(\frac{0}{0} \ \& \ \frac{\infty}{\infty}\right)$

Example (2.1): Find $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \frac{0}{0}$

Solution:

$$\lim_{x \to 0} \frac{(1+x)^{\frac{1}{2}}-1}{x} = \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1} = \frac{1}{2}$$

Example (2.2): Find $\lim_{x \to 0} \frac{1-\cos(x)}{x+x^2} = \frac{0}{0}$

Solution:

$$\lim_{x \to 0} \frac{1-\cos(x)}{x+x^2} = \lim_{x \to 0} \frac{\sin(x)}{1+2x} = \frac{0}{1} = 0$$

Example (2.3): Find $\lim_{x \to 3} \frac{x^4-81}{x-3} = \frac{0}{0}$

Solution:

$$\lim_{x \to 3} \frac{x^4-81}{x-3} = \lim_{x \to 3} \frac{4x^3}{1} = 108$$

Example (2.4): Find $\lim_{x \to \infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$

Solution:

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

Exercises (2.1): Find

1) $\lim\limits_{x \to a} \dfrac{\sec(x) - \sec(a)}{x - a}$ 2) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$

3) $\lim\limits_{\theta \to 0} \dfrac{\sin(2\theta) - 2\sin(\theta)}{\sin(3\theta) - 3\sin(\theta)}$ 4) $\lim\limits_{x \to -\frac{\pi}{2}} \dfrac{\tan(x)}{1 + \tan(x)}$

ii. The Form $(0.\infty \ \& \ \infty - \infty)$

Example (2.5): Find $\lim\limits_{x \to \infty} x^2 e^{-x} = 0.\infty$

Solution:

$$\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

Example (2.6): Find $\lim\limits_{x \to 0} \left(\csc(x) - \dfrac{1}{x}\right) = \infty - \infty$

Solution:

$$\lim_{x \to 0} \left(\csc(x) - \frac{1}{x}\right) = \lim_{x \to 0} \left(\frac{x - \sin(x)}{x\sin(x)}\right) = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1 - \cos(x)}{x\cos(x) + \sin(x)} = \lim_{x \to 0} \frac{\sin(x)}{\cos(x) + \cos(x) - x\sin(x)} = \frac{0}{2} = 0$$

iii. The Form $(0^0 \ , \ \infty^0 \ , \ 1^\infty)$

Example (2.7): Find $\lim\limits_{x \to 0} (\cos(x))^{\frac{1}{x^2}} = 1^\infty$

Solution:

$$\text{Let } y = (\cos(x))^{\frac{1}{x^2}} \Rightarrow \ln(y) = \frac{1}{x^2}\ln(\cos(x)) \Rightarrow \lim_{x \to 0} \ln(y) = \lim_{x \to 0} \frac{1}{x^2}\ln(\cos(x))$$

$$= \lim_{x \to 0} \frac{-\sin(x)}{2x\cos(x)} = \lim_{x \to 0} \frac{-\cos(x)}{2(\cos(x) - x\sin(x))} = -\frac{1}{2}$$

$$\Rightarrow \lim_{x \to 0} \ln(y) = -\frac{1}{2} \Rightarrow \ln\left(\lim_{x \to 0} y\right) = -\frac{1}{2}$$

$$\Rightarrow e^{\ln\left(\lim\limits_{x \to 0} y\right)} = e^{-\frac{1}{2}} \Rightarrow \lim_{x \to 0} y = e^{-\frac{1}{2}}$$

$$\therefore \lim_{x \to 0} (\cos(x))^{\frac{1}{x^2}} = e^{-\frac{1}{2}}$$

Example (2.8): Find $\lim_{x\to\frac{\pi}{2}}(\sin(x)-\cos(x))^{\tan(x)}=1^{\infty}$

Solution:

Let $y=(\sin(x)-\cos(x))^{\tan(x)} \Rightarrow \ln(y)=\tan(x)\ln(\sin(x)-\cos(x))$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\ln(y)=\lim_{x\to\frac{\pi}{2}}\tan(x)\ln(\sin(x)-\cos(x))$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\sin(x)\ln(\sin(x)-\cos(x))}{\cos(x)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\sin(x)\left(\frac{\cos(x)+\sin(x)}{\sin(x)-\cos(x)}\right)+\cos(x)\ln(\sin(x)-\cos(x))}{-\sin(x)}=-1$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\ln(y)=-1 \Rightarrow \lim_{x\to\frac{\pi}{2}}y=e^{-1}$$

$$\therefore \lim_{x\to\frac{\pi}{2}}(\sin(x)-\cos(x))^{\tan(x)}=\frac{1}{e}$$

Exercises (2.2): Prove that

1) $\lim_{x\to\frac{\pi}{2}}(\tan(x))^{\cos(x)}=1$

2) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

Exercises (2.3): Find

1) $\lim_{x\to\frac{\pi}{2}}(2x-\pi)\sec(x)$

2) $\lim_{x\to 0}(\sec^3(x))^{\cot^2(x)}$

3) $\lim_{x\to 1}(1-x)^{\ln(x)}$